Research Article

Strong Convergence Theorems for Quasi-Bregman Nonexpansive Mappings in Reflexive Banach Spaces

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We study a strong convergence for a common fixed point of a finite family of quasi-Bregman nonexpansive mappings in the framework of real reflexive Banach spaces. As a consequence, convergence for a common fixed point of a finite family of Bergman relatively nonexpansive mappings is discussed. Furthermore, we apply our method to prove strong convergence theorems of iterative algorithms for finding a common solution of a finite family equilibrium problem and a common zero of a finite family of maximal monotone mappings. Our theorems improve and unify most of the results that have been proved for this important class of nonlinear mappings.

1. Introduction

Throughout this paper, E is a real reflexive Banach space with the dual space E^*. The norm and the dual pair between E^* and E are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. We also assume that $f : E \to (-\infty, +\infty]$ is a proper, lower semicontinuous, and convex function. Denote the domain of f by dom f; that is, dom $f = \{x \in E : f(x) < \infty\}$. Let $x \in \text{int}(\text{dom } f)$. The subdifferential of f at $x \in E$ is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x \rangle \le f(y), \forall y \in E\}. \tag{1}$$

The *Fenchel conjugate* of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by $f^*(y) = \sup\{\langle y, x \rangle - f(x) : x \in E\}$.

Let $x \in \text{Dom}(f)$ and $y \in E$. The right-hand derivative of f at x in the direction of y is defined by $f^{\circ}(x, z) = \lim_{t \to 0^+}(f(x + ty) - f(x))/t$. The function f is called *Gâteaux differentiable* at x if $\lim_{t \to 0^+}(f(x + ty) - f(x))/t$ exists for any y and hence $f^{\circ}(x, y)$ coincides with $\nabla f(x)$, the value of the gradient ∇f of f at x. The function f is said to be *Gâteaux differentiable* if it is Gâteaux differentiable for any $x \in \text{int dom } f$. Furthermore, f is said to be *Fréchet differentiable* at x if this limit is attained uniformly in $\|y\| = 1$ and it is called

uniformly Fréchet differentiable on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$.

Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The function $D_f : \text{dom } f \times \text{int}(\text{dom } f) \to [0, +\infty)$ defined by

$$D_f(x, y) := f(x) - f(y) - \langle \nabla f(y), x - y \rangle \tag{2}$$

is called the *Bregman distance* with respect to f [1].

When E is a smooth Banach space, setting $f(x) = \|x\|^2$ for all $x \in E$, we have $\nabla f(x) = 2Jx$, for all $x \in E$, where J is the normalized duality mapping from E into 2^{E^*}, and hence $D_f(x, y)$ reduces to $D_f(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$, for all $x, y \in E$, which is the Lyapunov function introduced by Alber [2]. If $E = H$, a Hilbert space, J is identity mapping and hence $D_f(x, y)$ becomes $D_f(x, y) = \|x - y\|^2$, for all $x, y \in H$.

Let C be a nonempty and convex subset of $\text{int}(\text{dom } f)$ and let $T : C \to \text{int}(\text{dom } f)$ be a mapping. T is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$, and T is said to be *quasinonexpansive* if $F(T) \ne \emptyset$ and $\|Tx - p\| \le \|x - p\|$, for all $x \in K$ and $p \in F(T)$, where $F(T)$ stands for the fixed point set of T; that is, $F(T) = \{x \in C : Tx = x\}$. A point $p \in C$ is said to be an asymptotic fixed point of T (see [3]) if C contains a sequence $\{x_n\}$ which converges weakly to

p such that $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T is denoted by $\widehat{F}(T)$.

A mapping $T : C \to \text{int}(\text{dom}\, f)$ with $F(T) \neq \emptyset$ is called

(i) *quasi-Bregman nonexpansive* [4] if

$$D_f(p, Tx) \leq D_f(p, x), \quad \forall x \in C, \ p \in F(T); \quad (3)$$

(ii) *Bregman relatively nonexpansive* [4] if

$$D_f(p, Tx) \leq D_f(p, x), \quad \forall x \in C, \ p \in F(T),$$
$$\widehat{F}(T) = F(T); \quad (4)$$

(iii) *Bregman firmly nonexpansive* [5] if, for all $x, y \in C$,

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty \rangle$$
$$\leq \langle \nabla f(x) - \nabla f(y), Tx - Ty \rangle \quad (5)$$

or, equivalently,

$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y)$$
$$\leq D_f(Tx, y) + D_f(Ty, x). \quad (6)$$

Iterative methods for approximating fixed points of non-expansive, quasinonexpansive mappings and their generalizations have been studied by various authors (see, e.g., [6–16] and the references therein) in Hilbert spaces. But extending this theory to Banach spaces encountered some difficulties because the useful examples of nonexpansive operators in Hilbert spaces are no longer nonexpansive in Banach spaces (e.g., the resolvent $R_A = (I + rA)^{-1}$, for $r > 0$, of a monotone mapping $A : C \to 2^H$ and the metric projection P_C onto a nonempty, closed, and convex subset C of H). To overcome these difficulties, Bregman [1] discovered techniques with the use of Bregman distance function $D_f(\cdot, \cdot)$ instead of norm in the process of designing and analyzing feasibility and optimization problems. This opened a growing area of research for solving variational inequalities and approximating solutions or fixed points of nonlinear mappings (see, e.g., [1, 17–21] and the references therein).

