Research Article

A Crank-Nicolson Difference Scheme for Solving a Type of Variable Coefficient Delay Partial Differential Equations

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A linearized Crank-Nicolson difference scheme is constructed to solve a type of variable coefficient delay partial differential equations. The difference scheme is proved to be unconditionally stable and convergent, where the convergence order is two in both space and time. A numerical test is provided to illustrate the theoretical results.

1. Introduction

In the past few years, many scholars pay their attention to the theory of delay differential equations (DDEs) [1, 2]. There are many research results on delay ordinary differential equations [3, 4]; however, only few scholars focus on studies of delay partial differential equations. As we know, since, in most cases, DDEs' exact solutions cannot be computed analytically, efficient numerical methods are needed to solve such equations.

In this paper, the numerical solutions of the following variable coefficient delay partial differential equations are considered:

$$r(x,t)\, u_t - d u_{xx} = f(u(x, t-s)), \quad (x,t) \in (0,1) \times (0,T],$$
$$(1)$$

$$u(x,t) = \phi(x,t), \quad x \in [0,1],\ t \in [-s,0], \quad (2)$$

$$u(0,t) = \alpha(t), \quad u(1,t) = \beta(t), \quad t \in (0,T], \quad (3)$$

where $d > 0$ is the constant diffusion coefficient, $s > 0$ is the delay term, $\Omega = [0,1] \times [-s,T]$, $r(x,t) \in C((0,1) \times (0,T])$, and $r(x,t) \geq c_0 > 0$. In the case of $r(x,t) = 1$, numerical solutions of (1)–(3) have been considered in [5–7]. A Crank-Nicolson scheme and a linearized compact difference scheme have been proposed by Zhang and Sun in [5] and [6],

respectively. Q. F. Zhang and C. J. Zhang considered a new linearized compact multisplitting scheme in [7]. We will construct a Crank-Nicolson scheme for solving (1)–(3). The unconditional stability and convergence will be shown in this paper, where the convergence order is two in both space and time. To testify the theoretical results, a numerical test is provided.

The paper is organized as follows. In Section 2, a linearized Crank-Nicolson scheme is constructed to solve (1)–(3). Section 3 considers the solvability, stability, and convergence of the Crank-Nicolson scheme. In Section 4, a numerical test is provided to illustrate the theoretical results. Section 5 gives a brief discussion of this paper.

2. Construction of the Linearized Crank-Nicolson Scheme

In this subsection, a linearized Crank-Nicolson scheme for solving (1)–(3) is constructed. In this paper, we make the following assumptions:

(H1) assume that (1)–(3) had a unique solution $u \in C^{4,3}(\Omega)$, u, and its partial derivatives are bounded by a constant c_1;

(H2) $f(u(x, t-s))$ has second derivatives, and we denote

$$c_2 = \max_{|\varepsilon_1| \leq \varepsilon_0, |\varepsilon_2| \leq \varepsilon_0, |\varepsilon_3| \leq \varepsilon_0} \left\{ \left| f\left(u\left(x,t-s\right)+\varepsilon_1\right)\right|, \right.$$

$$\left| f'\left(u\left(x,t-s\right)+\varepsilon_2\right)\right|, \quad (4)$$

$$\left. \left| f''\left(u\left(x,t-s\right)+\varepsilon_3\right)\right| \right\},$$

where $\varepsilon_0 > 0$, and c_2 are constants.

Two positive integers M and j are taken; then let $h = 1/M$, $\tau = s/j$, $x_i = ih$, $t_k = k\tau$ and $t_{k+1/2} = (t_k + t_{k+1})/2$. Define $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$, where $\Omega_h = \{x_i \mid 0 \leq i \leq M\}$ and $\Omega_\tau = \{t_k \mid -j \leq k \leq N\}$, $N = [T/\tau]$. Denote $U_i^k = u(x_i, t_k)$, $0 \leq i \leq M, -j \leq k \leq N$. Let

$$\mathscr{W} = \left\{ v_i^k \mid 0 \leq i \leq M, -j \leq k \leq N \right\} \quad (5)$$

be the grid function space defined on $\Omega_{h\tau}$. Introduce the following notations:

$$v_i^{k+1/2} = \frac{v_i^k + v_i^{k+1}}{2}, \qquad \delta_t v_i^{k+1/2} = \frac{v_i^{k+1} - v_i^k}{\tau},$$

$$(6)$$

$$\delta_x v_{i+1/2}^k = \frac{v_{i+1}^k - v_i^k}{h}, \qquad \delta_x^2 v_i^k = \frac{v_{i+1}^k - 2v_i^k + v_{i-1}^k}{h^2}.$$

