

## Research Article

# Partial Synchronizability Characterized by Principal Quasi-Submatrices Corresponding to Clusters

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A partial synchronization problem in an oscillator network is considered. The concept on a principal quasi-submatrix corresponding to the topology of a cluster is proposed for the first time to study partial synchronization. It is shown that partial synchronization can be realized under the condition depending on the principal quasi-submatrix, but not distinctly depending on the intercluster couplings. Obviously, the dimension of any principal quasi-submatrix is usually far less than the one of the network topology matrix. Therefore, our criterion provides us a novel index of partial synchronizability, which reduces the network size when the network is composed of a great amount of nodes. Numerical simulations are carried out to confirm the validity of the method.

## 1. Introduction

Since the pioneering work of Pecora and Carroll [1], extensive researches on chaos synchronization have been carried out due to their potential applications in various disciplines such as physics, engineering, and biology [2–5]. For example, wireless sensor networks are widely researched due to the important applications in the real scenarios. One of the biggest challenges in investigating wireless sensor networks is how to obtain higher synchronization accuracy with minimal overhead [6]. And many other network-induced phenomena have also been discussed extensively in engineering [7].

Up to date, there have been many synchronization types being proposed and discussed, including complete synchronization [8, 9], partial synchronization [10], and inner and outer synchronization [11]. Generally speaking, there exists an intimate relationship between the phenomena of synchronization and invariant manifolds of coupled systems [9–12]. Thus, when we carry out researches on partial synchronization of coupled systems, it is always supposed that the corresponding full or partial synchronization manifolds

are invariant manifolds. The established tools for the stability of invariant synchronization manifolds mainly consist of local linearization method and Lyapunov function method. One of the prominent results of the former is the master stability function method [9]. The method has obtained two significant factors determining the local stability of the synchronous state, that is, the maximum Lyapunov exponent of the node dynamics and the eigenvalues of the topology matrix. One of the prominent results of the latter is the study of the global attractiveness of the synchronization manifold. A crucial requirement for the method is the condition of the node dynamics satisfying  $f \in \text{QUAD}(\Delta, P, \Omega)$  [10]. In some sense, the condition means that the system can synchronize when the coupling is made sufficiently large. Recently, some new conditions have been obtained for synchronization of networks without Lipschitz condition or QUAD condition [12].

The type of synchronization concerned in this paper is partial synchronization. It means that the coupled oscillators split into subgroups called clusters, and all the oscillators in the same cluster behave in the same fashion. Many relative

studies have been carried out [13–18]. By using pinning control strategy, partial synchronization of coupled stochastic delayed neural networks was discussed in [14]. Similar control strategy is also proposed to select controlled communities by analyzing the information of each community such as in-degrees and out-degrees [15]. Afterwards, some simple intermittent pinning controls and centralized adaptive intermittent controls are proposed [16]. However, a suitable control law must be presented in order to use pinning control strategy. Some other researches focused on partial synchronization induced by the coupling configuration. An arbitrarily selected partial synchronization manifold was constructed for a network with cooperative and competitive couplings [17]. Recently, a sufficient and necessary condition for partial synchronization manifolds being invariant manifolds was obtained in networks coupled linearly and symmetrically [10]. More significantly, some sufficient conditions for the global attractiveness of the partial synchronization manifold were derived by decomposing the whole space into a direct sum of the synchronization manifold and the transverse space. The results are meaningful and interesting. Based on the method in [10], partial synchronization bifurcations [19] were analyzed for a globally coupled network with a parameter, which topology is not complex. Nevertheless, all the eigenvalues of the topology matrix are essential for that method, which needs a large quantity of computation when the network size is very large.

In this paper, a novel criterion on partial synchronization is proposed through the analysis of principal quasi-submatrices corresponding to the clusters. Previous researches have obtained several criteria on partial synchronization [10]. However, these criteria depend heavily on the topology matrix of the whole network. For many complex networks in real world, it is tedious to obtain the eigenvalues and eigenvectors of the topology matrix of the whole network. Therefore, this paper aims to propose a novel criterion, which is not distinctly dependent on the intercluster couplings and the topology matrix of the whole network. Instead, it is sufficient for partial synchronization to verify that the conditions on the intracluster couplings are satisfied. Therefore, it will be advantageous to discuss partial synchronization in the complex network with a large number of nodes because the dimension of any principal quasi-submatrix is usually far less than the one of the network topology matrix. At first, the tedious work of solving the eigenvectors corresponding to the second-largest eigenvalue is avoided. Secondly, partial synchronization is studied by the inner topologies of the individual clusters. Obviously, the network size reduction provides convenience for the study of partial synchronization in networks with great mounts of oscillators. In order to confirm its effectiveness, some numerical simulations are carried out to study partial synchronization in a star-global network. The numerical simulations are in good agreement with the theoretical analysis.

The present paper is built up as follows. Some concerned concepts and a lemma for the invariance of the partial synchronization manifold are introduced in Section 2. Principal quasi-submatrices corresponding to the clusters are proposed, and our main results are then established

in Section 3. Numerical examples are presented to confirm the effectiveness of the results in Section 4. Finally, a brief discussion of the obtained results is given in Section 5.

## 2. Preliminaries

In this section, we introduce some basic concepts of invariant synchronization manifolds and a related lemma, which are required throughout the paper.