The existence and approximation of fixed points of Bregman firmly nonexpansive mappings were studied in [5]. It is also known that if T is Bregman firmly nonexpansive and f is *Legendre function* which is bounded, uniformly Fréchet differentiable, and *totally convex* on bounded subsets of E, then $F(T) = \widehat{F}(T)$ and $F(T)$ is closed and convex (see [5]). It also follows that every Bregman firmly nonexpansive mapping is Bregman relatively nonexpansive and hence quasi-Bregman nonexpansive mapping.

A *Bregman projection* [1] of $x \in \text{int}(\text{dom}\, f)$ onto the nonempty closed and convex set $C \subset \text{dom}\, f$ is the unique vector $P_C^f(x) \in C$ satisfying

$$D_f\left(P_C^f(x), x\right) = \inf\left\{D_f(y, x) : y \in C\right\}. \quad (7)$$

Remark 1. If E is a smooth Banach space and $f(x) = \|x\|^2$ for all $x \in E$, then the Bregman projection $P_C^f(x)$ reduces to the generalized projection $\Pi_C(x)$ (see, e.g., [2]) which is defined by

$$\phi\left(\Pi_C(x), x\right) = \min_{y \in C}\phi(y, x), \quad (8)$$

where $\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2$.

Very recently, by using Bregman projection, Reich and Sabach [4] introduced an algorithm for finding a common zero of many finitely maximal monotone mappings $A_i : E \to E$ $(i = 1, 2, \ldots, N)$ satisfying $\mathscr{F} := \bigcap_{i=1}^{N} A_i^{-1}(0) \neq \emptyset$ in a reflexive Banach space E as follows:

$$x_0 \in E, \quad \text{chosen arbitrarily,}$$
$$y_n^i = \text{Res}_{\lambda_n^i A_i}^f\left(x_n + e_n^i\right),$$
$$C_n^i = \left\{z \in E : D_f\left(z, y_n^i\right) \leq D_f\left(z, x_n + e_n^i\right)\right\},$$
$$C_n := \bigcap_{i=1}^{N} C_n^i,$$
$$Q_n = \left\{z \in E : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \leq 0\right\},$$
$$x_{n+1} = P_{C_n \cap Q_n}^f(x_0), \quad \forall n \geq 0,$$

(9)

where $\text{Res}_{rA}^f := (\nabla f + rA)^{-1}(\nabla f)$. Under suitable conditions, they proved that if, for each $i = 1, 2, \ldots, N$, we have that $\liminf_{n \to \infty}\lambda_n^i > 0$ and the sequences of errors $\{e_n^i\} \subset E$ satisfy $\liminf_{n \to \infty}e_n^i = 0$, then the sequence $\{x_n\}$ converges strongly to $P_{\mathscr{F}}^f(x_0)$.

In [22], Reich and Sabach proposed an algorithm for finding a common fixed point of many finitely Bregman firmly nonexpansive mappings $T_i : C \to C$ $(i = 1, 2, \ldots, N)$ satisfying $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ in a reflexive Banach space E as follows:

$$x_0 \in E, \quad \text{chosen arbitrarily,}$$
$$Q_0^i = E,$$
$$y_n^i = T_i\left(x_n + e_n^i\right),$$
$$Q_{n+1}^i = \left\{z \in Q_n^i : \langle \nabla f(x_n + e_n^i) - \nabla f(y_n^i), z - y_n^i \rangle \leq 0\right\},$$
$$Q_n = \bigcap_{i=1}^{N} Q_n^i,$$
$$x_{n+1} = P_{Q_{n+1}}^f(x_0), \quad \forall n \geq 0.$$

(10)

They proved that, under suitable conditions, the sequence $\{x_n\}$ generated by (10), converges strongly to $\bigcap_{i=1}^{N} F(T_i)$ and applied it to the solution of convex feasibility and equilibrium problems.

Remark 2. But it is worth mentioning that the iteration processes (9) and (10) seem difficult in the sense that, at each stage of iteration, the set(s) C_n and/or Q_n are/is computed and the next iterate is taken as the Bregman projection of x_0 onto the intersection of C_n and/or Q_n. This seems difficult to do in applications.

In this paper, we investigate an iterative scheme for finding a common fixed point of a finite family of quasi-Bregman nonexpansive mappings in reflexive Banach spaces. We prove strong convergence theorems for the sequences produced by the method. Furthermore, we apply our method to prove strong convergence theorems for finding a solution of a finite family of equilibrium problems and for finding a common zero of a finite family of maximal monotone mappings. Our results improve and generalize many known results in the current literature (see, e.g., [4, 23])

2. Preliminaries

Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The function f is said to be *essentially smooth* if ∂f is both locally bounded and single-valued on its domain. It is called *essentially strictly convex*, if $(\partial f)^{-1}$ is locally bounded on its domain and is strictly convex on every convex subset of $\mathrm{dom}\,\partial f$. f is said to be a *Legendre*, if it is both essentially smooth and essentially strictly convex. When the subdifferential of f is single-valued, it coincides with the gradient $\partial f = \nabla f$ (see [24]).

We note that if E is a reflexive Banach space, then we have the following.

   (i) f is essentially smooth if and only if f^* is essentially strictly convex (see [25, Theorem 5.4]).

   (ii) $(\partial f)^{-1} = \partial f^*$ (see [26]).

   (iii) f is Legendre if and only if f^* is Legendre (see [25, Corollary 5.5]).

   (iv) If f is Legendre, then ∇f is a bijection satisfying $\nabla f = (\nabla f^*)^{-1}$, $\mathrm{ran}\nabla f = \mathrm{dom}\,\nabla f^* = \mathrm{int}\,\mathrm{dom}\,f^*$, and $\mathrm{ran}\nabla f^* = \mathrm{dom}\,\nabla f = \mathrm{int}\,\mathrm{dom}\,f$ (see [25, Theorem 5.10]).

