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Research Article

Weak Convergence Theorems for Bregman Relatively Nonexpansive Mappings in Banach Spaces

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We study Mann type iterative algorithms for finding fixed points of Bregman relatively nonexpansive mappings in Banach spaces. By exhibiting an example, we first show that the class of Bregman relatively nonexpansive mappings embraces properly the class of Bregman strongly nonexpansive mappings which was investigated by Martín-Márques et al. (2013). We then prove weak convergence theorems for the sequences produced by the methods. Some application of our results to the problem of finding a zero of a maximal monotone operator in a Banach space is presented. Our results improve and generalize many known results in the current literature.

1. Introduction

Let E be a (real) Banach space with norm $\|\cdot\|$ and dual space E^*. For any x in E, we denote the value of x^* in E^* at x by $\langle x, x^*\rangle$. When $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in E, we denote the strong convergence of $\{x_n\}_{n\in\mathbb{N}}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. Let C be a nonempty subset of E. Let $T : C \to E$ be a map. We denote by $F(T) = \{x \in C : Tx = x\}$ the set of *fixed points* of T. We call the map T

(i) *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all x, y in C,

(ii) *quasi-nonexpansive* if $F(T) \ne \emptyset$ and $\|Tx - y\| \le \|x - y\|$ for all x in C and y in $F(T)$.

The nonexpansivity plays an important role in the study of the *Mann iteration*, given by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) Tx_n, \tag{1}$$

where the sequence $\{\beta_n\}_{n\in\mathbb{N}}$ satisfies some appropriate conditions. Construction of fixed points of nonexpansive mappings via Mann's algorithm [1] has been extensively investigated in the literature (see, e.g., [2] and the references therein).

Let $g : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function on a Banach space E. The *Bregman distance* [3] (see also [4, 5]) corresponding to g is the function $D_g : E \times E \to \mathbb{R}$ defined by

$$D_g(x, y) = g(x) - g(y) - \langle x - y, \nabla g(y)\rangle, \quad \forall x, y \in E. \tag{2}$$

It follows from the strict convexity of g that $D_g(x, y) \ge 0$ for all x, y in E. However, D_g might not be symmetric and D_g might not satisfy the triangular inequality.

When E is a smooth Banach space, setting $g(x) = \|x\|^2$ for all x in E, we have that $\nabla g(x) = 2Jx$ for all x in E. Here J is the normalized duality mapping from E into E^*. Hence, $D_g(\cdot,\cdot)$ reduces to the usual map $\phi(\cdot,\cdot)$ as

$$D_g(x, y) = \phi(x, y) := \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{3}$$

If E is a Hilbert space, then $D_g(x, y) = \|x - y\|^2$. For more details, we refer the readers to [6].

Let E be a smooth, strictly convex, and reflexive Banach space and let J be the normalized duality mapping of E. Let C be a nonempty, closed, and convex subset of E. The generalized projection Π_C from E onto C is defined and denoted by

$$\Pi_C(x) = \arg\min_{y\in C}\phi(y,x), \tag{4}$$

where $\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$. Let C be a nonempty, closed, and convex subset of a smooth Banach space E; let T be a mapping from C into itself. A point $p \in C$ is said to be an *asymptotic fixed point* [7] of T if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in C which converges weakly to p and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\widehat{F}(T)$. A point $p \in C$ is called a *strong asymptotic fixed point* of T if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in C which converges strongly to p and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. We denote the set of all strong asymptotic fixed points of T by $\widetilde{F}(T)$.

Let C be a nonempty, closed, and convex subset of a reflexive Banach space E. Let $g : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous, and convex function. Recall that a mapping $T : C \to C$ is said to be *Bregman quasi-nonexpansive*, if $F(T) \neq \emptyset$ and

$$D_g(p, Tx) \le D_g(p, x), \quad \forall x \in C,\ p \in F(T). \tag{5}$$

A mapping $T : C \to C$ is said to be *Bregman relatively nonexpansive* if the following conditions are satisfied:

(1) $F(T)$ is nonempty;

(2) $D_g(p, Tv) \le D_g(p, v)$, $\forall p \in F(T)$, $v \in C$;

(3) $\widehat{F}(T) = F(T)$.

A mapping $T : C \to C$ is said to be *Bregman weak relatively nonexpansive* if the following conditions are satisfied:

(1) $F(T)$ is nonempty;

(2) $D_g(p, Tv) \le D_g(p, v)$, $\forall p \in F(T)$, $v \in C$;

(3) $\widetilde{F}(T) = F(T)$.

A mapping $T : C \to C$ is said to be *Bregman strongly nonexpansive* (BSNE) if the following conditions are satisfied:

(1) $F(T)$ is nonempty;

(2) $D_g(p, Tv) \le D_g(p, v)$, $\forall p \in F(T)$, $v \in C$;

(3) $\widehat{F}(T) = F(T)$;

(4) for any bounded sequence $\{x_n\}_{n\in\mathbb{N}} \subset C$ and any $p \in F(T)$ we have

$$\lim_{n\to\infty}\left[D_g(p, x_n) - D_g(p, Tx_n)\right] = 0 \Longrightarrow \lim_{n\to\infty} D_g(Tx_n, x_n) = 0. \tag{6}$$

It is obvious that any Bregman strongly nonexpansive mapping is a Bregman relatively nonexpansive mapping, but the converse is not true in general. In the following, we show that there exists a Bregman relatively nonexpansive mapping which is not a Bregman strongly nonexpansive mapping.