In past years, many studies [9, 10, 12] of synchronization phenomena focused on oscillator networks coupled linearly and symmetrically. The system can be described by the following ordinary differential equations:

$$\dot{x}_i = f(x_i, t) + \varepsilon \sum_{j=1}^m a_{ij} \Gamma x_j, \quad i = 1, \dots, m, \quad (1)$$

where  $x_i = (x_i^1, \dots, x_i^n)^T$  is the  $n$ -dimensional state variable of the  $i$ th oscillator,  $m > 1$  is the network size,  $t \in [0, +\infty)$  is a continuous time,  $f: R^n \times [0, +\infty) \rightarrow R^n$  is a continuous map,  $\varepsilon > 0$  is the coupling strength, and  $\Gamma = \text{diag}\{\gamma_1, \dots, \gamma_n\}$  is a nonnegative matrix determining the interaction of variables. The coupling weight matrix  $A = (a_{ij})_{m \times m}$  is assumed to satisfy that  $a_{ij} = a_{ji} \geq 0$ , for  $i \neq j$ , and  $\sum_{j=1}^m a_{ij} = s$  for  $i = 1, \dots, m$ .

In order to study partial synchronization of the system (1), the set of nodes  $\{1, \dots, m\}$  is divided into  $d$  nonempty subsets (clusters). Let  $G = \{G^1, \dots, G^d\}$  be the partition, and denote  $K = (K_1, K_2, \dots, K_d)$ , where  $K_k \geq 1$  is the cardinal number of the cluster  $G^k$ ,  $k = 1, \dots, d$ . Without loss of generality, suppose that  $G^1 = \{1, \dots, K_1\}, \dots, G^d = \{\sum_{p=1}^{d-1} K_p + 1, \dots, m\}$ . Based on the partition  $G$ , we rewrite the coupling matrix  $A$  as a block matrix,

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1d} \\ A_{21} & A_{22} & \cdots & A_{2d} \\ \cdots & \cdots & \cdots & \cdots \\ A_{d1} & A_{d2} & \cdots & A_{dd} \end{bmatrix}, \quad (2)$$

where  $A_{kl} \in R^{K_k \times K_l}$ ,  $k, l = 1, \dots, d$ .

We will discuss sufficient conditions for the  $K_k$  nodes in the cluster  $G^k$  to synchronize with each other,  $k = 1, \dots, d$ . Before that, the concepts of partial synchronization manifolds and transverse spaces introduced in the following [10]:

$$\mathbb{M}_K^k = \left\{ (x_1^T, \dots, x_m^T)^T \mid \left( x_{\sum_{p=0}^{k-1} K_p + 1}^T, \dots, x_{\sum_{p=0}^k K_p}^T \right)^T \in \mathbb{M}_K^k, \right. \\ \left. k = 1, \dots, d \right\}, \quad (3)$$

where  $K_0 = 0$ ,

$$\mathbb{M}_K^k = \left\{ \left( x_{\sum_{p=0}^{k-1} K_p + 1}^T, \dots, x_{\sum_{p=0}^k K_p}^T \right)^T \right. \\ \left. \in R^{nK_k} \mid x_{\sum_{p=0}^{k-1} K_p + 1} = \dots = x_{\sum_{p=0}^k K_p} \right\}, \quad (4)$$

are called a partial synchronization manifold. We also call

$$\mathbb{L}_K = \left\{ (x_1^\top, \dots, x_m^\top)^\top \mid \left( x_{\sum_{p=0}^{k-1} K_p+1}^\top, \dots, x_{\sum_{p=0}^k K_p}^\top \right)^\top \in \mathbb{L}_K^k, \right. \\ \left. k = 1, \dots, d \right\}, \quad (5)$$

where

$$\mathbb{L}_K^k = \left\{ \left( x_{\sum_{p=0}^{k-1} K_p+1}^\top, \dots, x_{\sum_{p=0}^k K_p}^\top \right)^\top \right. \\ \left. \in R^{nK_k} \mid \sum_{i=1}^{K_k} x_{\sum_{p=0}^{k-1} K_p+i} = 0 \right\}, \quad (6)$$

a transverse space for  $\mathbb{M}_K$ .

In case  $n = 1$ , the four sets mentioned above are denoted by  $M_K, M_K^k, L_K,$  and  $L_K^k$ , respectively. In case  $d = 1$ , the synchronization manifold  $\mathbb{M}_K$  is called a full synchronization manifold. For simplicity, we denote the full synchronization manifold  $\mathbb{M}_K$  by  $\mathbb{M}$ , and the corresponding transverse space  $\mathbb{L}_K$  by  $\mathbb{L}$ . Sometimes, the manifold  $\mathbb{M}_K$  is also denoted by  $\mathbb{M}_K(G)$  to emphasize its partition  $G$ .

*Definition 1.* The synchronization manifold  $\mathbb{M}_K$  is said to be globally attractive for the system (1), or the system (1) is said to realize partial synchronization with the partition  $G = \{G^1, G^2, \dots, G^d\}$  if, for any initial condition  $(x_1^\top(0), x_2^\top(0), \dots, x_m^\top(0))^\top \in R^{mn}$ ,

$$\lim_{t \rightarrow +\infty} \sum_{k=1}^d \sum_{i \in G^k} \left\| x_i(t) - x_{\sum_{p=0}^{k-1} K_p+1}(t) \right\| = 0, \quad (7)$$

where  $\|\cdot\|$  denotes 2-norm of vectors. In case of  $d = 1$ , the system (1) is said to realize full synchronization, if the full synchronization manifold  $\mathbb{M}$  is globally attractive.