When E is a smooth and strictly convex Banach space, one important and interesting example of Legendre function is $f(x) := (1/p)\|x\|^p$ ($1 < p < \infty$). In this case the gradient ∇f of f coincides with the generalized duality mapping of E; that is, $\nabla f = J_p$ ($1 < p < \infty$). In particular, $\nabla f = I$, the identity mapping in Hilbert spaces.

Let E be a Banach space and let $B_r := \{z \in E : \|z\| \le r\}$, for all $r > 0$ and $S_E = \{x \in E : \|x\| = 1\}$. Then a function $f : E \to \mathbb{R}$ is said to be *uniformly convex* on bounded subsets of E [27, page 203] if $\rho_r(t) > 0$, for all $r, t > 0$, where $\rho_r : [0, \infty) \to [0, \infty]$ is defined by

$$\rho_r(t) := \inf_{x,y \in B_r, \|x-y\|=t, \alpha \in (0,1)} (\alpha f(x) + (1-\alpha) f(y)$$
$$- f(\alpha x + (1-\alpha) y)) \quad (11)$$
$$\times (\alpha(1-\alpha))^{-1},$$

for all $t \ge 0$. The function ρ_r is called the *gauge of uniform convexity of f*. The function f is also said to be *uniformly smooth* on bounded subsets of E [27, page 207] if $\lim_{t \to 0^+}(\sigma_r(t)/t) = 0$, for all $r > 0$, where $\sigma_r : [0, \infty) \to [0, \infty]$ is defined by

$$\sigma_r(t) := \sup_{x \in B_r, y \in S_E, \alpha \in (0,1)} (\alpha f(x + (1-\alpha)ty)$$
$$+ (1-\alpha) f(x - \alpha ty) - f(x)) \quad (12)$$
$$\times (\alpha(1-\alpha))^{-1},$$

for all $t \ge 0$.

In the sequel, we will need the following lemmas.

Lemma 3 (see [28]). *Let E be a Banach space, let $r > 0$ be a constant, and let $f : E \to \mathbb{R}$ be a uniformly convex on bounded subsets of E. Then*

$$f\left(\sum_{k=0}^n \alpha_k x_k\right) \le \sum_{k=0}^n \alpha_k f(x_k) - \alpha_i \alpha_j \rho_r(\|x_i - y_j\|), \quad (13)$$

for all $i, j \in \{0, 1, 2, \ldots, n\}$, $x_k \in Br$, $\alpha_k \in (0,1)$, and $k = 0, 1, 2, \ldots, n$ with $\sum_{k=0}^n \alpha_k = 1$, where ρ_r is the gauge of uniform convexity of f.

Lemma 4 (see [24]). *Let $f : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous, and convex function; then $f^* : E^* \to (-\infty, +\infty]$ is a proper, weak* lower semicontinuous, and convex function. Thus, for all $z \in E$, we have*

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^N t_i \nabla f(x_i)\right)\right) \le \sum_{i=1}^N t_i D_f(z, x_i). \quad (14)$$

Lemma 5 (see [29]). *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable on $\mathrm{int}(\mathrm{dom}\,f)$ such that ∇f^* is bounded on bounded subsets of $\mathrm{dom}\,f^*$. Let $x^* \in X$ and $\{x_n\} \subset E$. If $\{D_f(x, x_n)\}$ is bounded, so is the sequence $\{x_n\}$.*

A function f on E is *coercive* [30] if the sublevel set of f is bounded; equivalently, $\lim_{\|x\| \to \infty} f(x) = \infty$. A function f on E is said to be *strongly coercive* [27] if $\lim_{\|x\| \to \infty} f(x)/\|x\| = \infty$.

Lemma 6 (see [27]). *Let E be a reflexive Banach space and let $f : E \to \mathbb{R}$ be a continuous convex function which is strongly coercive. Then the following assertions are equivalent.*

   (1) *f is bounded on bounded subsets and uniformly smooth on bounded subsets of E;*

   (2) *f^* is Fréchet differentiable and ∇f^* is uniformly norm-to-norm continuous on bounded subsets of E^*;*

   (3) *$\mathrm{dom}\,f = E^*$, f^* is strongly coercive and uniformly convex on bounded subsets of E^*.*

Lemma 7 (see [5]). *Let $f : E \to (-\infty, +\infty]$ be a Legendre function. Let C be a nonempty closed convex subset of $\mathrm{int}(\mathrm{dom}\,f)$ and let $T : C \to C$ be a quasi-Bregman nonexpansive mapping. Then $F(T)$ is closed and convex.*

Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The modulus of total convexity of f at $x \in \text{dom } f$ is the function $\nu_f(x, \cdot) : [0, +\infty) \to [0, +\infty]$ defined by

$$\nu_f(x, t) := \inf \left\{ D_f(y, x) : y \in \text{dom } f, \ \|y - x\| = t \right\}. \quad (15)$$

The function f is called *totally convex* at x if $\nu_f(x, t) > 0$, whenever $t > 0$. The function f is called totally convex if it is totally convex at any point $x \in \text{int}(\text{dom } f)$ and is said to be totally convex on bounded sets if $\nu_f(B, t) > 0$ for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $\nu_f : \text{int dom } f \times [0, +\infty) \to [0, +\infty]$ defined by

$$\nu_f(B, t) := \inf \left\{ \nu_f(x, t) : x \in B \cap \text{dom } f \right\}. \quad (16)$$

We know that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets (see [18, Theorem 2.10]). The following lemmas will be useful in the proof of our main results.