Example 1. Let $E = l^2$, where

$$\begin{gathered} l^2 = \left\{\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots) : \sum_{n=1}^{\infty}\|\sigma_n\|^2 < \infty\right\}, \\ \|\sigma\| = \left(\sum_{n=1}^{\infty}\|\sigma_n\|^2\right)^{1/2}, \quad \forall \sigma \in l^2, \\ \langle \sigma, \eta\rangle = \sum_{n=1}^{\infty}\sigma_n\eta_n, \quad \forall \delta = (\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots), \\ \eta = (\eta_1, \eta_2, \ldots, \eta_n, \ldots) \in l^2. \end{gathered} \tag{7}$$

Let $\{x_n\}_{n\in\mathbb{N}\cup\{0\}} \subset E$ be a sequence defined by

$$\begin{gathered} x_0 = (1, 0, 0, 0, \ldots) \\ x_1 = (1, 1, 0, 0, 0, \ldots) \\ x_2 = (1, 0, 1, 0, 0, 0, \ldots) \\ x_3 = (1, 0, 0, 1, 0, 0, 0, \ldots) \\ \vdots \\ x_n = (\sigma_{n,1}, \sigma_{n,2}, \ldots, \sigma_{n,k}, \ldots) \\ \vdots \end{gathered} \tag{8}$$

where

$$\sigma_{n,k} = \begin{cases} 1 & \text{if } k = 1,\ n+1, \\ 0 & \text{if } k \neq 1,\ k \neq n+1, \end{cases} \tag{9}$$

for all $n \in \mathbb{N}$. It is clear that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to x_0. Indeed, for any $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots) \in l^2 = (l^2)^*$, we have

$$\Lambda(x_n - x_0) = \langle x_n - x_0, \Lambda\rangle = \sum_{k=2}^{\infty}\lambda_k\sigma_{n,k} \longrightarrow 0 \tag{10}$$

as $n \to \infty$. It is also obvious that $\|x_n - x_m\| = \sqrt{2}$ for any $n \neq m$ with n, m sufficiently large. Thus, $\{x_n\}_{n\in\mathbb{N}}$ is not a Cauchy sequence. Let k be an even number in $\mathbb{N}$ and let $g : E \to \mathbb{R}$ be defined by

$$g(x) = \frac{1}{k}\|x\|^k, \quad x \in E. \tag{11}$$

It is easy to show that $\nabla g(x) = J_k(x)$ for all $x \in E$, where

$$J_k(x) = \left\{x^* \in E^* : \langle x, x^*\rangle = \|x\|\,\|x^*\|,\ \|x^*\| = \|x\|^{k-1}\right\}. \tag{12}$$

It is also obvious that

$$J_k(\lambda x) = \lambda^{k-1}J_k(x), \quad \forall x \in E,\ \forall \lambda \in \mathbb{R}. \tag{13}$$

Now, we define a mapping $T : E \to E$ by

$$T(x) = \begin{cases} \frac{-n}{n+1}x, & \text{if } x = x_n; \\ -x, & \text{if } x \neq x_n. \end{cases} \quad (14)$$

It is clear that $F(T) = \{0\}$ and for any $n \in \mathbb{N}$

$$\begin{aligned} D_g(0, Tx_n) &= g(0) - g(Tx_n) - \langle 0 - Tx_n, \nabla g(Tx_n)\rangle \\ &= -\frac{n^k}{(n+1)^k} g(x_n) \\ &\quad - \left\langle \frac{n}{n+1}x_n, -\frac{n^{k-1}}{(n+1)^{k-1}} \nabla g(x_n) \right\rangle \\ &= -\frac{n^k}{(n+1)^k} g(x_n) + \frac{n^k}{(n+1)^k} \langle x_n, \nabla g(x_n)\rangle \\ &= \frac{n^k}{(n+1)^k} [-g(x_n) + \langle x_n, \nabla g(x_n)\rangle] \\ &= \frac{n^k}{(n+1)^k} D_g(0, x_n) \\ &\leq D_g(0, x_n). \end{aligned} \quad (15)$$

If $x \neq x_n$, then we have

$$\begin{aligned} D_g(0, Tx) &= g(0) - g(Tx) - \langle 0 - Tx, \nabla g(Tx)\rangle \\ &= -g(x) - \langle x, -\nabla g(x)\rangle \\ &= -g(x) - \langle -x, \nabla g(x)\rangle \\ &= D_g(0, x). \end{aligned} \quad (16)$$

Therefore, T is a Bregman quasi-nonexpansive mapping. Next, we claim that for any subsequence $\{x_{n_j}\}_{j\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$, $\lim_{j\to\infty}\|x_{n_j} - Tx_{n_j}\| \neq 0$. If not, then there exists a subsequence $\{x_{n_j}\}_{j\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $\lim_{j\to\infty}\|x_{n_j} - Tx_{n_j}\| = 0$. This implies that $\lim_{j\to\infty}\|x_{n_j} - Tx_{n_j}\| = \lim_{j\to\infty}\|x_{n_j} - (-n_j/(n_j+1))x_{n_j}\| = \lim_{j\to\infty}\|x_{n_j} + (n_j/(n_j+1))x_{n_j}\| = 0$, which is impossible. Now, we claim that T is a Bregman relatively nonexpansive mapping. Indeed, for any sequence $\{z_n\}_{n\in\mathbb{N}} \subset E$ such that $z_n \rightharpoonup z_0$ and $\|z_n - Tz_n\| \to 0$ as $n \to \infty$, since $\{x_n\}_{n\in\mathbb{N}}$ is not a Cauchy sequence, there exists a sufficiently large number $N \in \mathbb{N}$ such that $z_n \neq x_m$, for any $n, m > N$. If we suppose that there exists $m \leq N$ such that $z_n = x_m$ for infinitely many $n \in \mathbb{N}$, then a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ would satisfy $z_{n_i} = x_m$, so $z_0 = \lim_{i\to\infty} z_{n_i} = x_m$ and $z_0 = \lim_{i\to\infty} Tz_{n_i} = Tx_m = (m/(m+1))x_m$ which is impossible due to the fact that $\|x_m - Tx_m\| \neq 0$ for all $m \in \mathbb{N}$. This implies that $Tz_n = -z_n$ for all $n > N$. It follows from $\|z_n - Tz_n\| \to 0$ that $2z_n \to 0$ and hence $z_n \to 0$, which implies that $z_0 = 0$. Since $z_0 \in F(T)$, we conclude that T is a Bregman relatively nonexpansive mapping. Finally, we show that T is not a Bregman strongly nonexpansive mapping. To this end, we consider the sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ defined by (8); then, we have