Synchronization manifolds are always supposed to be invariant in order to discuss the attractiveness. Therefore, it is necessary to recall the definition of an invariant manifold [20] for the ordinary differential equations,

$$\dot{x} = X(x, t), \quad x \in R^N, \quad X : R^N \times [0, +\infty) \rightarrow R^N. \quad (8)$$

Denote  $\mathcal{M}$  as the manifold of codimension  $p$  defined by a vector equation  $H(x) = 0, H = (h_1, h_2, \dots, h_p), 1 \leq p \leq N - 1$ .  $\mathcal{M}$  is called an invariant manifold of the system (8) if

$$(\text{grad } H \cdot X)|_{H=0} \equiv 0, \quad (9)$$

which implies that the vector field (8) is tangent to  $\mathcal{M}$ . For more details of the existence of invariant manifolds, one can refer to the papers by Golubitsky and coworkers [21, 22]. The following lemma gives a sufficient and necessary condition for a partial synchronization manifold being an invariant manifold.

**Lemma 2** (see [10]). *Let  $K = (K_1, K_2, \dots, K_d)$ . The synchronization manifold  $\mathbb{M}_K$  is an invariant manifold of the system (1) if and only if the coupling matrix  $A$  has the form (2) with every  $A_{ij}$  being equal row sum.*

*Remark 3.* Based on Lemma 2, we can find all invariant synchronization manifolds for a given coupling matrix. As we know, each diagonal block reveals the intracluster information communication, and each nondiagonal block represents the information communication among different clusters. By the condition that every  $A_{ij}$  is equal row sum, we mean that every node in the same cluster receive an equal amount of information communication from every other cluster.

### 3. Main Results

Noticing the sufficient and necessary condition in Lemma 2, we assume that every submatrix  $A_{kl} \in R^{K_k \times K_l}$  is an equal row sum matrix with row sum  $s_{kl}, k, l = 1, \dots, d$ . Since  $A^s$  is an equal row sum matrix, that is,  $\sum_{j=1}^m a_{ij} = \sum_{k=1}^d s_{ik} = s$ , where the denotation  $\hat{i}$  represents  $k$ , for all  $i \in G^k$ , we define the matrices  $\bar{A}_{kk} = (\bar{a}_{ij})_{K_k \times K_k}$  as

$$\bar{A}_{kk} = A_{kk} + (s - s_{kk})E_{K_k} = A_{kk} + \sum_{l=1, l \neq k}^d s_{kl}E_{K_k}, \quad (10)$$

where  $E_{K_k} \in R^{K_k \times K_k}$  is the identity matrix,  $k = 1, \dots, d$ . It is easy to conclude from (10) that  $\sum_{j \in G^{\hat{i}}} \bar{a}_{ij} = s$  and

$$\bar{a}_{ij} = \begin{cases} a_{ij}, & i \neq j; \\ a_{ij} + \sum_{l=1, l \neq \hat{i}}^d s_{il}, & i = j. \end{cases} \quad (11)$$

Noticing that  $A_{kk}$  is a principal submatrix of  $A$ , we call the matrix  $\bar{A}_{kk}$  a principal quasi-submatrix corresponding to the cluster  $G^k, k = 1, \dots, d$ .

Now, we are now in a position to carry out the following theorem with the help of Lyapunov function method.

**Theorem 4.** *Let  $K = (K_1, K_2, \dots, K_d), P = \text{diag}(p_1, \dots, p_n)$  be a positive-definite diagonal matrix, and let  $\Delta = \text{diag}(\delta_1, \dots, \delta_n)$  be a diagonal matrix. Suppose  $\delta_j \leq 0$  if  $j \notin J$ , where  $J = \{j : 1 \leq j \leq n, \gamma_j \neq 0\}$ . Then under the following three conditions.*

- (i) *Every submatrix  $A_{kl}$  in the block matrix (2) has equal row sum  $s_{kl}$ , and every principal submatrix  $A_{kk}$  is irreducible.*
- (ii) *There exists a constant  $\epsilon > 0$  such that, for any  $u, v \in R^n$  and all  $t \geq 0$ ,*

$$(u - v)^\top P \{ [f(u, t) - f(v, t)] - \Delta(u - v) \} \\ \leq -\epsilon(u - v)^\top (u - v). \quad (12)$$

- (iii) *For all  $j = 1, \dots, n, k = 1, \dots, d$ , the matrices  $\epsilon\gamma_j \bar{A}_{kk} + \delta_j E_{K_k}$  are negative semidefinite in the transverse space  $L_K^k$ , that is,*

$$z^\top (\epsilon\gamma_j \bar{A}_{kk} + \delta_j E_{K_k}) z \leq 0, \quad z \in L_K^k, \quad (13)$$

or, in particular,

$$\varepsilon\gamma_j\lambda_k^{(2)} + \delta_j \leq 0, \quad (14)$$

where  $\lambda_k^{(2)}$  is the second-largest eigenvalue of  $\bar{A}_{kk}$ .

The synchronization manifold  $\mathbb{M}_K$  is globally attractive for the system (1).