Lemma 8 (see [31]). *The function $f : E \to (-\infty, +\infty)$ is totally convex on bounded subsets of E if and only if, for any two sequences $\{x_n\}$ and $\{y_n\}$ in $\text{int}(\text{dom } f)$ and $\text{dom } f$, respectively, such that the first one is bounded,*

$$\lim_{n \to \infty} D_f(y_n, x_n) = 0 \implies \lim_{n \to \infty} \|y_n - x_n\| = 0. \quad (17)$$

Lemma 9 (see [27, 32]). *Let $f : E \to \mathbb{R}$ be a strongly coercive and uniformly convex on bounded subsets of E; then f^* is bounded on bounded sets and uniformly Fréchet differentiable on bounded subsets of E^*.*

Lemma 10 (see [33]). *Let $f : E \to (-\infty, +\infty]$ be uniformly Fréchet differentiable and bounded on bounded sets of E; then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Lemma 11 (see [18]). *Let C be a nonempty, closed, and convex subset of E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$; then*

(i) $z = P_C^f(x)$ if and only if $\langle \nabla f(x) - \nabla f(z), y - z \rangle \leq 0, \ \forall y \in C$.

(ii) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x), \ \forall y \in C$.

Let $f : E \to \mathbb{R}$ be a Legendre and Gâteaux differentiable function. Following [2, 34], we make use of the function $V_f : E \times E^* \to [0, +\infty)$ associated with f, which is defined by

$$V_f(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*), \quad \forall x \in E, \ x^* \in E^*. \quad (18)$$

Then V_f is nonnegative and

$$V_f(x, x^*) = D_f(x, \nabla f^*(x^*)) \quad \forall x \in E, \ x^* \in E^*. \quad (19)$$

Moreover, by the subdifferential inequality,

$$V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x \rangle \leq V_f(x, x^* + y^*), \quad (20)$$

where $\forall x \in E$ and $x^*, y^* \in E^*$ (see [35]).

Lemma 12 (see [36]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n) a_n + \alpha_n \delta_n, \quad n \geq n_0, \quad (21)$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfying the following conditions: $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_{n \to \infty} \delta_n \leq 0$. Then, $\lim_{n \to \infty} a_n = 0$.

Lemma 13 (see [37]). *Let $\{a_n\}$ be sequences of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists an increasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1}, \quad a_k \leq a_{m_k+1}. \quad (22)$$

In fact, m_k is the largest number n in the set $\{1, 2, \ldots, k\}$ such that the condition $a_n \leq a_{n+1}$ holds.

3. Main Results

We now prove the following theorem.

Theorem 14. *Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let C be a nonempty, closed, and convex subset of $\text{int}(\text{dom } f)$ and let $T_i : C \to C$, for $i = 1, 2, \ldots, N$, be a finite family of quasi-Bregman nonexpansive mappings such that $\widehat{F}(T_i) = F(T_i)$, for $i = 1, 2, \ldots, N$. Assume that $\mathscr{F} := \bigcap_{i=1}^{N} F(T_i)$ is nonempty. For $u, x_0 \in C$ let $\{x_n\}$ be a sequence generated by*

$$y_n = P_C^f \nabla f^* \left(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(x_n) \right),$$

$$x_{n+1} = \nabla f^* \left(\beta_0 \nabla f(y_n) + \sum_{i=1}^{N} \beta_i \nabla f(T_i y_n) \right), \quad \forall n \geq 0,$$

$$\quad (23)$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\beta_i\}_{i=0}^{N} \subset [c, d] \subset (0, 1)$ satisfying $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sum_{i=0}^{N} \beta_i = 1$. Then, $\{x_n\}$ converges strongly to $p = P_{\mathscr{F}}^f(u)$.

Proof. Lemma 7 ensures that each $F(T_i)$, for $i \in \{1, 2, \ldots, N\}$, and $\mathscr{F}$ are closed and convex. Thus, $P_{\mathscr{F}}^f$ is well defined. Let $p = P_{\mathscr{F}}^f(u)$. Then, from (23), Lemmas 11 and 4, and property of D_f we get that

$$D_f(p, y_n) = D_f \left(p, P_C^f \nabla f^* \left(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(x_n) \right) \right)$$

$$\leq D_f \left(p, \nabla f^* \left(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(x_n) \right) \right)$$

$$= \alpha_n D_f(p, u) + (1 - \alpha_n) D_f(p, x_n).$$

$$\quad (24)$$

Moreover, from (23), (18), and (19) we get that

$$D_f(p, x_{n+1}) = D_f\left(p, \nabla f^*\left(\beta_0 \nabla f(y_n) + \sum_{i=1}^{N} \beta_i \nabla f(T_i y_n)\right)\right)$$

$$= V\left(p, \beta_0 \nabla f(y_n) + \sum_{i=1}^{N} \beta_i \nabla f(T_i y_n)\right)$$

$$= f(p) - \left\langle p, \beta_0 \nabla f(y_n) + \sum_{i=1}^{N} \beta_i \nabla f(T_i y_n)\right\rangle$$

$$+ f^*\left(\beta_0 \nabla f(y_n) + \sum_{i=1}^{N} \beta_i \nabla f(T_i y_n)\right).$$

$$(25)$$

Since f is uniformly Fréchet differentiable function we have that f is uniformly smooth and hence by Theorem 3.5.5 of [27] we get that f^* is uniformly convex. This, with Lemmas 3 and (24), gives that