$$\begin{aligned} &D_g(0, x_n) - D_g(0, Tx_n) \\ &\quad = g(0) - g(x_n) - \langle 0 - x_n, \nabla g(x_n)\rangle \\ &\qquad - [g(0) - g(Tx_n) - \langle 0 - Tx_n, \nabla g(Tx_n)\rangle] \\ &\quad = -g(x_n) + \langle x_n, \nabla g(x_n)\rangle + g(Tx_n) - \langle Tx_n, \nabla g(Tx_n)\rangle \\ &\quad = -\frac{1}{k}\|x_n\|^2 + \langle x_n, \nabla g(x_n)\rangle + \frac{n^k}{k(n+1)^k}\|x_n\|^2 \\ &\qquad - \frac{n^k}{(n+1)^k} \langle x_n, \nabla g(x_n)\rangle \\ &\quad = -\frac{1}{k}\left(1 - \frac{n^k}{(n+1)^k}\right)\|x_n\|^2 \\ &\qquad + \left(1 - \frac{n^k}{(n+1)^k}\right)\langle x_n, \nabla g(x_n)\rangle. \end{aligned} \quad (17)$$

This implies that

$$\lim_{n\to\infty} D_g(0, x_n) - D_g(0, Tx_n) = 0. \quad (18)$$

On the other hand, we have

$$\|x_n - Tx_n\| = \left\|x_n - \frac{-n}{n+1}x_n\right\| = \frac{2n+1}{n+1}\|x_n\| \quad (19)$$

which implies that

$$\lim_{n\to\infty} \|x_n - Tx_n\| \neq 0. \quad (20)$$

Therefore, T is not a Bregman strongly nonexpansive mapping.

We refer the readers to see some other examples of Bregman relatively nonexpansive mappings in [8].

A Banach space E is said to satisfy the *Opial property* [9] if for any weakly convergent sequence $\{x_n\}_{n\in\mathbb{N}}$ in E with weak limit x, we have

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\| \quad (21)$$

for all y in E with $y \neq x$. It is well known that all Hilbert spaces, all finite dimensional Banach spaces, and the Banach spaces l^p $(1 \leq p < \infty)$ satisfy the Opial property. Working with the Bregman distance D_g, the following Bregman Opial-like inequality holds for every Banach space E such that ∇g is weakly sequentially continuous:

$$\limsup_{n\to\infty} D_g(x, x_n) < \limsup_{n\to\infty} D_g(y, x_n), \quad (22)$$

whenever $x_n \rightharpoonup x \neq y$. See Lemma 3 for details. The Opial property of Hilbert spaces and some other special Banach spaces is a powerful tool in establishing fixed point theorems for nonexpansive and, more generally, quasi-nonexpansive mappings. The Bregman-Opial property suggests introducing the notions of Bregman nonexpansive-like mappings and developing fixed point theorems and convergence results for the Mann iterations for these mappings.

Let E be a reflexive Banach space with the dual space E^* and let $A : E \to 2^{E^*}$ be a set-valued mapping. We define the domain and range of A by $\operatorname{dom} A = \{x \in E : Ax \neq \emptyset\}$ and $\operatorname{ran} A = \cup_{x \in E} Ax$, respectively. The graph of A is denoted by $G(A) = \{(x, x^*) \in E \times E^* : x^* \in Ax\}$. The mapping $A \subset E \times E^*$ is said to be *monotone* [10] if $\langle x - y, x^* - y^* \rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in A$. It is also said to be *maximal monotone* [11] if its graph is not contained in the graph of any other monotone operators on E. If $A \subset E \times E^*$ is maximal monotone, then we can show that the set $A^{-1}0 = \{z \in E : 0 \in Az\}$ is closed and convex. Let $g : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous, and convex function. Let A be a maximal monotone operator from E to E^*. For any $r > 0$, let the mapping $\operatorname{Res}^g_{rA} : E \to \operatorname{dom} A$ be defined by

$$\operatorname{Res}^g_{rA} = (\nabla g + rA)^{-1} \nabla g. \tag{23}$$

The mapping $\operatorname{Res}^g_{rA}$ is called the *g-resolvent* of A (see [12]). It is well known that $A^{-1}(0) = F(\operatorname{Res}^g_{rA})$ for each $r > 0$ (for more details, see, e.g., [13]).

Examples and some important properties of such operators are discussed in [14].

In this paper, using Bregman functions, we study Mann type iterative algorithms for finding fixed points of Bregman relatively nonexpansive mappings in Banach spaces. We prove weak convergence theorems for the sequences produced by the methods. Some application of our results to the problem of finding a zero of a maximal monotone operator in a Banach space is presented. Our results improve and generalize many known results in the current literature; see, for example, [15].