For convenience of the proof, the following notations are introduced.

$$\begin{aligned} \bar{x}_k(t) &= \frac{1}{K_k} \sum_{i \in G^k} x_i(t), & \bar{x}(t) &= [\bar{x}_1^\top(t), \dots, \bar{x}_m^\top(t)]^\top, \\ \delta x_i(t) &= x_i(t) - \bar{x}_i(t), \\ \delta x(t) &= [\delta x_1^\top(t), \dots, \delta x_m^\top(t)]^\top, \\ \delta \bar{x}_k^s(t) &= \left[ \delta x_{\sum_{p=0}^{k-1} K_p+1}^s(t), \dots, \delta x_{\sum_{p=0}^k K_p}^s(t) \right]^\top, \\ \delta \bar{x}_k(t) &= [\delta \bar{x}_k^{1\top}(t), \dots, \delta \bar{x}_k^{m\top}(t)]^\top, \end{aligned} \quad (15)$$

where  $k = 1, \dots, d$ ,  $i = 1, \dots, m$ ,  $s = 1, \dots, n$ .

Then any vector  $x = (x_1^\top, \dots, x_m^\top)^\top \in R^{mm}$  can be decomposed into

$$x = \bar{x} \oplus \delta x, \quad \bar{x} \in \mathbb{M}_K, \quad \delta x \in \mathbb{L}_K. \quad (16)$$

Therefore, the attractiveness of the invariant synchronization manifold  $\mathbb{M}_K$  is equivalent to  $\delta x \rightarrow 0$  when  $t \rightarrow +\infty$ ; that is, the dynamical flow in the transverse subspace  $\mathbb{L}_K$  converges to zero.

*Proof.* Noticing the condition (i), one gets

$$\begin{aligned} \sum_{j=1}^m a_{ij} \Gamma x_j(t) &= \sum_{l=1}^d \sum_{j \in G^l} a_{ij} \Gamma [\delta x_j(t) + \bar{x}_l(t)] \\ &= \sum_{l=1}^m a_{ij} \Gamma \delta x_j(t) + \sum_{l=1}^d s_{\bar{l}} \Gamma \bar{x}_l(t). \end{aligned} \quad (17)$$

Therefore,

$$\begin{aligned} \frac{d\delta x_i(t)}{dt} &= \frac{dx_i(t)}{dt} - \frac{1}{K_i} \sum_{p \in G^i} \frac{dx_p(t)}{dt} \\ &= f(x_i(t), t) - f(\bar{x}_i(t), t) + \varepsilon \sum_{j=1}^m a_{ij} \Gamma \delta x_j(t) + J_{\bar{i}}, \end{aligned} \quad (18)$$

where

$$\begin{aligned} J_{\bar{i}} &= f(\bar{x}_i(t), t) - \frac{1}{K_i} \sum_{p \in G^i} \left[ f(x_p(t), t) + \varepsilon \sum_{q=1}^m a_{pq} \Gamma x_q(t) \right] \\ &\quad + \varepsilon \sum_{l=1}^d s_{\bar{l}} \Gamma \bar{x}_l(t). \end{aligned} \quad (19)$$

In order to utilize the QUAD( $\Delta, P, R^n$ ) condition, a Lyapunov function is defined as follows:

$$V(\delta x(t)) = \frac{1}{2} \sum_{i=1}^m \delta x_i^\top(t) P \delta x_i(t). \quad (20)$$

One can conclude from  $\sum_{i \in G^k} \delta x_i(t) = 0$  that

$$\sum_{i=1}^m \delta x_i^\top(t) P J_{\bar{i}} = \sum_{k=1}^d \left[ \sum_{i \in G^k} \delta x_i^\top(t) \right] P J_{\bar{i}} = 0, \quad (21)$$

and then,

$$\begin{aligned} \frac{dV(\delta x_i(t))}{dt} &= \sum_{i=1}^m \delta x_i^\top(t) P \left[ f(x_i(t), t) - f(\bar{x}_i(t), t) + \varepsilon \sum_{j=1}^m a_{ij} \Gamma \delta x_j(t) \right] \\ &\leq -\varepsilon \sum_{i=1}^m \delta x_i^\top(t) \delta x_i(t) \\ &\quad + \sum_{i=1}^m \delta x_i^\top(t) P \left[ \varepsilon \sum_{j=1}^m a_{ij} \Gamma \delta x_j(t) + \Delta \delta x_i(t) \right]. \end{aligned} \quad (22)$$

Denote the second term in the right hand of (22) as  $S$  for convenience; then,

$$\begin{aligned} S &= \sum_{k=1}^d \sum_{i \in G^k} \delta x_i^\top(t) P \left[ \varepsilon \sum_{l=1}^d \sum_{j \in G^l} a_{ij} \Gamma \delta x_j(t) + \Delta \delta x_i(t) \right] \\ &= \sum_{k=1}^d \sum_{i \in G^k} \delta x_i^\top(t) P \left[ \varepsilon \sum_{j \in G^k} a_{ij} \Gamma \delta x_j(t) + \Delta \delta x_i(t) \right. \\ &\quad \left. + \varepsilon \sum_{l=1, l \neq k}^d \sum_{j \in G^l} a_{ij} \Gamma \delta x_j(t) \right]. \end{aligned} \quad (23)$$

Since  $i, j \in G^k$  holds for the first term in the right hand above, replacing  $a_{ij}$  with  $\bar{a}_{ij}$  according to (11) gives rise to that

$$\begin{aligned} S &= \sum_{k=1}^d \sum_{i \in G^k} \delta x_i^\top(t) P \left[ \varepsilon \sum_{j \in G^k} \bar{a}_{ij} \Gamma \delta x_j(t) + \Delta \delta x_i(t) \right] \\ &\quad + \varepsilon \sum_{k=1}^d \sum_{i \in G^k} \delta x_i^\top(t) P \sum_{l=1, l \neq k}^d \left[ -s_{\bar{l}} \Gamma \delta x_i(t) + \sum_{j \in G^l} a_{ij} \Gamma \delta x_j(t) \right] \\ &= S_1 + S_2. \end{aligned} \quad (24)$$

