

Research Article

Existence for Nonlinear Evolution Equations and Application to Degenerate Parabolic Equation

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We consider an abstract Cauchy problem for a doubly nonlinear evolution equation of the form $(d/dt) \mathcal{A}(u) + \mathcal{B}(u) \ni f(t)$ in V' , $t \in (0, T]$, where V is a real reflexive Banach space, \mathcal{A} and \mathcal{B} are maximal monotone operators (possibly multivalued) from V to its dual V' . In view of some practical applications, we assume that \mathcal{A} and \mathcal{B} are subdifferentials. By using the back difference approximation, existence is established, and our proof relies on the continuity of \mathcal{A} and the coerciveness of \mathcal{B} . As an application, we give the existence for a nonlinear degenerate parabolic equation.

1. Introduction

Let V be a real reflexive Banach space, and let \mathcal{A} , \mathcal{B} be maximal monotone operators (possibly multivalued) from V to its dual V' . In this paper, we consider the abstract evolution equation:

$$\begin{aligned} \frac{d}{dt} \mathcal{A}(u(t)) + \mathcal{B}(u(t)) \ni f(t) \quad t \in (0, T], \\ \mathcal{A}(u(0)) \ni v_0, \end{aligned} \quad (1)$$

where $f : [0, T] \rightarrow V'$ and $v_0 \in V'$ are given. Inspired by some practical applications (see [1–5]), \mathcal{A} and \mathcal{B} in our work are assumed to be subdifferentials of proper, convex, and lower semicontinuous functions on V .

During the past decades, the problem has been investigated in many papers, such as [1, 2, 6–16]. For the case that $\mathcal{A} = I$ and \mathcal{B} is a subdifferential operator, the existence and uniqueness in the Hilbert space framework (i.e., $V = H$) were established in [11, 13, 15, 16], and the unique solvability in the $V - V'$ setting was given by Akagi and Ôtani [2]. Assuming that \mathcal{A} is continuous, \mathcal{B} is continuous and elliptic in some sense, Alt and Luckhaus proved the existence in [8], and Otto established the L^1 -contraction and uniqueness in [17]. For the case that \mathcal{A} is Lipschitz continuous and \mathcal{B} is coercive, the existence theory was given in [14]. In fact, if \mathcal{A} is Lipschitz

continuous and invertible, the problem (1) can be rewritten as

$$\frac{du}{dt} + \mathcal{B}(v(t)) \ni f(t), \quad v(t) \in \mathcal{A}^{-1}(u(t)), \quad t \in (0, T]. \quad (2)$$

Under the condition that \mathcal{A}^{-1} is a bi-Lipschitz subdifferential operator, this problem was investigated and solved in [11]. Then the result was extended to a more general case in [12], where \mathcal{A}^{-1} is a maximal monotone operator.

More generally, both \mathcal{A} and \mathcal{B} are possibly nonlinear, and such equations are said to be doubly nonlinear. On the assumption that one of \mathcal{A} , \mathcal{B} is a subdifferential operator and the other is strongly monotone, the existence was established in [10]. In addition, many practical applications (see [1–5]) suggested that both \mathcal{A} and \mathcal{B} are subdifferential operators. For the case that \mathcal{A} and \mathcal{B} are subdifferentials of functions on a Hilbert space, the existence was given in [6]. Supposing that \mathcal{A} is a subdifferential operator in a Hilbert space H and \mathcal{B} is a subdifferential operator in a real reflexive Banach space V , respectively, Barbu [9] and Akagi [1, 7] obtained the existence with some appropriate assumptions imposed on \mathcal{B} .

In some papers (such as [6, 11, 12]), the problems investigated are time dependent; that is, \mathcal{B} (possibly together with \mathcal{A}) is time dependent. In this paper, we aim to extend the first existence theory in [14] to the case that \mathcal{A} is only continuous

but not Lipschitz continuous, and \mathcal{B} is coercive with some other appropriate conditions. Our basic assumptions and the existence theory are stated in Section 2, and the preliminaries are introduced in Section 3. In Section 4, we use the backward difference quotient to approximate the time derivative as in [8] and solve the problem by means of convex analysis and uniform estimation, in which we make good use of the properties of subdifferentials and maximal monotone operator. In Section 5, as an application of the abstract existence theorem, we give the existence for a nonlinear degenerate parabolic equation.

2. Basic Assumptions and Existence Theorem

2.1. Basic Assumptions. To state our assumptions clearly, we introduce some notations.

Let X be a real reflexive Banach space and let $\mathcal{F}(X)$ be the set of all proper, convex, and lower semicontinuous functions $\psi : X \rightarrow (-\infty, +\infty]$, where proper means that $\psi \not\equiv +\infty$.

For any $\psi \in \mathcal{F}(X)$, its subdifferential of ψ at u , denoted by $\partial_X \psi(u)$, is given by

$$\begin{aligned} \partial_X \psi(u) &= \{v \in X' : \langle v, x - u \rangle \leq \psi(x) - \psi(u), \forall x \in D(\psi)\}, \end{aligned} \tag{3}$$

where $D(\psi) = \{x \in X : \psi(x) < +\infty\}$. Then, we define the subdifferential operator $\partial_X \psi : X \rightarrow 2^{X'}; u \mapsto \partial_X \psi(u)$ with the domain $D(\partial_X \psi) = \{x \in D(\psi) : \partial_X \psi(x) \neq \emptyset\}$.