2. Properties of Bregman Functions and Bregman Distances

Let E be a (real) Banach space, and let $g : E \to \mathbb{R}$. For any x in E, the *gradient* $\nabla g(x)$ is defined to be the linear functional in E^* such that

$$\langle y, \nabla g(x) \rangle = \lim_{t \to 0} \frac{g(x + ty) - g(x)}{t}, \quad \forall y \in E. \tag{24}$$

The function g is said to be *Gâteaux differentiable* at x if $\nabla g(x)$ is well defined, and g is *Gâteaux differentiable* if it is Gâteaux differentiable everywhere on E. We call g Fréchet differentiable at x (see, e.g., [16, page 13] or [17, page 508]) if, for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned} &|g(y) - g(x) - \langle y - x, \nabla g(x) \rangle| \\ &\quad \leq \epsilon \|y - x\| \quad \text{whenever } \|y - x\| \leq \delta. \end{aligned} \tag{25}$$

The function g is said to be *Fréchet differentiable* if it is Fréchet differentiable everywhere. It is well known that if a continuous convex function $g : E \to \mathbb{R}$ is Gâteaux differentiable, then ∇g is norm-to-weak* continuous (see, e.g., [16, Proposition 1.1.10]). If g is also Fréchet differentiable, then ∇g is norm-to-norm continuous (see, [17, page 508]).

Let E be a Banach space, $r > 0$, and $B_r := \{z \in E : \|z\| \leq r\}$. A function $g : E \to \mathbb{R}$ is said to be

(i) *strongly coercive* if

$$\lim_{\|x_n\| \to +\infty} \frac{g(x_n)}{\|x_n\|} = +\infty; \tag{26}$$

(ii) *locally bounded* if $g(B_r)$ is bounded for all $r > 0$;

(iii) *locally uniformly smooth* on E ([18, pages 207, 221]) if the function $\sigma_r : [0, +\infty) \to [0, +\infty]$, defined by

$$\begin{aligned} \sigma_r(t) = \sup_{x \in B_r, y \in S_E, \alpha \in (0,1)} &(\alpha g(x + (1 - \alpha) ty) + (1 - \alpha) \\ &\times g(x - \alpha ty) - g(x)) (\alpha (1 - \alpha))^{-1}, \end{aligned} \tag{27}$$

satisfies

$$\lim_{t \downarrow 0} \frac{\sigma_r(t)}{t} = 0, \quad \forall r > 0; \tag{28}$$

(iv) *locally uniformly convex* on E (or *uniformly convex on bounded subsets* of E ([18, pages 203, 221])) if the gauge $\rho_r : [0, +\infty) \to [0, +\infty]$ of *uniform convexity* of g, defined by

$$\begin{aligned} \rho_r(t) &= \inf_{x, y \in B_r, \|x - y\| = t, \alpha \in (0,1)} (\alpha g(x) + (1 - \alpha) g(y) \\ &\quad - g(\alpha x + (1 - \alpha) y)) (\alpha (1 - \alpha))^{-1}, \end{aligned} \tag{29}$$

satisfies

$$\rho_r(t) > 0, \quad \forall r, \ t > 0. \tag{30}$$

For a locally uniformly convex map $g : E \to \mathbb{R}$, we have

$$\begin{aligned} g(\alpha x + (1 - \alpha) y) &\leq \alpha g(x) + (1 - \alpha) g(y) \\ &\quad - \alpha (1 - \alpha) \rho_r(\|x - y\|), \end{aligned} \tag{31}$$

for all x, y in B_r and for all α in $(0, 1)$.

Let E be a Banach space and $g : E \to \mathbb{R}$ a strictly convex and Gâteaux differentiable function. By (2), the Bregman distance satisfies [3]

$$\begin{aligned} D_g(x, z) &= D_g(x, y) + D_g(y, z) \\ &\quad + \langle x - y, \nabla g(y) - \nabla g(z) \rangle, \quad \forall x, y, z \in E. \end{aligned} \tag{32}$$

In particular,

$$\begin{aligned} D_g(x, y) &= -D_g(y, x) + \langle y - x, \nabla g(y) - \nabla g(x) \rangle, \\ &\quad \forall x, y \in E. \end{aligned} \tag{33}$$

Lemma 2 (see [8, 16]). *Let E be a Banach space and $g : E \to \mathbb{R}$ a Gâteaux differentiable function which is locally uniformly convex on E. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be bounded sequences in E. Then the following assertions are equivalent:*

(1) $\lim_{n\to\infty} D_g(x_n, y_n) = 0$;

(2) $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

In the following, derive an Opial-like inequality for the Bregman distance. For original Opial's inequality, we refer the readers to Lemma 1 of [9].

Lemma 3. *Let E be a Banach space and let $g : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function such that ∇g is weakly sequentially continuous. Suppose that $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in E such that $x_n \rightharpoonup x$ for some x in E. Then*

$$\limsup_{n\to\infty} D_g(x, x_n) < \limsup_{n\to\infty} D_g(y, x_n), \tag{34}$$

for all y in the interior of dom g *with $y \neq x$.*

Proof. In view of the definition of Bregman distance (see (2)), we obtain

$$\begin{aligned} D_g(x, x_n) - D_g(y, x_n) &= g(x) - g(x_n) - \langle x - x_n, \nabla g(x_n)\rangle \\ &\quad - [g(y) - g(x_n) - \langle y - x_n, \nabla g(x_n)\rangle] \\ &= g(x) - g(y) + \langle x - y, \nabla g(x)\rangle \\ &\quad - \langle x - y, \nabla g(x)\rangle + \langle y - x, \nabla g(x_n)\rangle \\ &= g(x) - g(y) + \langle x - y, \nabla g(x)\rangle \\ &\quad + \langle y - x, \nabla g(x_n) - \nabla g(x)\rangle \\ &= -D_g(y, x) + \langle y - x, \nabla g(x_n) - \nabla g(x)\rangle. \end{aligned} \tag{35}$$