As a result of the condition (13), one obtains that

$$\begin{aligned} S_1 &= \sum_{k=1}^d \sum_{i \in G^k} \delta x_i^\top(t) P \left[ \varepsilon \sum_{j \in G^k} \bar{a}_{ij} \Gamma \delta x_j(t) + \Delta \delta x_i(t) \right] \\ &= \sum_{k=1}^d \sum_{s=1}^n p_s \delta \tilde{x}_k^{s\top}(t) (\varepsilon \gamma_s \bar{A}_{kk} + \delta_s E_{K_k}) \delta \tilde{x}_k^s(t) \leq 0. \end{aligned} \quad (25)$$

Since  $s_{\bar{l}} = \sum_{j \in G^l} a_{ij}$ , the sum  $S_2$  can be decomposed into

$$\begin{aligned} S_2 &= \varepsilon \sum_{k=1}^d \sum_{i \in G^k} \delta x_i^\top(t) P \sum_{l=1, l \neq k}^d \sum_{j \in G^l} a_{ij} \Gamma (\delta x_j(t) - \delta x_i(t)) \\ &= \varepsilon \sum_{k=1}^{d-1} \sum_{l=k+1}^d \sum_{i \in G^k} \sum_{j \in G^l} a_{ij} \delta x_i^\top(t) P \Gamma (\delta x_j(t) - \delta x_i(t)) \\ &\quad + \varepsilon \sum_{l=1}^{d-1} \sum_{k=l+1}^d \sum_{i \in G^k} \sum_{j \in G^l} a_{ij} \delta x_i^\top(t) P \Gamma (\delta x_j(t) - \delta x_i(t)). \end{aligned} \quad (26)$$

Renaming in the second term  $k$  by  $l$ ,  $i$  by  $j$ , and vice versa [23] and utilizing the symmetry of  $a_{ij}$ , or  $A_{lk}^\top = A_{kl}$ , one gets

$$\begin{aligned} S_2 &= \varepsilon \sum_{k=1}^{d-1} \sum_{l=k+1}^d \sum_{i \in G^k} \sum_{j \in G^l} a_{ij} \delta x_i^\top(t) P \Gamma (\delta x_j(t) - \delta x_i(t)) \\ &\quad + \varepsilon \sum_{k=1}^{d-1} \sum_{l=k+1}^d \sum_{j \in G^l} \sum_{i \in G^k} a_{ji} \delta x_j^\top(t) P \Gamma (\delta x_i(t) - \delta x_j(t)) \\ &= -\varepsilon \sum_{k=1}^{d-1} \sum_{l=k+1}^d \sum_{i \in G^k} \sum_{j \in G^l} a_{ij} (\delta x_j(t) - \delta x_i(t))^\top P \Gamma \\ &\quad \times (\delta x_j(t) - \delta x_i(t)) \\ &\leq 0. \end{aligned} \quad (27)$$

Therefore, one obtains that  $S \leq 0$  and

$$\frac{dV(\delta x_i(t))}{dt} \leq -\varepsilon \sum_{i=1}^m \delta x_i^\top(t) \delta x_i(t) \leq -2\varepsilon \frac{V(\delta x_i(t))}{\max_i p_i}, \quad (28)$$

which implies that the partial synchronization manifold  $\mathbb{M}_K$  is globally attractive for the system (1).

The remainder of the proof is to show that condition (14) is also sufficient for  $S_1 \leq 0$ .

It is well known that a symmetric matrix  $\bar{A}_{kk}$  has the decomposition  $\bar{A}_{kk} = U_k \Lambda_k U_k^\top$ , where  $\Lambda_k = \text{diag}\{\lambda_k^{(1)}, \dots, \lambda_k^{(K_k)}\}$  is a real diagonal matrix and  $U_k \in R^{K_k \times K_k}$  is a unitary matrix, that is,  $U_k U_k^\top = E_{K_k}$ . The diagonal elements of  $\Lambda_k$  are the eigenvalues of  $\bar{A}_{kk}$  satisfying  $s = \lambda_k^{(1)} > \lambda_k^{(2)} \geq \dots \geq \lambda_k^{(K_k)}$ . The  $i$ th column of  $U_k$  is the eigenvector of  $\bar{A}_{kk}$  corresponding to the eigenvalue  $\lambda_k^{(i)}$ ,  $i = 1, \dots, K_k$ . By changes of variables

$\delta \tilde{x}_k^s(t) = U \eta_k^s(t)$ , the quadratic form (25) can be diagonalized as follows:

$$\begin{aligned} S_1 &= \sum_{k=1}^d \sum_{s=1}^n p_s \delta \tilde{x}_k^{s\top}(t) (\varepsilon \gamma_s \bar{A}_{kk} + \delta_s E_{K_k}) \delta \tilde{x}_k^s(t) \\ &= \sum_{k=1}^d \sum_{s=1}^n p_s \eta_k^{s\top}(t) (\varepsilon \gamma_s \Lambda_k + \delta_s E_{K_k}) \eta_k^s(t). \end{aligned} \quad (29)$$

Noticing that the first column of  $U_k$  is  $[1, \dots, 1]^\top$ , one can conclude from  $\sum_{i \in G^k} \delta x_i(t) = 0$  and  $\eta_k^s(t) = U^\top \delta \tilde{x}_k^s(t)$  that  $\eta_k^1(t) = 0$ . Therefore, condition (14) is sufficient for  $S_1 \leq 0$ .