Let V be a real reflexive Banach space, and let H be a real Hilbert space, where V is densely and compactly embedded in H . Denote the injection by $i : V \rightarrow H$.

Our basic assumptions are as follows.

(A) $\mathcal{A} = \partial_V(\Phi_A \circ i)$, where $\Phi_A \in \mathcal{F}(H)$, $\Phi_A(0) \leq 0$, and $\partial_H \Phi_A$ is continuous.

(B) $\mathcal{B} = \partial_V \Phi_B$, where $\Phi_B \in \mathcal{F}(V)$ with $D(\partial_V \Phi_B) \neq \emptyset$, satisfying that

(B1) Φ_B is coercive; that is,

$$\lim_{\|u\|_V \rightarrow +\infty} \frac{\Phi_B(u)}{\|u\|_V} = +\infty; \tag{4}$$

(B2) there exists a nondecreasing function $l(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$, such that

$$\|v\|_{V'} \leq l(\Phi_B(u)), \quad \forall u \in D(\partial_V \Phi_B), \quad \forall v \in \partial_V \Phi_B(u). \tag{5}$$

(F) $f \in W^{1,2}(0, T; V')$.

(V) There exists $u_0 \in \text{int}(D(\mathcal{A})) \cap D(\mathcal{B})$ such that $v_0 = \mathcal{A}(u_0)$.

Remark 1. (1) Condition (A) implies that the operator A is single valued.

(2) $\Phi_A(0) \leq 0$ is equivalent to $\Phi_A(0) < +\infty$.

(3) (B1) implies the coerciveness of \mathcal{B} ; that is,

$$\lim_{\|u\|_V \rightarrow +\infty} \frac{\langle v, u \rangle}{\|u\|_V} = +\infty. \tag{6}$$

2.2. Existence Theorem

Theorem 2. Assume (A), (B), (F), and (V) are all satisfied. Then, there exists at least one solution triplet

$$\begin{aligned} u \in L^\infty(0, T; V), \quad v \in L^\infty(0, T; V') \cap W^{1,2}(0, T; V'), \\ w \in L^\infty(0, T; V'), \end{aligned} \tag{7}$$

such that u is a solution of (1); that is, $v(t) = \mathcal{A}(u(t))$, $w(t) \in \mathcal{B}(u(t))$, and

$$\frac{d}{dt} v(t) + w(t) = f(t). \tag{8}$$

Moreover, $v(t) \rightarrow v_0$ strongly in V' as $t \rightarrow 0$.

3. Preliminaries

The proofs related to this section can be found in [3, 18–23].

3.1. Lower Semicontinuous Functions

Lemma 3. Let $\psi \in \mathcal{F}(X)$. Then ψ is bounded from below by an affine function; that is, there exist $x_0^* \in X'$ and $\beta \in \mathbb{R}$ such that

$$\psi(x) \geq \langle x_0^*, x \rangle + \beta, \quad \forall x \in X. \tag{9}$$

Let ψ be a function from X to $(-\infty, +\infty]$, then its conjugate function ψ^* , originally developed by Fenchel, is defined as

$$\psi^*(v) := \sup \{ \langle v, u \rangle - \psi(u) : u \in X \}, \quad v \in X'. \tag{10}$$

Lemma 4. Let X be a real reflexive Banach space.

(1) For any $\psi : X \rightarrow (-\infty, +\infty]$, ψ^* is convex and lower semicontinuous.

(2) For any $\psi \in \mathcal{F}(X)$, ψ^* is proper.

Remark 5. Let $\psi \in \mathcal{F}(X)$ with $\psi(0) \leq 0$. Then, $\psi^*(v) \geq 0$ for any $v \in D(\psi^*)$. In fact, from the definition of ψ^* ,

$$\psi^*(v) = \sup \{ \langle v, u \rangle - \psi(u) : u \in X \} \geq -\psi(0) \geq 0. \tag{11}$$

3.2. Maximal Monotone Operators. Let X be a real reflexive Banach space. An operator $\mathcal{T} : X \rightarrow 2^{X'}$ is called monotone, if

$$\begin{aligned} \langle v_1 - v_2, u_1 - u_2 \rangle \geq 0, \\ \forall u_j \in D(\mathcal{T}), \quad \forall v_j \in \mathcal{T}(u_j), \end{aligned} \tag{12}$$

$$j = 1, 2,$$

where $D(\mathcal{T}) = \{x \in X : \mathcal{T}(x) \neq \emptyset\}$. In addition, \mathcal{T} is called maximal if it has no proper monotone extension in X ; that is, for any $x \in X$ and any $y \in X'$,

$$\langle v - y, u - x \rangle \geq 0, \quad \forall u \in D(\mathcal{T}), \quad v \in \mathcal{T}(u) \tag{13}$$

only if $y \in \mathcal{T}(x)$.