Since $x_n \rightharpoonup x$ as $n \to \infty$ and ∇g is weakly sequentially continuous, we deduce that

$$\begin{aligned} &\limsup_{n\to\infty} D_g(x, x_n) \\ &\quad = \limsup_{n\to\infty} \left[D_g(x, x_n) - D_g(y, x_n) + D_g(y, x_n)\right] \\ &\quad = \limsup_{n\to\infty} \left[D_g(x, x_n) - D_g(y, x_n)\right] + \limsup_{n\to\infty} D_g(y, x_n) \\ &\quad = -D_g(y, x) + \limsup_{n\to\infty} D_g(y, x_n). \end{aligned} \tag{36}$$

Taking into account that $D_g(y, x) > 0$ for $y \neq x$, we obtain that

$$\limsup_{n\to\infty} D_g(x, x_n) < \limsup_{n\to\infty} D_g(y, x_n), \tag{37}$$

which completes the proof. □

We call a function $g : E \to (-\infty, +\infty]$ *lower semicontinuous* if $\{x \in E : g(x) \le r\}$ is closed for all r in $\mathbb{R}$. For a lower semicontinuous convex function $g : E \to \mathbb{R}$, the *subdifferential* ∂g of g is defined by

$$\partial g(x) = \{x^* \in E^* : g(x) + \langle y - x, x^*\rangle \le g(y),\ \forall y \in E\} \tag{38}$$

for all x in E. It is well known that $\partial g \subset E \times E^*$ is maximal monotone [19, 20]. For any lower semicontinuous convex function $g : E \to (-\infty, +\infty]$, the *conjugate function* g^* of g is defined by

$$g^*(x^*) = \sup_{x\in E} \{\langle x, x^*\rangle - g(x)\}, \quad \forall x^* \in E^*. \tag{39}$$

It is well known that

$$\begin{gathered} g(x) + g^*(x^*) \ge \langle x, x^*\rangle, \quad \forall (x, x^*) \in E \times E^*, \\ (x, x^*) \in \partial g \text{ is equivalent to } g(x) + g^*(x^*) = \langle x, x^*\rangle. \end{gathered} \tag{40}$$

We also know that if $g : E \to (-\infty, +\infty]$ is a proper lower semicontinuous convex function, then $g^* : E^* \to (-\infty, +\infty]$ is a proper weak* lower semicontinuous convex function. Here, saying g is *proper*, we mean that $\operatorname{dom} g := \{x \in E : g(x) < +\infty\} \neq \emptyset$.

The following definition is slightly different from that in Butnariu and Iusem [16].

Definition 4 (see [17]). Let E be a Banach space. A function $g : E \to \mathbb{R}$ is said to be a Bregman function if the following conditions are satisfied:

(1) g is continuous, strictly convex, and Gâteaux differentiable;

(2) the set $\{y \in E : D_g(x, y) \le r\}$ is bounded for all x in E and $r > 0$.

The following lemma follows from Butnariu and Iusem [16] and Zălinescu [18].

Lemma 5. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a strongly coercive Bregman function. Then*

(1) *$\nabla g : E \to E^*$ is one-to-one, onto, and norm-to-weak* continuous;*

(2) *$\langle x - y, \nabla g(x) - \nabla g(y)\rangle = 0$ if and only if $x = y$;*

(3) *$\{x \in E : D_g(x, y) \le r\}$ is bounded for all y in E and $r > 0$;*

(4) *$\operatorname{dom} g^* = E^*$, g^* is Gâteaux differentiable and $\nabla g^* = (\nabla g)^{-1}$.*

The following two results follow from [18, Proposition 3.6.4].

Proposition 6. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex function which is locally bounded. The following assertions are equivalent:*

(1) *g is strongly coercive and locally uniformly convex on E;*

(2) $\mathrm{dom}\, g^* = E^*$, g^* *is locally bounded and locally uniformly smooth on* E*;*

(3) $\mathrm{dom}\, g^* = E^*$, g^* *is Fréchet differentiable and* ∇g^* *is uniformly norm-to-norm continuous on bounded subsets of* E^*.

Proposition 7. *Let* E *be a reflexive Banach space and* $g : E \to \mathbb{R}$ *a continuous convex function which is strongly coercive. The following assertions are equivalent:*

(1) g *is locally bounded and locally uniformly smooth on* E*;*

(2) g^* *is Fréchet differentiable and* ∇g^* *is uniformly norm-to-norm continuous on bounded subsets of* E*;*

(3) $\mathrm{dom}\, g^* = E^*$, g^* *is strongly coercive and locally uniformly convex on* E.

Lemma 8 (see [17, 21]). *Let* E *be a reflexive Banach space, let* $g : E \to \mathbb{R}$ *be a strongly coercive Bregman function, and let* V *be the function defined by*

$$V(x, x^*) = g(x) - \langle x, x^* \rangle + g^*(x^*), \quad \forall x \in E,\ \forall x^* \in E^*. \tag{41}$$

The following assertions hold:

(1) $D_g(x, \nabla g^*(x^*)) = V(x, x^*)$ *for all* x *in* E *and* x^* *in* E^**;*

(2) $V(x, x^*) + \langle \nabla g^*(x^*) - x, y^* \rangle \le V(x, x^* + y^*)$ *for all* x *in* E *and* x^*, y^* *in* E^*.