$$D_f(p, x_{n+1}) \leq f(p) - \beta_0 \langle p, \nabla f(y_n)\rangle$$

$$- \sum_{i=1}^{N} \beta_i \langle p, \nabla f(T_i y_n)\rangle$$

$$+ \beta_0 f^*(\nabla f(y_n)) + \sum_{i=1}^{N} \beta_i f^*(\nabla f(T_i y_n))$$

$$- \beta_0 \beta_i \rho_r^*\left(\|\nabla f(y_n) - \nabla f(T_i x_n)\|\right)$$

$$= \beta_0 V(p, \nabla f(y_n)) + \sum_{i=1}^{N} \beta_i V(p, \nabla f(T y_n))$$

$$- \beta_0 \beta_i \rho_r^*\left(\|\nabla f(y_n) - \nabla f(T_i x_n)\|\right)$$

$$= \beta_0 D_f(p, y_n) + \sum_{i=1}^{N} \beta_i D_f(p, T y_n)$$

$$- \beta_0 \beta_i \rho_r^*\left(\|\nabla f(y_n) - \nabla f(T_i x_n)\|\right)$$

$$\leq \beta_0 D_f(p, y_n) + \sum_{i=1}^{N} \beta_i D_f(p, y_n)$$

$$- \beta_0 \beta_i \rho_r^*\left(\|\nabla f(y_n) - \nabla f(T_i x_n)\|\right)$$

$$\leq D_f(p, y_n) - \beta_0 \beta_i \rho_r^*\left(\|\nabla f(y_n) - \nabla f(T_i y_n)\|\right)$$

$$\leq D_f(p, y_n),$$

$$(26)$$

$$\leq \alpha_n D_f(p, u) + (1 - \alpha_n) D_f(p, x_n), \qquad (27)$$

for each $i \in \{1, 2, \ldots, N\}$. Thus, by induction,

$$D_f(p, x_{n+1}) \leq \max\left\{D_f(p, u), D_f(p, x_0)\right\}, \quad \forall n \geq 0,$$

$$(28)$$

which implies that $\{x_n\}$ is bounded. Now, let $z_n = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(x_n))$. Then we note that $y_n = P_C^f z_n$.

Since f is strongly coercive, uniformly convex, uniformly Fréchet differentiable, and bounded on bounded subsets of E by Lemmas 10, 9, and 6 we have that ∇f and ∇f^* are bounded on bounded sets and hence $\{z_n\}$ and $\{y_n\}$ are bounded. In addition, using (19), (20), and property of D_f we obtain that

$$D_f(p, y_n) \leq D_f(p, z_n) = V(p, \nabla f(z_n))$$

$$\leq V(p, \nabla f(z_n) - \alpha_n(\nabla f(u) - \nabla f(p)))$$

$$+ \alpha_n \langle \nabla f(u) - \nabla f(p), z_n - p\rangle$$

$$= D_f(p, \nabla f^*(\alpha_n \nabla f(p) + (1 - \alpha_n)\nabla f(x_n)))$$

$$+ \alpha_n \langle \nabla f(u) - \nabla f(p), z_n - p\rangle$$

$$\leq \alpha_n D_f(p, p) + (1 - \alpha_n) D_f(p, x_n)$$

$$+ \alpha_n \langle \nabla f(u) - \nabla f(p), z_n - p\rangle$$

$$= (1 - \alpha_n) D_f(p, x_n)$$

$$+ \alpha_n \langle \nabla f(u) - \nabla f(p), z_n - p\rangle.$$

$$(29)$$

Furthermore, from (26) and (29) we have that

$$D_f(p, x_{n+1}) \leq (1 - \alpha_n) D_f(p, x_n)$$

$$+ \alpha_n \langle \nabla f(u) - \nabla f(p), z_n - p\rangle \qquad (30)$$

$$- \beta_0 \beta_i \rho_r^*\left(\|\nabla f(y_n) - \nabla f(T_i y_n)\|\right)$$

$$\leq (1 - \alpha_n) D_f(p, x_n)$$

$$+ \alpha_n \langle \nabla f(u) - \nabla f(p), z_n - p\rangle. \qquad (31)$$

Now, we consider two cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{D_f(p, x_n)\}$ is nonincreasing for all $n \geq n_0$. In this situation, $\{D_f(p, x_n)\}$ is convergent. Then, from (30) we have that

$$\beta_0 \beta_i \rho_r^*\left(\|\nabla f(y_n) - \nabla f(T_i y_n)\|\right) \longrightarrow 0, \qquad (32)$$

which implies, by the property of ρ_r^*, that

$$\nabla f(y_n) - \nabla f(T_i y_n) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \qquad (33)$$

Now, since f is a strongly coercive and uniformly convex on bounded subsets of E, f^* is uniformly Fréchet differentiable on bounded subsets of E^* (see [27, Proposition 3.6.2]) and f is Legendre; we have by Lemma 10 that

$$y_n - T_i y_n \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \qquad (34)$$

for each $i \in \{1, 2, \ldots, N\}$.