Lemma 6. Let X be a real reflexive Banach space, and let $\mathcal{F} : X \rightarrow 2^{X'}$ be maximal monotone. Let $x_i \in X$ and $y_i \in \mathcal{F}(x_i)$ be such that $x_i \rightarrow x$, $y_i \rightarrow y$, and $\limsup \langle y_i, x_i \rangle \leq \langle y, x \rangle$. Then $y \in \mathcal{F}(x)$.

Lemma 7. Let X be a reflexive Banach space. Let $\mathcal{F}_1, \mathcal{F}_2 : X \rightarrow 2^{X'}$ be maximal monotone operators such that $\text{int}(D(\mathcal{F}_1)) \cap D(\mathcal{F}_2) \neq \emptyset$. Then $\mathcal{F}_1 + \mathcal{F}_2$ is maximal monotone from X to $2^{X'}$.

Lemma 8. Let $\mathcal{F}_1, \mathcal{F}_2 : X \rightarrow 2^{X'}$ be maximal monotone operators such that

(a) \mathcal{F}_2 is regular; that is, for all $x \in D(\mathcal{F}_2)$ and all $y \in R(\mathcal{F}_2)$, we have

$$\sup \{ \langle v - y, x - u \rangle : u \in X, v \in \mathcal{F}_2(u) \} < +\infty; \quad (14)$$

(b) $D(\mathcal{F}_1) \cap D(\mathcal{F}_2) \neq \emptyset$ and $R(\mathcal{F}_2) = X'$;

(c) $\mathcal{F}_1 + \mathcal{F}_2$ is maximal monotone.

Then, $R(\mathcal{F}_1 + \mathcal{F}_2) = X'$.

3.3. Subdifferentials. For any $\psi \in \mathcal{F}(X)$, its subdifferential $\partial_X \psi$, defined as (3), has the following properties.

Lemma 9. Let X be a reflexive Banach space and $\psi \in \mathcal{F}(X)$. Then,

- (1) $\partial_X \psi : X \rightarrow 2^{X'}$ is maximal monotone;
- (2) $v \in \partial_X \psi(u) \Leftrightarrow u \in \partial_{X'} \psi^*(v) \Leftrightarrow \psi(u) + \psi^*(v) = \langle v, u \rangle$.

Lemma 10. Let X be a reflexive Banach space and $A = \partial_X \psi$, where $\psi \in \mathcal{F}(X)$. Then the following conditions are equivalent:

$$\lim_{\substack{\|x\|_V \rightarrow +\infty \\ x \in D(\psi)}} \frac{\psi(x)}{\|x\|_V} = +\infty; \quad (15)$$

$$R(A) = X', \quad A^{-1} \text{ is bounded.}$$

Let $T > 0$ and $1 < p < +\infty$, and let ψ be a function on $[0, T] \times X$ such that

- (1) there are two constants α and β , for all $x \in V$ and all $t \in [0, T]$,

$$\psi(t, x) + \alpha \|x\|_X + \beta \geq 0; \quad (16)$$

- (2) $\psi(t, \cdot) \in \mathcal{F}(X)$ for each $t \in [0, T]$ and the function $t \rightarrow \psi(t, v(t))$ is measurable for each $v \in L^p(0, T; X)$.

Then, we can define a function \mathcal{F}_ψ on $L^p(0, T; X)$ as follows:

$$\mathcal{F}_\psi(u) = \begin{cases} \int_0^T \psi(t, u(t)) dt, & \text{if } \psi(\cdot, u(\cdot)) \in L^1(0, T), \\ +\infty, & \text{otherwise,} \end{cases} \quad (17)$$

which is proper, convex, and lower semicontinuous and $\mathcal{F}_\psi > -\infty$ on $L^p(0, T; X)$. For any $u \in L^p(0, T; X)$, we call $v \in \partial \mathcal{F}_\psi(u)$ in the sense of $L^{p'}(0, T; X')$ if $v \in L^{p'}(0, T; X')$ and $v \in \partial_Y \mathcal{F}_\psi(u)$, where $(1/p) + (1/p') = 1$ and $Y = L^p(0, T; X)$. Then, we have the following conclusion (Proposition 1.1 of [24]).

Lemma 11. Assume that for each $t \in [0, T]$ and each $z \in X$ with $\psi(t, z) < \infty$, there exists a function $v \in L^p(0, T; X)$ such that $v(t) = z$, $\psi(\cdot, v(\cdot)) \in L^1(0, T)$, v is right-continuous at t , and

$$\limsup_{s \downarrow t} \psi(s, v(s)) \leq \psi(t, z). \quad (18)$$

Let u be a function in $L^p(0, T; X)$ such that $\psi(\cdot, u(\cdot)) \in L^1(0, T)$ and let f be a function in $L^{p'}(0, T; X')$. Then, $f \in \partial \mathcal{F}_\psi(u)$ in the sense of $L^{p'}(0, T; X')$, if and only if $f(t) \in \partial_X \psi(u(t))$ for a.e. $t \in [0, T]$.