It also follows from the definition that V is convex in the second variable x^* and

$$V(x, \nabla g(y)) = D_g(x, y). \tag{42}$$

Let E be a Banach space and let C be a nonempty and convex subset of E. Let $g : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function. Then, we know from [22] that, for x in E and x_0 in C, one has

$$D_g(x_0, x) = \min_{y \in C} D_g(y, x) \quad \text{iff } \langle y - x_0, \nabla g(x) - \nabla g(x_0) \rangle \le 0, \quad \forall y \in C. \tag{43}$$

Further, if C is a nonempty, closed, and convex subset of a reflexive Banach space E and $g : E \to \mathbb{R}$ is a strongly coercive Bregman function, then, for each x in E, there exists a unique x_0 in C such that

$$D_g(x_0, x) = \min_{y \in C} D_g(y, x). \tag{44}$$

The *Bregman projection* proj_C^g from E onto C defined by $\mathrm{proj}_C^g(x) = x_0$ has the following property:

$$D_g(y, \mathrm{proj}_C^g x) + D_g(\mathrm{proj}_C^g x, x) \le D_g(y, x), \quad \forall y \in C, \quad \forall x \in E. \tag{45}$$

See [16] for details.

Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a lower-semicontinuous, strictly convex, and Gâteaux differentiable function. Let C be a nonempty, closed, and convex subset of E and let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence in E. For any x in E, we set

$$\mathrm{Br}(x, \{x_n\}_{n\in\mathbb{N}}) = \limsup_{n\to\infty} D_g(x, x_n). \tag{46}$$

The *Bregman asymptotic radius* of $\{x_n\}_{n\in\mathbb{N}}$ relative to C is defined by

$$\mathrm{Br}(C, \{x_n\}_{n\in\mathbb{N}}) = \inf\{\mathrm{Br}(x, \{x_n\}_{n\in\mathbb{N}}) : x \in C\}. \tag{47}$$

The *Bregman asymptotic center* of $\{x_n\}_{n\in\mathbb{N}}$ relative to C is the set

$$\mathrm{BA}(C, \{x_n\}_{n\in\mathbb{N}}) = \{x \in C : \mathrm{Br}(x, \{x_n\}_{n\in\mathbb{N}}) = \mathrm{Br}(C, \{x_n\}_{n\in\mathbb{N}})\}. \tag{48}$$

Proposition 9. *Let* C *be a nonempty, closed, and convex subset of a reflexive Banach space* E*, and let* $g : E \to \mathbb{R}$ *be strictly convex, Gâteaux differentiable, and locally bounded on* E*. If* $\{x_n\}_{n\in\mathbb{N}}$ *is a bounded sequence of* C*, then* $BA(C, \{x_n\}_{n\in\mathbb{N}})$ *is singleton.*

Proof. In view of the definition of Bregman asymptotic radius, we may assume that $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to z in C. By Lemma 3, we conclude that $BA(C, \{x_n\}_{n\in\mathbb{N}}) = \{z\}$. □

Lemma 10 (see [23]). *Let* C *be a nonempty, closed, and convex subset of a reflexive Banach space* E*. Let* $g : E \to \mathbb{R}$ *be strictly convex, continuous, strongly coercive, Gâteaux differentiable, and locally bounded on* E*. Let* $T : C \to E$ *be a Bregman quasi-nonexpansive mapping. Then* $F(T)$ *is closed and convex.*

3. Weak Convergence Theorems for Bregman Relatively Nonexpansive Mappings

In this section, we prove weak convergence theorems concerning Bregman relatively nonexpansive mappings in a reflexive Banach space. We propose the following Bregman Mann's type iteration.

Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function. Let C be a nonempty, closed, and convex subset of E. Let $T : C \to C$ be a Bregman relatively nonexpansive mapping. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence defined by

$$x_{n+1} = \mathrm{proj}_C^g(\nabla g^*[\gamma_n \nabla g(x_n) + (1 - \gamma_n)\nabla g(Tx_n)]), \tag{49}$$

where $\{\gamma_n\}_{n\in\mathbb{N}}$ is an arbitrary sequence in $[0, 1]$.

Lemma 11. *Let* C *be a nonempty, closed, and convex subset of a reflexive Banach space* E*. Let* $g : E \to \mathbb{R}$ *be a strictly convex and Gâteaux differentiable function. Let* $T : C \to C$ *be a Bregman quasi-nonexpansive mapping with a nonempty fixed point set* $F(T)$*.Let* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence defined by* (49) *such that $\{\gamma_n\}_{n\in\mathbb{N}}$ is an arbitrary sequence in $[0,1]$. Then the following assertions hold:*

(1) $D_g(z,x_{n+1}) \le D_g(z,x_n)$ *for all z in $F(T)$ and $n = 1,2,\ldots$;*

(2) $\lim_{n\to\infty} D_g(z,x_n)$ *exists for any z in $F(T)$.*

Proof. Let $z \in F(T)$. In view of (49), we have

$$\begin{aligned}
&D_g(z,x_{n+1}) \\
&= D_g\left(z, \operatorname{proj}_C^g\left(\nabla g^*\left[\gamma_n \nabla g(x_n) + (1-\gamma_n)\nabla g(Tx_n)\right]\right)\right) \\
&= D_g\left(z, \nabla g^*\left[\gamma_n \nabla g(x_n) + (1-\gamma_n)\nabla g(Tx_n)\right]\right) \\
&= V\left(z, \nabla\gamma_n \nabla g(x_n) + (1-\gamma_n)\nabla g(Tx_n)\right) \\
&\le \gamma_n V\left(z,\nabla g(x_n)\right) + (1-\gamma_n) V\left(z,\nabla g(Tx_n)\right) \\
&= \gamma_n D_g(z,x_n) + (1-\gamma_n) D_g(z,Tx_n) \\
&\le \gamma_n D_g(z,x_n) + (1-\gamma_n) D_g(z,x_n) \\
&= D_g(z,x_n).
\end{aligned} \tag{50}$$