Furthermore, Lemma 11, property of D_f, and the fact that $\alpha_n \to 0$, as $n \to \infty$, imply that

$$
\begin{aligned}
D_f\left(x_n, y_n\right) = D_f\left(x_n, P_C^f z_n\right) &\leq D_f\left(x_n, z_n\right) \\
&= D_f\left(x_n, \nabla f^*\left(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(x_n)\right)\right) \\
&\leq \alpha_n D_f\left(x_n, u\right) + (1-\alpha_n) D_f\left(x_n, x_n\right) \\
&\leq \alpha_n D_f\left(x_n, u\right) \\
&\quad + (1-\alpha_n) D_f\left(x_n, x_n\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,
\end{aligned}
\tag{35}
$$

and hence by Lemma 8 we get that

$$
x_n - y_n \longrightarrow 0, \quad x_n - z_n \longrightarrow 0, \quad n \longrightarrow \infty. \tag{36}
$$

Since $\{z_n\}$ is bounded and E is reflexive, we choose a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ such that $z_{n_j} \rightharpoonup z$ and $\limsup_{n\to\infty}\langle \nabla f(u) - \nabla f(p), z_n - p\rangle = \lim_{j\to\infty}\langle \nabla f(u) - \nabla f(p), z_{n_j} - p\rangle$. Then, from (36) we get that

$$
y_{n_j} \rightharpoonup z, \quad \text{as } j \longrightarrow \infty. \tag{37}
$$

Thus, from (34) and the fact that $\widehat{F}(T_i) = F(T_i)$ we obtain that $z \in F(T_i)$, for each $i \in \{1, 2, \ldots, N\}$ and hence $z \in \bigcap_{i=1}^{N} F(T_i)$.

Therefore, by Lemma 11, we immediately obtain that $\limsup_{n\to\infty}\langle \nabla f(u) - \nabla f(p), z_n - p\rangle = \lim_{j\to\infty}\langle \nabla f(u) - \nabla f(p), z_{n_j} - p\rangle = \langle \nabla f(u) - \nabla f(p), z - p\rangle \leq 0$. It follows from Lemma 12 and (31) that $D_f(p, x_n) \to 0$, as $n \to \infty$. Consequently, by Lemma 8 we obtain that $x_n \to p$.

Case 2. Suppose that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that

$$
D_f\left(p, x_{n_j}\right) < D_f\left(p, x_{n_j+1}\right), \tag{38}
$$

for all $j \in \mathbb{N}$. Then, by Lemma 13, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, $D_f(p, x_{m_k}) \leq D_f(p, x_{m_k+1})$ and $D_f(p, x_k) \leq D_f(p, x_{m_k+1})$, for all $k \in \mathbb{N}$. Then from (30) and the fact that $\alpha_n \to 0$ we obtain that

$$
\rho_r^*\left(\left\|\nabla f\left(y_{m_k}\right) - \nabla f\left(T_i y_{m_k}\right)\right\|\right) \longrightarrow 0, \quad \text{as } k \longrightarrow \infty, \tag{39}
$$

for each $i \in \{1, 2, \ldots, N\}$. Thus, following the method of proof of Case 1, we obtain that $y_{m_k} - T_i y_{m_k} \to 0$, $x_{m_k} - y_{m_k} \to 0$, $x_{m_k} - z_{m_k} \to 0$, as $k \to \infty$, and hence we obtain that

$$
\limsup_{k\to\infty}\left\langle \nabla f(u) - \nabla f(p), z_{m_k} - p\right\rangle \leq 0. \tag{40}
$$

Then from (31) we have that

$$
\begin{aligned}
D_f\left(p, x_{m_k+1}\right) &\leq \left(1-\alpha_{m_k}\right) D_f\left(p, x_{m_k}\right) \\
&\quad + \alpha_{m_k}\left\langle \nabla f(u) - \nabla f(p), z_{m_k} - p\right\rangle.
\end{aligned}
\tag{41}
$$

Now, since $D_f(p, x_{m_k}) \leq D_f(p, x_{m_k+1})$, inequality (41) implies that

$$
\begin{aligned}
\alpha_{m_k} D_f\left(p, x_{m_k}\right) &\leq D_f\left(p, x_{m_k}\right) - D_f\left(p, x_{m_k+1}\right) \\
&\quad + \alpha_{m_k}\left\langle \nabla f(u) - \nabla f(p), z_{m_k} - p\right\rangle \\
&\leq \alpha_{m_k}\left\langle \nabla f(u) - \nabla f(p), z_{m_k} - p\right\rangle.
\end{aligned}
\tag{42}
$$

In particular, since $\alpha_{m_k} > 0$, we get

$$
D_f\left(p, x_{m_k}\right) \leq \left\langle \nabla f(u) - \nabla f(p), z_{m_k} - p\right\rangle. \tag{43}
$$

Then, from (40) we obtain $D_f(p, x_{m_k}) \to 0$, as $k \to \infty$. This together with (41) gives $D_f(p, x_{m_k+1}) \to 0$, as $k \to \infty$. But $D_f(p, x_k) \leq D_f(p, x_{m_k+1})$, for all $k \in N$; thus we obtain that $x_k \to p$. Therefore, from the above two cases, we can conclude that $\{x_n\}$ converges strongly to $p = P_{\mathscr{F}}^f(u)$ and the proof is complete. $\square$

If, in Theorem 14, we consider a single quasi-Bregman nonexpansive mapping, we get the following corollary.

Corollary 15. *Let $f : E \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let C be a nonempty, closed, and convex subset of $\mathrm{int}(\mathrm{dom}\, f)$ and let $T : C \to C$ be a quasi-Bregman nonexpansive mapping such that $\widehat{F}(T) = F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$
\begin{aligned}
y_n &= P_C^f \nabla f^*\left(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(x_n)\right), \\
x_{n+1} &= \nabla f^*\left(\beta \nabla f(y_n) + (1-\beta)\nabla f(T y_n)\right), \quad \forall n \geq 0,
\end{aligned}
\tag{44}
$$

where $\{\alpha_n\} \subset (0, 1)$ and $\beta \in (0, 1)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty} = \infty$. Then $\{x_n\}$ converges strongly to $p = P_{F(T)}^f(u)$.