Considering (1) at the point $(x_i, t_{k+1/2})$, we have

$$r\left(x_i, t_{k+1/2}\right) \frac{\partial u}{\partial t}\left(x_i, t_{k+1/2}\right) - d\frac{\partial^2 u}{\partial x^2}\left(x_i, t_{k+1/2}\right)$$

$$= f\left(u\left(x_i, t_{k+(1/2)-j}\right)\right), \quad 1 \leq i \leq M-1, \ 0 \leq k \leq N-1.$$

$$(7)$$

From Taylor expansion,

$$\frac{\partial u}{\partial t}\left(x_i, t_{k+1/2}\right) = \delta_t U_i^{k+1/2} - \frac{\tau^2}{24}\frac{\partial^3 u}{\partial t^3}\left(x_i, \eta_i^k\right),$$

$$\eta_i^k \in \left(t_k, t_{k+1}\right),$$

$$\frac{\partial^2 u}{\partial x^2}\left(x_i, t_{k+1/2}\right)$$

$$= \frac{1}{2}\left[\frac{\partial^2 u}{\partial x^2}\left(x_i, t_k\right) + \frac{\partial^2 u}{\partial x^2}\left(x_i, t_{k+1}\right)\right]$$

$$- \frac{\tau^2}{8}\frac{\partial^4 u}{\partial x^2 \partial t^2}\left(x_i, \gamma_i^k\right)$$

$$= \frac{1}{2}\left(\delta_x^2 U_i^k + \delta_x^2 U_i^{k+1}\right)$$

$$- \frac{h^2}{24}\left[\frac{\partial^4 u}{\partial x^4}\left(\xi_i^k, t_k\right) + \frac{\partial^4 u}{\partial x^4}\left(\xi_i^{k+1}, t_{k+1}\right)\right]$$

$$- \frac{\tau^2}{8}\frac{\partial^4 u}{\partial x^2 \partial t^2}\left(x_i, \gamma_i^k\right),$$

$$\xi_i^k, \xi_i^{k+1} \in \left(x_{i-1}, x_{i+1}\right), \quad \gamma_i^k \in \left(t_k, t_{k+1}\right),$$

$$f\left(u\left(x_i, t_{k+(1/2)-j}\right)\right)$$

$$= f\left(u\left(x_i, t_{k-j}\right)\right)$$

$$+ \frac{\tau}{2}f'\left(u\left(x_i, t_{k-j}\right)\right)u_t'\left(x_i, t_{k-j}\right) + \zeta_i^{k-j}$$

$$= f\left(U_i^{k-j}\right) + \frac{\tau}{2}f'\left(U_i^{k-j}\right)\delta_t U_i^{k+(1/2)-j}$$

$$+ \zeta_i^{k-j}, \quad \zeta_i^{k-j} \in \left(t_{k-j}, t_{k+1-j}\right),$$

$$(8)$$

where $|\zeta_i^{k-j}| \leq c_3 \tau^2$. Substituting (8) into (7) and denoting $r_i^{k+1/2} = r(x_i, t_{k+1/2})$, we obtain

$$r_i^{k+1/2}\delta_t U_i^{k+1/2} - d\delta_x^2 U_i^{k+1/2}$$

$$= f\left(U_i^{k-j}\right) + \frac{\tau}{2}f'\left(U_i^{k-j}\right)\delta_t U_i^{k+(1/2)-j} + R_i^k,$$

$$(9)$$

where

$$\left|R_i^k\right| \leq c_4\left(\tau^2 + h^2\right), \quad 1 \leq i \leq M-1, \ 0 \leq k \leq N-1. \quad (10)$$

Discretizing the initial and boundary conditions of (2) and (3), we obtain

$$U_i^k = \phi\left(x_i, t_k\right), \quad 0 \leq i \leq M, \ -j \leq k \leq 0, \quad (11)$$

$$U_0^k = \alpha\left(t_k\right), \quad U_M^k = \beta\left(t_k\right), \quad 1 \leq k \leq N. \quad (12)$$