This implies that $\{D_g(z,x_n)\}_{n\in\mathbb{N}}$ is a bounded and nonincreasing sequence for all z in $F(T)$. Thus we have $\lim_{n\to\infty} D_g(z,x_n)$ that exists for any z in $F(T)$. □

Theorem 12. *Let C be a nonempty, closed, and convex subset of a reflexive Banach space E. Let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function which is locally bounded, locally uniformly convex, and locally uniformly smooth on E. Let $T : C \to C$ be a Bregman relatively nonexpansive mapping. Let $\{\gamma_n\}_{n\in\mathbb{N}}$ be a sequence in $[0,1]$ satisfying the control condition*

$$\sum_{n=1}^{\infty} \gamma_n (1-\gamma_n) = +\infty. \tag{51}$$

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by the algorithm (49). Then $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to a fixed point of T.

Proof. The boundedness of the sequence $\{x_n\}_{n\in\mathbb{N}}$ follows from Lemma 11 and Definition 4. Since T is a Bregman quasi-nonexpansive mapping, for any q in $F(T)$, we have

$$D_g(q,Tx_n) \le D_g(q,x_n), \quad \forall n \in \mathbb{N}. \tag{52}$$

This, together with Definition 4 and the boundedness of $\{x_n\}_{n\in\mathbb{N}}$, implies that $\{Tx_n\}_{n\in\mathbb{N}}$ is bounded. The function g is bounded on bounded subsets of E and therefore ∇g is also bounded on bounded subsets of E^* (see, e.g., [16, Proposition 1.1.11] for more details). This implies that the sequences $\{\nabla g(x_n)\}_{n\in\mathbb{N}}$ and $\{\nabla g(Tx_n)\}_{n\in\mathbb{N}}$ are bounded in E^*.

In view of Proposition 7, we have that $\operatorname{dom} g^* = E^*$ and g^* is strongly coercive and uniformly convex on bounded subsets of E^*. Let $s_2 = \sup\{\|\nabla g(x_n)\|, \|\nabla g(Tx_n)\| : n \in \mathbb{N}\} < \infty$ and let $\rho_{s_2}^* : E^* \to \mathbb{R}$ be the gauge of uniform convexity of the conjugate function g^*.

Claim. For any p in $F(T)$ and n in $\mathbb{N}$,

$$\begin{aligned}
D_g(p,x_{n+1}) \le D_g(p,x_n) - \gamma_n(1-\gamma_n)\rho_{s_2}^* \\
\times \left(\|\nabla g(x_n) - \nabla g(Tx_n)\|\right).
\end{aligned} \tag{53}$$

Let $p \in F(T)$. For each n in $\mathbb{N}$, it follows from the definition of Bregman distance (2), Lemma 8, (32), and (49) that

$$\begin{aligned}
&D_g(p,x_{n+1}) \\
&= g(p) - g(x_{n+1}) - \langle p - x_{n+1}, \nabla g(x_{n+1})\rangle \\
&= g(p) + g^*(\nabla g(x_{n+1})) - \langle x_{n+1}, \nabla g(x_{n+1})\rangle \\
&\quad - \langle p, \nabla g(x_{n+1})\rangle + \langle x_{n+1}, \nabla g(x_{n+1})\rangle \\
&= g(p) + g^*\left((1-\gamma_n)\nabla g(x_n) + \gamma_n \nabla g(Tx_n)\right) \\
&\quad - \langle p, (1-\gamma_n)\nabla g(x_n) + \gamma_n \nabla g(Tx_n)\rangle \\
&\le (1-\gamma_n) g(p) + \gamma_n g(p) + (1-\gamma_n) g^*(\nabla g(x_n)) \\
&\quad + \gamma_n g^*(\nabla g(Tx_n)) - \gamma_n(1-\gamma_n)\rho_{s_2}^* \\
&\quad \times \left(\|\nabla g(x_n) - \nabla g(Tx_n)\|\right) \\
&\quad - (1-\gamma_n)\langle p, \nabla g(x_n)\rangle - \gamma_n \langle p, \nabla g(Tx_n)\rangle \\
&= (1-\gamma_n)\left[g(p) + g^*(\nabla g(x_n)) - \langle p, \nabla g(x_n)\rangle\right] \\
&\quad + \gamma_n\left[g(p) + g^*(\nabla g(Tx_n)) - \langle p, \nabla g(Tx_n)\rangle\right] \\
&\quad - \gamma_n(1-\gamma_n)\rho_{s_2}^*\left(\|\nabla g(x_n) - \nabla g(Tx_n)\|\right) \\
&= (1-\gamma_n)\left[g(p) - g(x_n) + \langle x_n, \nabla g(x_n)\rangle \right. \\
&\qquad \left. - \langle p, \nabla g(x_n)\rangle\right] \\
&\quad + \gamma_n\left[g(p) - g(Tx_n) + \langle Tx_n, \nabla g(Tx_n)\rangle \right. \\
&\qquad \left. - \langle p, \nabla g(Tx_n)\rangle\right] \\
&\quad - \gamma_n(1-\gamma_n)\rho_{s_2}^*\left(\|\nabla g(x_n) - \nabla g(Tx_n)\|\right) \\
&= (1-\gamma_n) D(p,x_n) + \gamma_n D(p,Tx_n) \\
&\quad - \gamma_n(1-\gamma_n)\rho_{s_2}^*\left(\|\nabla g(x_n) - \nabla g(Tx_n)\|\right) \\
&\le (1-\gamma_n) D(p,x_n) + \gamma_n D(p,x_n) \\
&\quad - \gamma_n(1-\gamma_n)\rho_{s_2}^*\left(\|\nabla g(x_n) - \nabla g(Tx_n)\|\right) \\
&= D(p,x_n) - \gamma_n(1-\gamma_n)\rho_{s_2}^*\left(\|\nabla g(x_n) - \nabla g(Tx_n)\|\right).
\end{aligned} \tag{54}$$