The proof is completed.  $\square$

Compared with the previous results [10], the conditions in Theorem 4 are not dependent on the intercluster couplings, which are eliminated in the proof through a set of mathematical skills in inequalities (25) and (27). The obtained results greatly reduce the network sizes theoretically. However, another question arises naturally: how to implement the proposed condition in reality? The following remarks might answer these questions.

*Remark 5.* In case that the network size is not very large and the coupling matrix is given, it is easy to verify all the conditions in Theorem 4. In case that the network consists of great mounts of oscillators, it should still be full of challenges to implement though our result reduces the network size greatly. And it might be hard to implement.

In order to make clear the implications of Theorem 4, several remarks on the conditions are given as follows.

*Remark 6.* (1) Many chaotic oscillators have been proved to satisfy condition (ii), such as Chua circuits [24] and standard Hopfield neural networks [25]. However, many other systems are not the case such as a lattice of  $x$ -coupled Rössler systems, in which the stability of synchronization regime is lost with the increasing of coupling [26].

(2) Providing that the trajectories of the uncoupled systems  $x_i = f(x_i, t)$ ,  $i = 1, \dots, m$  are eventually dissipative, that is, the trajectories will be in the absorbing domain  $\mathcal{B}$  eventually, it has been proved that each trajectory of the coupled system (1) is also eventually dissipative and will be in the absorbing domain  $\underbrace{\mathcal{B} \times \dots \times \mathcal{B}}_m$  [17] eventually. Therefore, condition (ii) holds when the time  $t$  is large enough. For example, the uncoupled Lorenz system

$$\begin{aligned} \dot{u} &= \sigma(v - u), \quad \sigma = 10, \\ \dot{v} &= ru - v - uv, \quad r = 28, \\ \dot{w} &= -bw + uv, \quad b = \frac{8}{3}, \end{aligned} \quad (30)$$

is eventually dissipative, where

$$\mathcal{B} = \left\{ (u)^2 + (v)^2 + (w - \sigma - r)^2 < \frac{b^2(\sigma + r)^2}{4(b-1)} = B \right\}. \quad (31)$$

It has been proved that  $x$ -coupled [27] or  $y$ -coupled Lorenz systems [28] satisfy condition (ii) when the time  $t$  is large enough.

(3) If there exists a  $k_0 \in \{1, 2, \dots, d\}$  such that  $K_{k_0} = 1$ , which implies that the subset  $G^{k_0}$  contains only one element, which does not synchronize with any other node. Without loss of generality, suppose that  $K_k = 1$  for  $k > \bar{d}$ . In this case, the corresponding principal quasi-submatrix  $\bar{A}_{kk} = s$  and the constant  $s$  can be regarded as the single eigenvalue of  $\bar{A}_{kk}$  since  $\bar{A}_{kk}\vec{v} = s\vec{v}$  for any  $\vec{v} \in R$ . Therefore, the second-largest eigenvalue  $\lambda_k^{(2)}$  does not exist. But notice the definition of partial synchronization in Section 2, synchronization in the cluster  $G^k$  always occurs, and conditions (13) and (14) should be regarded to hold for any  $\varepsilon > 0$ ,  $k > \bar{d}$ .

(4) Since  $A_{kk}$  is irreducible and diffusive, the largest eigenvalue of  $A_{kk}$  is zero, which is simple. Therefore, the largest eigenvalue of  $\bar{A}_{kk}$  is  $s$ , and it is simple also. In case  $s = 0$ , it can be seen that  $\lambda_k^{(2)} < 0$  and condition (14) is equivalent to

$$\varepsilon \geq \frac{\max_{j \in J} \{0, \delta_j / \gamma_j\}}{\min_{1 \leq k \leq \bar{d}} |\lambda_k^{(2)}|}. \quad (32)$$

Condition (14) provides us a novel index of partial synchronizability.

## 4. Numerical Examples

Consider the system (1) with  $2m$   $y$ -coupled Lorenz systems

$$\dot{x}_i = f(x_i, t) + \varepsilon \sum_{j=1}^{2m} a_{ij}(\theta) \Gamma x_j, \quad i = 1, \dots, 2m, \quad (33)$$

where  $f(x_i, t)$ ,  $x_i = (u_i, v_i, w_i)^T$  defined by the system (30),  $\Gamma = \text{diag}(0, 1, 0)$ . Let  $P = \text{diag}(B/2(b-\varepsilon)\sigma + \varepsilon/\sigma, 1, 1)$ , and  $\Delta = \text{diag}(0, 2b-\varepsilon-1 + [B-2(b-\varepsilon)(\sigma-\varepsilon)]^2/2B(b-\varepsilon), 0)$ , and the  $f$  in (27) has been proved to satisfy condition (ii) when the time  $t$  is large enough. Through translating time, the network system (27) satisfies condition (ii) in Theorem 4.