Remark 12. Assume $T > 0$ and $\psi \in \mathcal{F}(X)$. Let $u \in L^2(0, T; X)$ and $v \in L^2(0, T; X')$. Then, $v \in \partial \mathcal{F}_\psi(u)$ in the sense of $L^2(0, T; X')$, if and only if $v(t) \in \partial_X \psi(u(t))$ for a.e. $t \in [0, T]$.

Next, we introduce some chain rules of subdifferentials in different forms.

By the definition of ψ^* and Lemma 9, we can easily verify the following chain rule in the form of difference quotient.

Lemma 13. Let $v(t) \in \partial_X \psi(u(t))$. Then, for each $h > 0$,

$$\begin{aligned} \partial_{-h} \psi(u(t)) &\leq \langle v(t), \partial_{-h} u(t) \rangle, \\ \partial_{-h} \psi^*(v(t)) &\leq \langle \partial_{-h} v(t), u(t) \rangle, \end{aligned} \quad (19)$$

where ∂_{-h} denotes the backward difference operator,

$$\partial_{-h} u(t) = \frac{u(t) - u(t-h)}{h}. \quad (20)$$

The following chain rule of integral form was proved in [25].

Lemma 14. Assume $p \in [1, +\infty)$, and $(1/p) + (1/q) = 1$. Let X be a real reflexive Banach space and $\psi \in \mathcal{F}(X)$. Let $u \in W^{1,p}(0, T; X)$ and $w \in L^q(0, T; X')$ be such that $w(t) \in \partial_X \psi(u(t))$ almost everywhere in $(0, T)$. Then $t \mapsto \psi(u(t))$ is absolutely continuous, and for all $v \in L^q(0, T; X')$ with $v(t) \in \partial_X \psi(u(t))$ almost everywhere in $(0, T)$,

$$\psi(u(t)) - \psi(u(s)) = \int_s^t \left\langle v, \frac{du}{dt} \right\rangle \quad \forall 0 \leq s \leq t \leq T. \quad (21)$$

Lemma 15. Let X be a real reflexive Banach space and let Y be a Hilbert space with X densely and compactly embedded in Y . Let $\Lambda : X \rightarrow Y$ be a linear continuous operator, and assume that $\psi \in \mathcal{F}(Y)$ is continuous at some point of $R(\Lambda)$ (the range of Λ). Then

$$\partial_X (\psi \circ \Lambda) = \Lambda' \circ \partial_Y \psi \circ \Lambda : X \rightarrow 2^{X'}, \quad (22)$$

where L' is the dual operator of Λ .

Remark 16. Since $\partial_H \Phi_A$ is continuous and V is densely and compactly embedded in H , $\mathcal{A} = \partial_V(\Phi_A \circ i) = i' \circ \partial_H \Phi_A \circ i$ is compact, where i is the injection from V to H .

Lemma 17. *Assume (A), (B), and (V) are satisfied. Then $\mathcal{A} + \mathcal{B}$ is maximal monotone from V to $2^{V'}$ and $R(\mathcal{A} + \mathcal{B}) = V'$.*

Proof. Since $\text{int}(D(\mathcal{A})) \cap D(\mathcal{B}) \neq \emptyset$, we get the maximal monotonicity of $\mathcal{A} + \mathcal{B}$ from Lemmas 9 and 7.

To prove $R(\mathcal{A} + \mathcal{B}) = V'$, we only need to verify that \mathcal{A} and \mathcal{B} satisfy Lemma 8. Applying (4) and Lemma 10, we can deduce that $R(\mathcal{B}) = V'$, and then the proof is completed if we could show that $\mathcal{B} = \partial_V \Phi_B$ is regular.

In fact, for any real reflexive Banach space X and any $\psi \in \mathcal{F}(X)$, $\partial_X \psi$ is regular.

Take $x \in D(\partial_X \psi)$ and $y \in R(\partial_X \psi)$. Since $R(\partial_X \psi) \subset D(\psi^*)$,

$$\begin{aligned} \langle v - y, x - u \rangle &\leq \psi(x) - \psi(u) - \langle y, x - u \rangle \\ &\leq \psi(x) + \psi^*(y) - \langle y, x \rangle \end{aligned} \quad (23)$$

holds for any $u \in D(\psi)$ and for any $v \in \partial_X \psi(u)$. Since the right-hand side of (23) is a constant independent of v and u ,

$$\sup \{ \langle v - y, x - u \rangle : u \in D(\partial_X \psi), v \in \partial_X \psi(u) \} < +\infty; \quad (24)$$

that is, $\partial_X \psi$ is regular. \square

4. Proof of Theorem 2

In this section, to prove Theorem 2, we use the backward difference to approximate the time derivative. Since we can establish the solvability of the resulting approximate equations from Lemma 17, then, combining convex analysis and uniform estimation, we verify the existence.