Thus we have

$$\begin{aligned}
&\gamma_n(1-\gamma_n)\rho_{s_2}^*\left(\|\nabla g(x_n) - \nabla g(T_n x_n)\|\right) \\
&\quad \le D_g(p,x_n) - D_g(p,x_{n+1}).
\end{aligned} \tag{55}$$

Since $\{D_g(p, x_n)\}_{n\in\mathbb{N}}$ converges, together with the control condition (60), we have

$$\liminf_{n\to\infty} \rho_{s_2}^* \left(\|\nabla g(x_n) - \nabla g(Tx_n)\|\right) = 0. \quad (56)$$

Therefore, from the property of $\rho_{s_2}^*$ we deduce that

$$\liminf_{n\to\infty} \|\nabla g(x_n) - \nabla g(Tx_n)\| = 0. \quad (57)$$

Since ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^* (see, e.g., [18]), we arrive at

$$\liminf_{n\to\infty} \|x_n - Tx_n\| = 0. \quad (58)$$

Since E is reflexive, then there exists a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{n_i} \rightharpoonup p \in C$ as $i \to \infty$. Since T is a Bregman relatively nonexpansive mapping, we deduce that $p \in F(T)$. We claim that $x_n \rightharpoonup p$ as $n \to \infty$. If not, then there exists a subsequence $\{x_{n_j}\}_{j\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $\{x_{n_j}\}_{j\in\mathbb{N}}$ converges weakly to some q in C with $p \neq q$. This implies that $q \in F(T)$. By Lemma 11, $\lim_{n\to\infty} D_g(z, x_n)$ exists for all z in $F(T)$. By the Bregman Opial-like property of E, we obtain that

$$\begin{aligned} \lim_{n\to\infty} D_g(p, x_n) &= \lim_{i\to\infty} D_g\left(p, x_{n_i}\right) < \lim_{i\to\infty} D_g\left(q, x_{n_i}\right) \\ &= \lim_{n\to\infty} D_g(q, x_n) = \lim_{j\to\infty} D_g\left(q, x_{n_j}\right) \\ &< \lim_{j\to\infty} D_g\left(p, x_{n_j}\right) = \lim_{n\to\infty} D_g(p, x_n). \end{aligned} \quad (59)$$

This is a contradiction. Thus we have $p = q$, and the desired assertion follows. □

Corollary 13. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function which is locally bounded, locally uniformly convex, and locally uniformly smooth on E. Let $T : E \to E$ be a Bregman relatively nonexpansive mapping. Let $\{\gamma_n\}_{n\in\mathbb{N}}$ be a sequence in $[0, 1]$ satisfying the control condition*

$$\sum_{n=1}^{\infty} \gamma_n (1 - \gamma_n) = +\infty. \quad (60)$$

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by

$$x_{n+1} = \nabla g^* \left[\gamma_n \nabla g(x_n) + (1 - \gamma_n) \nabla g(Tx_n)\right], \quad (61)$$

where ∇g is the right-hand derivative of g. Then $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to a fixed point of T.

4. Applications (Approximating Zeros of Maximal Monotone Operators)

As an application of our main result, we include a concrete example in support of Theorem 12. Using Theorem 12, we obtain the following strong convergence theorem for maximal monotone operators.

Theorem 14. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a strongly coercive Bregman function which is bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subsets of E. Let A be a maximal monotone operator from E to E^* such that $Z := A^{-1}(0) \neq \emptyset$. Let $r > 0$ and $\operatorname{Res}_{rA}^g = (\nabla g + rA)^{-1} \nabla g$ be the g-resolvent of A. Let $\{\gamma_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence in $[0, 1]$ which satisfies the control condition*

$$\sum_{n=1}^{\infty} \gamma_n (1 - \gamma_n) = +\infty. \quad (62)$$

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by

$$x_{n+1} = \nabla g^* \left[\gamma_n \nabla g(x_n) + (1 - \gamma_n) \nabla g\left(\operatorname{Res}_{rA}^g x_n\right)\right], \quad (63)$$

where ∇g is the right-hand derivative of g. Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined in (63) converges weakly to an element in Z as $n \to \infty$.

Proof. Letting $T = \operatorname{Res}_{rA}^g$, in Theorem 12, from (49), we obtain (63). We need only to show that T satisfies all the conditions in Theorem 12. In view of [8, Lemma 3.2], we conclude that T is a Bregman relatively nonexpansive mapping. Thus, we obtain

$$\begin{gathered} D_g\left(p, \operatorname{Res}_{rA}^g v\right) \leq D_g(p, v), \quad \forall v \in E, p \in F\left(\operatorname{Res}_{rA}^g\right), \\ \widetilde{F}\left(\operatorname{Res}_{rA}^g\right) = F\left(\operatorname{Res}_{rA}^g\right) = A^{-1}(0), \end{gathered} \quad (64)$$

where $\widetilde{F}(\operatorname{Res}_{rA}^g)$ is the set of all strong asymptotic fixed points of $\operatorname{Res}_{rA}^g$. Therefore, in view of Theorem 12, we have the conclusions of Theorem 14. This completes the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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