**4.1. A Star-Global Network.** Design the following topology matrix of the system (33) as

$$A(\theta) = \begin{bmatrix} A_{11}(\theta) & \theta E_m \\ \theta E_m & A_{22}(\theta) \end{bmatrix}_{2m \times 2m}, \quad (34)$$

where  $\theta \geq 0$  is the coupling weight parameter of couplings between the two clusters, and the identity matrix  $E_m$  implies that the  $i$ th ( $1 \leq i \leq m$ ) oscillator in the first cluster is coupled with the  $(m+i)$ th oscillator in the second cluster. As

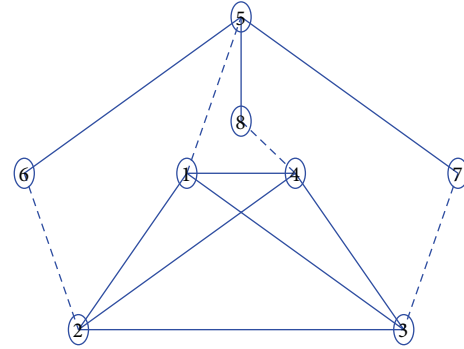


FIGURE 1: Topology structure of a star-global network composed of 8 oscillators. The solid lines represent the inner couplings of each clusters; the dashed lines represent the outer couplings between different clusters.

a special case, the submatrices  $A_{11}(\theta)$  and  $A_{22}(\theta)$  are taken as follows:

$$A_{11}(\theta) = \begin{bmatrix} 1-m-\theta & 1 & \cdots & 1 \\ 1 & 1-m-\theta & \cdots & 1 \\ & \cdots & \cdots & \\ 1 & 1 & \cdots & 1-m-\theta \end{bmatrix}, \quad (35)$$

$$A_{22}(\theta) = \begin{bmatrix} 1-m-\theta & 1 & 1 & \cdots & 1 \\ 1 & -1-\theta & 0 & \cdots & 0 \\ & \cdots & \cdots & & \\ 1 & 0 & 0 & \cdots & -1-\theta \end{bmatrix}.$$

Obviously,  $A_{11}(\theta)$  and  $A_{22}(\theta)$  are the topology matrices of a globally coupled network and a star-coupled network, respectively. Therefore, we call the network a star-global (coupled) network.

**4.2. Numerical Simulations.** As an example, the topology structure of a star-global network with  $m = 4$  is shown in Figure 1. Due to the specific topological structure of the network, the following partitions satisfy condition (i),

$$\begin{aligned} G_1 &= \{1, 2, 3, 4, 5, 6, 7, 8\}, \\ G_2 &= \{1, 2, 3, 4, 5, 6, 7, 8\}, \\ G_3 &= \{1, 5, 2, 3, 4, 6, 7, 8\}. \end{aligned} \quad (36)$$

**Remark 7.** According to the criterion in the previous researches [10], we should firstly find the eigenvalues and the corresponding eigenvectors of the  $8 \times 8$  matrix  $A(\theta)$ . As we know, it should be of a vast amount of calculations. However, based on Theorem 4, it is enough for partial synchronization with the partition  $G_2(G_3)$  if we can find the eigenvalues of two  $4 \times 4$  matrices (four  $2 \times 2$  matrices). Obviously, Theorem 4 reduces the calculations greatly.

The principal quasi-submatrices corresponding to the clusters in the partitions  $G_1$  or  $G_2$ , are  $A(\theta)$  or  $A_{11}(\theta)|_{\theta=0}$  and  $A_{22}(\theta)|_{\theta=0}$ , respectively. And denote the ones in the partition

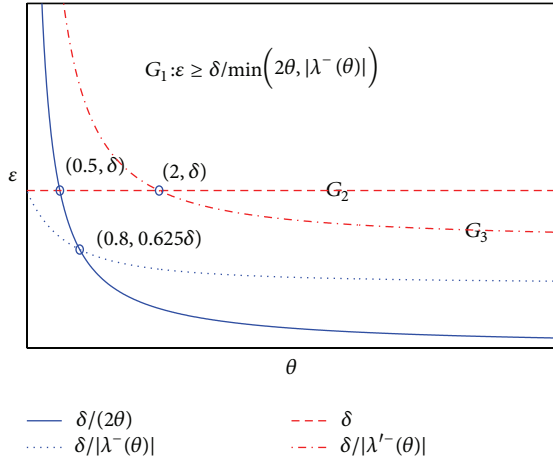


FIGURE 2: Threshold of the coupling strength  $\varepsilon$  and the coupling weight  $\theta$  for partial synchronization with different partitions.

$G_3$  as  $\bar{A}_{11}(\theta)$  and  $\bar{A}_{22}(\theta)$ . Further analysis gives rise to the eigenvalues sets of the quasi-submatrices,

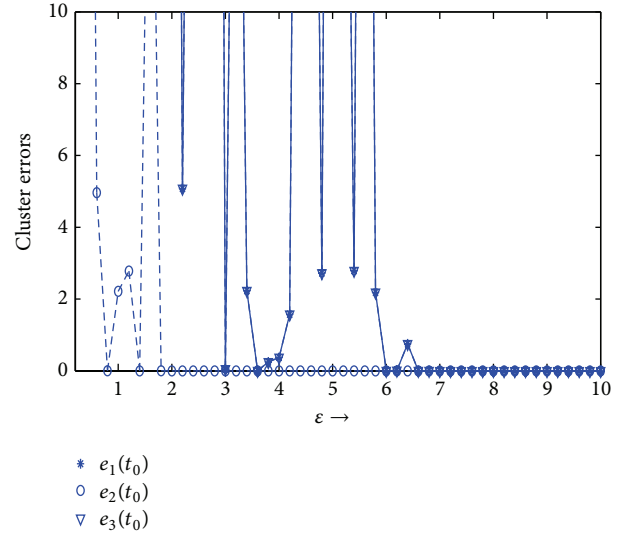
$$\begin{aligned} \sigma(A(\theta)) &= \{0, -4, -2\theta, -4 - 2\theta, \lambda^\pm, \lambda^\pm\}, \\ \sigma(A_{11}(\theta)|_{\theta=0}) &= \{0, -4, -4, -4\}, \\ \sigma(A_{22}(\theta)|_{\theta=0}) &= \{0, -1, -1, -4\}, \\ \sigma(\bar{A}_{11}(\theta)) &= \{0, -2\theta\}, \\ \sigma(\bar{A}_{22}(\theta)) &= \{0, -2\theta, \lambda'^\pm, \lambda'^\pm\}, \end{aligned} \quad (37)$$