4.1. Approximate Problems and Approximate Solutions. Let N be a positive integer, and $h = T/N$.

To prove Theorem 2, we approximate the time derivative in (1) by ∂_{-h} and approximate f by f_h :

$$\begin{aligned} f_h(t) &= \frac{1}{h} \int_{(k-1)h}^{kh} f(s) ds, \\ (k-1)h < t \leq kh, \quad k &= 1, 2, \dots, N. \end{aligned} \quad (25)$$

These lead to the approximate problem

$$\begin{aligned} \partial_{-h} \mathcal{A}(u(t)) + \mathcal{B}(u(t)) &\ni f_h(t) \quad \text{in } V', \quad t \in (0, T], \\ \mathcal{A}(u_h(t)) &= v_0, \quad t \in (-h, 0], \end{aligned} \quad (26)$$

and we can solve the solution $u_h(t)$ inductively for $t \in (-h, kh]$, $k = 0, 1, \dots, N$, as follows.

For $t \in (-h, 0]$, we set $u_h(t) = u_0$, $v_h(t) = v_0 = \mathcal{A}(u_0)$. Suppose that we have a solution of (26) with $T = kh$, $0 \leq k < N$, which implies that we have a solution triplet

$$\begin{aligned} u_h(t) &\in V, \quad v_h(t) = \mathcal{A}(u_h(t)), \\ w_h(t) &\in \mathcal{B}(u_h(t)), \\ t &\in (-h, kh]. \end{aligned} \quad (27)$$

Consider the problem (26) with $T = (k+1)h$, which is equivalent to

$$\begin{aligned} \mathcal{A}(u_h(t)) + h\mathcal{B}(u_h(t)) &\ni hf_h(t) + v_h(t-h) \quad \text{in } V', \\ -h < t &\leq (k+1)h. \end{aligned} \quad (28)$$

Since for $t \in (kh, (k+1)h]$, $hf_h(t) + v_h(t-h) \in V'$, by Lemma 17, this problem has at least one solution $u_h(t)$ for $t \in (kh, (k+1)h]$. Then we can solve $u_h(t)$ inductively for $t \in (-h, kh]$, $k = 0, 1, \dots, N$, and consequently the problem (26) has at least one solution triplet:

$$\begin{aligned} u_h(t) &\in V, \quad v_h(t) = \mathcal{A}(u_h(t)), \\ w_h(t) &\in \mathcal{B}(u_h(t)), \end{aligned} \quad (29)$$

such that

$$\partial_{-h} v_h(t) + w_h(t) = f_h(t) \quad \text{in } V', \quad 0 < t \leq T, \quad (30)$$

$$v_h(0) = v_0. \quad (31)$$

Obviously, the triplet $(u_h(t), v_h(t), w_h(t))$ is piecewise constant; that is, the triplet is constant in each interval $((k-1)h, kh]$, $k = 0, 1, \dots, N$.

In the following, we aim to obtain some uniform estimates on the approximation solutions (see Section 4.2) and then solve the problem (1) by taking the limit of an appropriate subsequence (see Section 4.3).

4.2. Uniform Estimates

Lemma 18. *There exists a constant $C > 0$, such that*

$$\sup_{0 \leq t \leq T} \|u_h(t)\|_V \leq C, \quad (32)$$

$$\sup_{0 \leq t \leq T} \|w_h(t)\|_{V'} \leq C, \quad (33)$$

$$\|\partial_{-h} v_h\|_{L^2(0, T; V')} \leq C. \quad (34)$$

Proof. Applying (30), for any $\tau = kh$, $1 \leq k \leq N$, we have

$$\begin{aligned} &\int_0^\tau \langle \partial_{-h} v_h(t), \partial_{-h} u_h(t) \rangle + \int_0^\tau \langle w_h(t), \partial_{-h} u_h(t) \rangle \\ &= \int_0^\tau \langle f_h(t), \partial_{-h} u_h(t) \rangle. \end{aligned} \quad (35)$$

In view of maximal monotonicity of subdifferentials, we have

$$\int_0^\tau \langle \partial_{-h} v_h(t), \partial_{-h} u_h(t) \rangle \geq 0. \quad (36)$$

By virtue of Lemma 13, the second term in (35) could be estimated as follows:

$$\begin{aligned} \int_0^\tau \langle w_h(t), \partial_{-h} u_h(t) \rangle &\geq \int_0^\tau \partial_{-h} \Phi_B(u_h(t)) dt \\ &= \Phi_B(u_h(\tau)) - \Phi_B(u_h(0)). \end{aligned} \quad (37)$$

As for the third term, applying integration by parts, we have

$$\begin{aligned} \int_0^\tau \langle f_h(t), \partial_{-h} u_h(t) \rangle &= \langle f_h(\tau), u_h(\tau) \rangle - \langle f_h(h), u_h(0) \rangle \\ &\quad - \int_0^{\tau-h} \langle \partial_h f_h(t), u_h(t) \rangle dt. \end{aligned} \quad (38)$$

Since $f_h \rightarrow f$ in $W^{1,2}(0, T; V')$ as $h \rightarrow 0$ and the embedding $W^{1,2}(0, T; V') \rightarrow C([0, T]; V')$ is continuous, there exists a constant $C > 0$ such that

$$\int_0^\tau \langle f_h(t), \partial_{-h} u_h(t) \rangle \leq C + C \sup_{0 \leq t \leq T} \|u_h(t)\|_{V'}. \quad (39)$$

Then, from the assumption on initial value (see (V) Section 2.1), it follows that

$$\sup_{0 \leq t \leq T} \Phi_B(u_h(t)) \leq C + C \sup_{0 \leq t \leq T} \|u_h(t)\|_{V'}. \quad (40)$$