If, in Theorem 14, we assume that each T_i, for $i \in \{1, 2, \ldots, N\}$, is Bregman relatively nonexpansive mapping, then we have that $\widehat{F}(T_i) = F(T_i)$ for each $i \in \{1, 2, \ldots, N\}$. Thus, we get the following corollary.

Corollary 16. *Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let C be a nonempty, closed, and convex subset of $\mathrm{int}(\mathrm{dom}\, f)$ and let $T_i : C \to C$, for $i = 1, 2, \ldots, N$, be a finite family of Bregman relatively nonexpansive mappings. Assume that $\mathscr{F} := \bigcap_{i=1}^{N} F(T_i)$ is nonempty. For $u, x_0 \in C$ let $\{x_n\}$ be a sequence generated by*

$$
\begin{aligned}
y_n &= P_C^f \nabla f^*\left(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(x_n)\right), \\
x_{n+1} &= \nabla f^*\left(\beta_0 \nabla f(y_n) + \sum_{i=1}^{N}\beta_i \nabla f(T_i y_n)\right), \quad \forall n \geq 0,
\end{aligned}
\tag{45}
$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\beta_i\}_{i=0}^N \subset [c,d] \subset (0,1)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$, and $\sum_{i=0}^N \beta_i = 1$. Then, $\{x_n\}$ converges strongly to $p = P_{\mathscr{F}}^f(u)$.

If, in Theorem 14, we assume that each T_i $(i = 1, 2, \ldots, N)$ is Bregman firmly nonexpansive, then we get the following corollary.

Corollary 17. *Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let C be a nonempty, closed, and convex subset of $\mathrm{int}(\mathrm{dom}\, f)$ and let $T_i : C \to C$, for $i = 1, 2, \ldots, N$, be a finite family of Bregman firmly nonexpansive mappings with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$y_n = P_C^f \nabla f^* \left(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(x_n)\right),$$

$$x_{n+1} = \nabla f^* \left(\beta_0 \nabla f(y_n) + \sum_{i=1}^N \beta_i \nabla f(T_i y_n)\right), \quad \forall n \geq 0,$$

(46)

where $\{\alpha_n\} \subset (0,1)$ and $\{\beta_i\}_{i=0}^N \subset [c,d] \subset (0,1)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$, and $\sum_{i=0}^N \beta_i = 1$. Then $\{x_n\}$ converges strongly to $p = P_{\mathscr{F}}^f(u)$.

4. Applications

4.1. Equilibrium Problems. Let $f : E \to (-\infty, +\infty]$ be a coercive Legendre function. Let C be a nonempty, closed, and convex subset of $\mathrm{int}(\mathrm{dom}\, f)$. Let $H : C \times C \to \mathbb{R}$ be a bifunction with the following conditions (see example [38]).

(B1) $H(x, x) = 0$, for all $x \in C$;

(B2) H is monotone; that is, $H(x, y) + H(y, x) \leq 0$, for all $x, y \in C$;

(B3) for all $x, y, z \in C$, $\limsup_{t \to 0^+} H(tz + (1-t)x, y) \leq H(x, y)$;

(B4) for each $x \in C$, $H(x, \cdot)$ is convex and lower semicontinuous.

The equilibrium problem corresponding to H is to find $x^* \in C$ such that

$$H(x^*, y) \geq 0, \quad \forall y \in C. \quad (47)$$

The set of solutions of (47) is denoted by $EP(H)$. Equilibrium problem is a unified model of several problems, namely, variational inequality problem, complementary problem, saddle point problem, optimization problem, fixed point problem, and so forth; see [20, 30, 38–43].

The resolvent of a bifunction $H : C \times C \to \mathbb{R}$ [39] is the operator $\mathrm{Res}_H^f : E \to 2^C$, defined by

$$\mathrm{Res}_H^f(x) = \{z \in C : H(z, y)$$

$$+ \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0, \forall y \in C\}.$$

(48)

We know the following lemma in [23].

Lemma 18. *Let $f : E \to (-\infty, +\infty]$ be a coercive Legendre function. Let C be a nonempty, closed, and convex subset of $\mathrm{int}(\mathrm{dom}\, f)$. If the bifunction $H : C \times C \to \mathbb{R}$ satisfies conditions (B1)–(B4), then*

(1) Res_H^f *is single-valued and $\mathrm{dom}(\mathrm{Res}_g^f)) = E$;*

(2) Res_H^f *is a Bregman firmly nonexpansive mapping;*

(3) $EP(H)$ *is a closed and convex subset of C and $EP(H) = F(\mathrm{Res}_H^f)$;*

(4) *For all $x \in E$ and for all $u \in F(\mathrm{Res}_H^f)$, we have*

$$D_f\left(u, \mathrm{Res}_H^f(x)\right) + D_f\left(\mathrm{Res}_H^f(x), x\right) \leq D_f(u, x). \quad (49)$$

In addition, if f is uniformly Fréchet differentiable and bounded on bounded subsets of E, then by Reich and Sabach [5] we have that $F(\mathrm{Res}_H^f) = \widehat{F}(\mathrm{Res}_H^f)$. Thus, considering $T_i = \mathrm{Res}_{H_i}^f$, for $i = 1, 2, \ldots, N$, in Corollary 17, we get the following result.