where  $\lambda^\pm = -(2\theta + 5 \pm \sqrt{9 + 4\theta^2})/2$ ,  $\lambda'^\pm = -(2\theta + 3 \pm \sqrt{9 + 4\theta^2})/2$ . Therefore, partial synchronization with the partition  $G_2$  occurs if  $\varepsilon \geq \delta$ ; partial synchronization with the partition  $G_3$  occurs if  $\varepsilon \geq \delta/|\lambda'^-(\theta)|$ ; and full synchronization with the partition  $G_1$  occurs if  $\varepsilon \geq \max\{\delta/2\theta, \delta/|\lambda^-(\theta)|\}$ . These are seen much more clearly in Figure 2.

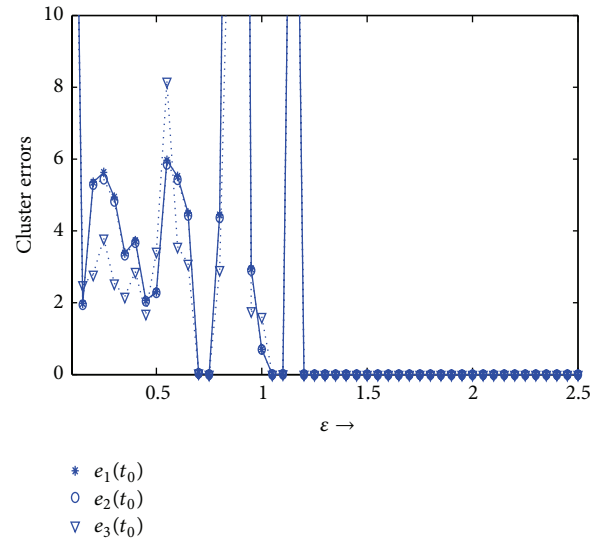
By fixing  $\theta \in (0, 0.5]$  and increasing  $\varepsilon$  gradually, partial synchronization with the partition  $G_2$  will firstly occur; then, the one with  $G_3$  and full synchronization occurs at the same time. Figure 2 also implies that the threshold for partial synchronization with the partition  $G_3$  is sufficient for full synchronization.

In order to validate the effectiveness of Figure 2 numerically, the following average cluster errors of the system (33) are defined to measure partial synchronization with partitions  $G_1, G_2$ , and  $G_3$ , respectively,

$$\begin{aligned} e_1 &= \frac{1}{8} \sum_{i=1}^8 \|x_i(t_0) - x_1(t_0)\|, \\ e_2 &= \frac{1}{8} \sum_{i=1}^4 \|x_i(t_0) - x_1(t_0)\| + \frac{1}{8} \sum_{i=5}^8 \|x_i(t_0) - x_5(t_0)\|, \\ e_3 &= \frac{1}{8} \|x_5(t_0) - x_1(t_0)\| \\ &\quad + \frac{1}{8} \sum_{i=2, i \neq 5}^8 \|x_i(t_0) - x_2(t_0)\|. \end{aligned} \quad (38)$$



(a)  $\theta = 0.2$



(b)  $\theta = 5$

FIGURE 3: Dependence of the average cluster errors on the coupling strength  $\varepsilon$  for the system (33) with the topology matrix (34). The errors  $e_1(t_0), e_2(t_0)$ , and  $e_3(t_0)$  measure the attractiveness of the manifolds  $\mathbb{M}_K(G_1), \mathbb{M}_K(G_2)$ , and  $\mathbb{M}_K(G_3)$ , respectively.

Choose the initial conditions  $(u_i(0), v_i(0), \text{ and } w_i(0))$  randomly on  $[-1, 1] \times [-1, 1] \times [-1, 1]$ , and pick  $t_0 = 500$ . Fix the coupling weight  $\theta$  at 0.2, and 5; the dependence of the cluster errors on  $\varepsilon$  is shown, respectively, in Figure 3. As can be seen, there is a good agreement between Figures 3 and 2.

### 5. Conclusions

In summary, this paper introduced a novel index of partial synchronizability of a network. It is shown that partial synchronization can be ensured by the conditions merely on the quasi-submatrices corresponding to the clusters. If a network is composed of a great amount of nodes, the enormous amount of calculation can be reduced by replacing the

coupling matrix with several quasi-submatrices. Numerically, different types of partial synchronization occur in a star-global network when the coupling strength is increased, the order of which is forecasted accurately by our result. It should be a meaningful and effective method to study partial synchronization with different partitions.

In the past decades, the networked control systems have attracted much attention due to their applications covering a wide range of industries. And the network-induced phenomena under consideration in engineering have been discussed widely, including missing measurements [29], fading measurements [30], and probabilistic sensor delays [31]. Therefore, the obtained approach for partial synchronization in this paper might be applicable to the complex networks with networked induced phenomena. The related studies should be one of the future research topics.

### Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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