From the coercivity of Φ_B (see (4)), we get (32) and (33) from (40) and (5). Applying $\partial_{-h} v_h(t) = f_h - w_h$, we get (34). \square

4.3. Completion of Proof

Lemma 19. *There exist $u \in L^\infty(0, T; V)$, $v(t) \in L^\infty(0, T; V') \cap W^{1,2}(0, T; V')$, and $w(t) \in L^\infty(0, T; V')$ such that for almost all $t \in [0, T]$, $v(t) = \mathcal{A}(u(t))$, $w(t) \in \mathcal{B}(u(t))$, and*

$$\frac{d}{dt} v(t) + w(t) = f(t) \quad \text{in } V'. \quad (41)$$

Moreover, $v(t) \rightarrow v_0$ strongly in V' as $t \rightarrow 0$.

Proof. From (32) and (33), there exist $u \in L^\infty(0, T; V)$ and $w \in L^\infty(0, T; V')$ such that

$$u_h(t) \rightharpoonup u \quad \text{weakly in } L^2(0, T; V), \quad (42)$$

$$w_h(t) \rightharpoonup w \quad \text{weakly in } L^2(0, T; V'). \quad (43)$$

Since \mathcal{A} is compact from V to V' (see Remark 16) and u_h is uniformly bounded in $L^\infty(0, T; V)$, $v_h(t)$ is precompact in V' for each $t \in [0, T]$. Then combining (34), there exist a subsequence of v_h (still denoted by v_h) and $v \in L^\infty(0, T; V')$ such that

$$v_h(t) \rightarrow v \quad \text{strongly in } C(0, T; V'). \quad (44)$$

In addition, from (34), there exists $\chi \in L^\infty(0, T; V')$ such that

$$\partial_{-h} v_h(t) \rightharpoonup \chi \quad \text{weakly in } L^2(0, T; V'), \quad (45)$$

and we can easily testify that $\chi = dv/dt$ combining (44).

In virtue of (30), for any $x \in L^2(0, T; V)$, we have

$$\int_0^T \langle \partial_{-h} v_h(t), x \rangle dt + \int_0^T \langle w_h(t), x \rangle dt = \int_0^T \langle f_h(t), x \rangle dt. \quad (46)$$

Applying (43) and (45) and letting $h \rightarrow 0$, it follows that

$$\int_0^T \left\langle \frac{d}{dt} v(t), x \right\rangle dt + \int_0^T \langle w(t), x \rangle dt = \int_0^T \langle f(t), x \rangle dt, \quad (47)$$

which implies that for almost all $t \in (0, T]$,

$$\frac{d}{dt} v(t) + w(t) = f(t) \quad \text{in } V'. \quad (48)$$

To complete the proof, we need to show that $v = \mathcal{A}(u)$, $w \in \mathcal{B}(u)$ for almost all $t \in [0, T]$. Since \mathcal{A}, \mathcal{B} are maximal monotone operators, then, combining Lemmas 6 and 11, it suffices to prove that

$$\limsup_{h \rightarrow 0} \int_0^T \langle v_h(t), u_h(t) \rangle dt \leq \int_0^T \langle v(t), u(t) \rangle dt, \quad (49)$$

$$\limsup_{h \rightarrow 0} \int_0^T \langle w_h(t), u_h(t) \rangle dt \leq \int_0^T \langle w(t), u(t) \rangle dt. \quad (50)$$

In view of (42) and (44), we have

$$\lim_{h \rightarrow 0} \int_0^T \langle v_h(t), u_h(t) \rangle dt = \int_0^T \langle v(t), u(t) \rangle dt, \quad (51)$$

which implies (49).

Applying (30), we have

$$\begin{aligned} \int_0^T \langle w_h(t), u_h(t) \rangle dt &= \int_0^T \langle f_h(t), u_h(t) \rangle dt \\ &\quad - \int_0^T \langle \partial_{-h} v_h(t), u_h(t) \rangle dt. \end{aligned} \quad (52)$$

Since $f_h \rightarrow f$ strongly in $L^2(0, T; V')$ and $u_h \rightarrow u$ weakly in $L^2(0, T; V)$, we have

$$\lim_{h \rightarrow 0} \int_0^T \langle f_h(t), u_h(t) \rangle dt = \int_0^T \langle f(t), u(t) \rangle dt. \quad (53)$$

For the second term, applying Lemma 13, we have

$$\begin{aligned} \liminf_{h \rightarrow 0} \int_0^T \langle \partial_{-h} v_h(t), u_h(t) \rangle dt \\ \geq \liminf_{h \rightarrow 0} \int_0^T \partial_{-h} \Phi_A^*(v_h(t)) dt \\ \geq \Phi_A^*(v(T)) - \Phi_A^*(v_0), \end{aligned} \quad (54)$$

since $v_h(T) \rightarrow v(T)$ strongly in V' and Φ_A^* is lower semicontinuous. Moreover, from Lemma 14, we can easily get

$$\Phi_A^*(v(T)) - \Phi_A^*(v_0) = \int_0^T \left\langle \frac{d}{dt} v(t), u(t) \right\rangle dt. \quad (55)$$

Therefore,

$$\begin{aligned} \limsup_{h \rightarrow 0} \int_0^T \langle w_h(t), u_h(t) \rangle dt \\ \leq \int_0^T \left\langle f(t) - \frac{d}{dt} v(t), u(t) \right\rangle dt. \end{aligned} \quad (56)$$

On the other hand, applying (47) on u , we have

$$\int_0^T \left\langle f(t) - \frac{d}{dt} v(t), u(t) \right\rangle dt = \int_0^T \langle w(t), u(t) \rangle dt. \quad (57)$$

Consequently, we obtain (50).