Theorem 19. *Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let C be a nonempty, closed, and convex subset of $\mathrm{int}(\mathrm{dom}\, f)$ and let $H_i : C \to C$, for $i = 1, 2, \ldots, N$, be bifunctions which satisfy the conditions (B1)–(B4) and $\mathscr{F} := \bigcap_{i=1}^N EP(H_i)$ is nonempty. For $x_0 \in C$ let $\{x_n\}$ be a sequence generated by*

$$y_n = \nabla f^* \left(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(x_n)\right),$$

$$x_{n+1} = \nabla f^* \left(\beta_0 \nabla f(y_n) + \sum_{i=1}^N \beta_i \nabla f(T_i y_n)\right), \quad \forall n \geq 0,$$

(50)

where $T_i = \mathrm{Res}_{H_i}^f$, $\{\alpha_n\} \subset (0,1)$, and $\{\beta_i\}_{i=0}^N \subset [c,d] \subset (0,1)$ satisfy $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$, and $\sum_{i=0}^N \beta_i = 1$. Then, $\{x_n\}$ converges strongly to $p = P_{\mathscr{F}}^f(u)$.

4.2. Zeroes of Maximal Monotone Operators. In this section we present an algorithm for finding a common zero of a finite family of maximal monotone mappings.

Let $A : E \to 2^{E^*}$ be a mapping with range $R(A) = \{x^* \in E^* : x^* \in Ax\}$ and domain $D(A) = \{x \in E : Ax \neq \emptyset\}$. Then, A is said to be *monotone* if, for any $x, y \in \mathrm{dom}(A)$, we have

$$x^* \in Ax, \ y^* \in Ay \Longrightarrow \langle x - y, x^* - y^* \rangle \geq 0. \quad (51)$$

It is said to be *maximal monotone* if A is monotone and the graph of A is not a proper subset of the graph of any other monotone mappings, where graph of A is given by $G(A) := \{(x, x^*) \in E \times E^* : x^* \in Ax\}$. It is known that if A is maximal monotone, then the set $A^{-1}(0) = \{z \in E : 0 \in Az\}$ is closed and convex.

Let $A : C \subseteq E \to 2^{E^*}$ be maximal monotone mapping and f is Legendre function which is bounded, uniformly Fréchet differentiable on bounded subsets of E. The resolvent

of A with respect to f is the mapping $\operatorname{Res}_A^f : E \to 2^E$ defined by

$$\operatorname{Res}_A^f = \left(\nabla f + A \right)^{-1} \circ \nabla f. \tag{52}$$

It is known that Res_A^f is single-valued, Bregman firmly nonexpansive and $F(\operatorname{Res}_A^f) = A^{-1}(0)$ (see [17]). Furthermore, the result by Reich and Sabach [5] implies that $F(\operatorname{Res}_A^f) = \widehat{F}(\operatorname{Res}_A^f)$. If, in Theorem 14, we assume that $T_i = \operatorname{Res}_{A_i}^f$, then we obtain an algorithm for finding a common zero of a finite family of maximal monotone mappings.

Theorem 20. *Let $f : E \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let C be a nonempty, closed, and convex subset of $\operatorname{int}(\operatorname{dom} f)$ and let $A_i : C \to 2^{E^*}$, $i = 1, 2, \ldots, N$, be N maximal monotone mappings with $\mathscr{F} := \bigcap_{i=1}^{N} A_i^{-1}(0) \neq \emptyset$. Then, for each $x_1, u \in E$, the sequence $\{x_n\}$ is defined by*

$$y_n = \nabla f^* \left(\alpha_n \nabla f \left(u \right) + \left(1 - \alpha_n \right) \nabla f \left(x_n \right) \right),$$

$$x_{n+1} = \nabla f^* \left(\beta_0 \nabla f \left(y_n \right) + \sum_{i=1}^{N} \beta_i \nabla f \left(T_i y_n \right) \right), \quad \forall n \geq 0, \tag{53}$$

where $T_i = \operatorname{Res}_{H_i}^f$, $\{\alpha_n\} \subset (0, 1)$, and $\{\beta_i\}_{i=0}^{N} \subset [c, d] \subset (0, 1)$ satisfy $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sum_{i=0}^{N} \beta_i = 1$. Then, $\{x_n\}$ converges strongly to $p = P_{\mathscr{F}}^f(u)$.

For a common point of a solution of equilibrium problem and a zero of maximal monotone mapping we have the following corollary.

Corollary 21. *Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let C be a nonempty, closed, and convex subset of $\operatorname{int}(\operatorname{dom} f)$ and let $H : C \to C$ be bifunction which satisfy the conditions (B1)–(B4) and let $A : C \to 2^{E^*}$ be a maximal monotone mapping. Assume that $\mathscr{F} := EP(H_i) \cap A^{-1}(0)$ is nonempty. For $x_0 \in C$ let $\{x_n\}$ be a sequence generated by*

$$y_n = \nabla f^* \left(\alpha_n \nabla f \left(u \right) + \left(1 - \alpha_n \right) \nabla f \left(x_n \right) \right),$$

$$x_{n+1} = \nabla f^* \left(\beta_0 \nabla f \left(y_n \right) + \beta_1 \nabla f \left(\operatorname{Res}_H^f y_n \right) \right. \tag{54}$$

$$\left. + \beta_2 \nabla f \left(\operatorname{Res}_A^f y_n \right) \right), \quad \forall n \geq 0,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\beta_i\}_{i=1}^{2} \subset [c, d] \subset (0, 1)$ satisfying $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sum_{i=0}^{2} \beta_i = 1$. Then, $\{x_n\}$ converges strongly to $p = P_{\mathscr{F}}^f(u)$.

Remark 22. Corollary 17 and Corollary 4.3 improve Theorem 1 of Reich and Sabach [23] and Theorem 4.2 of Reich and Sabach [4], respectively, in the sense that at each stage the computation of C_n or Q_n is not required.

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