As the end of the proof, since $v_h \rightarrow v$ in $C(0, T; V')$ and

$$\|v_h(t) - v_0\|_{V'} \leq \int_0^t \|\partial_{-h} v_h(\tau)\|_{V'} d\tau \leq C\sqrt{t} \rightarrow 0, \quad (58)$$

as $t \rightarrow 0$, we have $v(t) \rightarrow v_0$ strongly in V' as $t \rightarrow 0$. \square

5. Application to an Initial Boundary Value Problem

The abstract existence can be applied to many models in fluid mechanics (see [26, 27]). We shall illustrate the application of Theorem 2 to establish the existence of a solution to a nonlinear parabolic initial-boundary-value problem with nonlinear degenerate terms under the time derivative. This problem includes a nonlinear dynamic boundary condition.

Let Ω be a bounded domain in \mathbb{R}^3 with smooth boundary $\Gamma = \partial\Omega$ and Γ has the partition $\Gamma = \Gamma_1 \cup \Gamma_2$. Consider

$$\begin{aligned} \frac{\partial}{\partial t} a(u) - \sum_{i=1}^3 \frac{\partial}{\partial x_i} \left(|\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \right) = f(x, t), \\ (x, t) \in \Omega \times (0, T], \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial t} c(u) + |\nabla u|^{p-2} \nabla u \cdot n + g(x) |u|^{r-2} u = 0, \\ (x, t) \in \Gamma_1 \times (0, T], \end{aligned} \quad (59)$$

$$u = 0, \quad (x, t) \in \Gamma_2 \times (0, T],$$

$$a(u) = a_0(x), \quad (x, t) \in \Omega \times \{0\},$$

$$c(u) = c_0(x), \quad (x, t) \in \Gamma_1 \times \{0\},$$

where $2 \leq p < +\infty$, $1 \leq r \leq p$, $a(\cdot)$ and $c(\cdot)$ are continuous and nondecreasing and $g(x) \in L^\infty(\Omega)$ with $g(x) \geq g_0 > 0$.

Remark 20. For the case $meas(\Gamma_1) > 0$, the problem (59) cannot be covered by [8].

5.1. Formulation of Abstract Form. Let $H = L^2(\Omega) \oplus L^2(\Gamma_1)$ and $V = \{u \in W^{1,p}(\Omega) : u = 0 \text{ on } \Gamma_2\}$, equipped with the norms $\|u\|_V = \|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^p(\Gamma_1)}$ and $\|(u_1, u_2)\|_H = \|u_1\|_{L^2(\Omega)} + \|u_2\|_{L^2(\Gamma_1)}$, respectively. Then V is embedded in H densely and compactly, and denote the injection by i . Assume that $a(\cdot), c(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ are nondecreasing and continuous, and $\phi_a, \phi_c : \mathbb{R} \rightarrow (-\infty, +\infty]$ satisfy

$$\phi_a(x) = \int_0^x a(s) ds, \quad \phi_c(x) = \int_0^x c(s) ds. \quad (60)$$

Define

$$\begin{aligned} \Phi_a(u) &= \int_\Omega \phi_a(u) dx, \quad \forall u \in L^2(\Omega), \\ \Phi_c(u) &= \int_{\Gamma_1} \phi_c(u) ds, \quad \forall u \in L^2(\Gamma_1), \end{aligned} \quad (61)$$

$$\Phi_A(u) = \Phi_a(u_1) + \Phi_c(u_2), \quad \forall u = (u_1, u_2) \in H,$$

$$\Phi_B(u) = \frac{1}{p} \int_\Omega |\nabla u|^p dx + \frac{1}{r} \int_{\Gamma_1} g(x) |u|^r ds, \quad \forall u \in V.$$

Let $\mathcal{A} = \partial_V(\Phi_A \circ i)$ and $\mathcal{B} = \partial_V \Phi_B$; then $\forall v \in V$,

$$\begin{aligned} \langle \mathcal{A}(u), v \rangle &= \int_\Omega a(u) v dx + \int_{\Gamma_1} c(u) v ds, \\ \langle \mathcal{B}(u), v \rangle &= \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx + \int_{\Gamma_1} g(x) |u|^{r-2} u v ds, \\ \langle F(t), v \rangle &= \int_\Omega f(x, t) v dx. \end{aligned} \quad (62)$$

Assuming $a_0 \in L^2(\Omega)$ and $v_0 \in L^2(\Gamma_1)$, the problem (59) can be rewritten as

$$(CP) \begin{cases} \frac{d}{dt} \mathcal{A}(u) + \mathcal{B}(u) \ni F(t) & t \in (0, T), \\ \mathcal{A}(u) \ni v_0 & t = 0, \end{cases} \quad (63)$$

where $v_0 = (a_0, c_0) \in H \subset V'$.

5.2. Existence of Solutions. Applying the existence theorem of abstract form, we could claim the solvability of the problem (59), by some appropriate assumptions.

(H1) $a(\cdot), c(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ are nondecreasing and continuous.

(H2) $\forall (u_1, u_2) \in H = L^2(\Omega) \oplus L^2(\Gamma_1)$, $a(u_1) \in L^2(\Omega)$, and $c(u_2) \in L^2(\Gamma_1)$.

(H3) $g \in L^\infty(\Gamma_1)$ with $g \geq g_0 > 0$.

Assume that (H1)–(H3) hold; then Φ_A and Φ_B satisfy Theorem 2. Therefore, for any v_0 and F satisfying Theorem 2, the abstract problem (63) is solvable, which implies the existence for the solution of (59) as follows.

Theorem 21. Assume (H1)–(H3). Let $2 \leq p < +\infty$, $1 \leq r \leq p$, and $p' = p/(p - 1)$. Assume that $f \in W^{1,2}(0, T; W^{-1,p'}(\Omega))$. Assume $(a_0, c_0) \in L^2(\Omega) \oplus L^2(\Gamma_1)$ satisfying

$$\begin{aligned} \exists u_0 \in \{u \in W^{1,p}(\Omega) : u = 0 \text{ on } \Gamma_2\} \text{ such that} \\ a(u_0) = a_0, \quad c(u_0) = c_0. \end{aligned} \tag{64}$$

Then there exists at least one solution $u(x, t)$ of (59); that is, u is a measurable function such that

$$\begin{aligned} u &\in L^\infty(0, T; W^{1,p}(\Omega)), \\ a(u) &\in L^\infty(0, T; W^{-1,p'}(\Omega)) \cap W^{1,2}(0, T; W^{-1,p'}(\Omega)), \\ c(u) &\in L^\infty(0, T; L^{p'}(\Gamma_1)) \cap W^{1,2}(0, T; L^{p'}(\Gamma_1)), \end{aligned} \tag{65}$$

and for a. e. $t \in (0, T)$,

$$\begin{aligned} \int_\Omega \frac{\partial}{\partial t} a(u) \phi dx + \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \phi dx \\ + \int_{\Gamma_1} \left(\frac{\partial}{\partial t} c(u) + g(x) |u|^{p-2} u \right) \phi ds \\ = \int_\Omega f(x, t) \phi dx \end{aligned} \tag{66}$$

for any $\phi \in W^{1,p}(\Omega)$. Moreover, $a(u(t)) \rightarrow a_0$ strongly in $W^{-1,p'}(\Omega)$ and $c(u(t)) \rightarrow c_0$ strongly in $L^{p'}(\Gamma_1)$ as $t \rightarrow 0$.

Remark 22. More generally, instead of the boundary condition on Γ_1 in the problem (59), we assume that (see [27])

$$\begin{aligned} \frac{\partial}{\partial t} c(u) + |\nabla u|^{p-2} \nabla u \cdot n + b(u) + g(x) |u|^{r-2} u \ni 0, \\ (x, t) \in \Gamma_1 \times (0, T], \end{aligned} \tag{67}$$

where $b(\cdot)$ is multivalued and maximal monotone. Assume that there exists a proper, convex, lower semicontinuous function ϕ_b defined on \mathbb{R} with $\phi_b(0) = 0$ such that $b(\cdot) = \partial\phi_b(\cdot)$. Define

$$\Phi_b(u) = \int_{\Gamma_1} \phi_b(u) ds \quad \forall u \in V, \tag{68}$$

and make a modification of Φ_B as follows:

$$\begin{aligned} \Phi_B(u) = \frac{1}{p} \int_\Omega |\nabla u|^p dx + \frac{1}{r} \int_{\Gamma_1} g(x) |u|^r ds + \Phi_b(u) \\ \forall u \in V. \end{aligned} \tag{69}$$

Then the problem can be solved by imposing some appropriate assumptions on b . For a simple case, we could suppose that

(b1) there exists a constant $b_0 \in \mathbb{R}$, such that $y \geq b_0$ for any $x \in \mathbb{R}$, and $y \in b(x)$;

(b2) there exists a nondecreasing function $l_b(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\|b(u)\|_{L^{p'}(\Gamma_1)} \leq l_b(\Phi_b(u)), \quad \forall u \in L^p(\Gamma_1). \tag{70}$$

Then Φ_B satisfy Theorem 2. In addition, (b1) and (b2) can be satisfied by extensive functions. For example, for $r_0 \in \mathbb{R}$, set

$$b(s) = \begin{cases} 1, & s > r_0, \\ [-1, 1], & s = r_0, \\ -1, & s < r_0, \end{cases} \tag{71}$$

and $\Phi_b(u) = \int_{\Gamma_1} |u - r_0| ds$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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