Replacing U_i^k by u_i^k and omitting R_i^k, we obtain the following Crank-Nicolson scheme:

$$r_i^{k+1/2}\delta_t u_i^{k+1/2} - d\delta_x^2 u_i^{k+1/2}$$

$$= f\left(u_i^{k-j}\right) + \frac{\tau}{2}f'\left(u_i^{k-j}\right)\delta_t u_i^{k+(1/2)-j},$$

$$(13)$$

$$u_i^k = \phi\left(x_i, t_k\right), \quad 0 \leq i \leq M, \ -j \leq k \leq 0,$$

$$u_0^k = \alpha\left(t_k\right), \quad u_M^k = \beta\left(t_k\right), \quad 1 \leq k \leq N.$$

3. The Solvability, Convergence, and Stability of the Crank-Nicolson Scheme

Define the following grid function space on Ω_h:

$$V = v \mid v = \left(v_0, v_1, \ldots, v_M\right), \quad v_0 = v_M = 0. \quad (14)$$

If $v \in V$, introducing the following notations:

$$\|v\| = \sqrt{h\sum_{i=1}^{M-1}\left(v_i\right)^2}, \qquad |v|_1 = \sqrt{h\sum_{i=1}^{M}\left(\frac{v_i - v_{i-1}}{h}\right)^2},$$

$$(15)$$

$$\|v\|_\infty = \max_{0 \leq i \leq M}\left|v_i\right|.$$

The following two inequalities are satisfied [8]:

$$\|v\|_\infty \leq \frac{1}{2}|v|_1, \quad (16)$$

$$\|v\| \leq \frac{1}{\sqrt{6}}|v|_1. \quad (17)$$

For the analysis of the difference scheme, the following Lemma is needed.

Lemma 1 (see [8]). *Let $\{F^k \mid k \geq 0\}$ be nonnegative sequence and satisfy*

$$F^{k+1} \leq A + B\tau \sum_{i=1}^{k} F^l, \quad k = 0, 1, \ldots; \tag{18}$$

then

$$F^{k+1} \leq A \exp(BK\tau), \quad k = 0, 1, 2, \ldots, \tag{19}$$

where A and B are nonnegative constants.

Theorem 2. *The difference scheme* (13) *has a unique solution, under the condition that h and τ are small enough.*

Proof. From the positive definiteness of the coefficient matrix of the scheme (13), we can easily obtain the results of Theorem 2 by the mathematical induction method. □

Denoting $e_i^k = U_i^k - u_i^k$, $0 \leq i \leq M$, $-j \leq k \leq N$, subtracting (13) from (9), (11), and (12), respectively, we obtain the following error equations:

$$r_i^{k+1/2} \delta_t e_i^{k+1/2} - d\delta_x^2 e_i^{k+1/2}$$
$$= f\left(U_i^{k-j}\right) - f\left(u_i^{k-j}\right)$$
$$+ \frac{\tau}{2}\left[f'\left(U_i^{k-j}\right)\delta_t U_i^{k+(1/2)-j}\right. \tag{20}$$
$$\left. - f'\left(u_i^{k-j}\right)\delta_t u_i^{k+(1/2)-j}\right] + R_i^k,$$

$$e_i^k = 0, \quad 0 \leq i \leq M, \ -j \leq k \leq 0, \tag{21}$$

$$e_0^k = 0, \quad e_M^k = 0, \ 1 \leq k \leq N. \tag{22}$$

Theorem 3. *Letting h and τ be small enough, one has*

$$\|e^k\|_\infty \leq C\left(\tau^2 + h^2\right), \quad 0 \leq k \leq N, \tag{23}$$

where $C > 0$ is independent of h and τ.

Proof. Multiplying (20) by $h\delta_t e_i^{k+1/2}$ and summing up for i from 1 to $M - 1$, we obtain

$$h\sum_{i=1}^{M-1} r_i^{k+1/2}\left(\delta_t e_i^{k+1/2}\right)^2 - dh\sum_{i=1}^{M-1}\delta_x^2 e_i^{k+1/2}\delta_t e_i^{k+1/2} = I_1 + I_2, \tag{24}$$

where

$$I_1 = h\sum_{i=1}^{M-1}\left\{f\left(U_i^{k-j}\right) - f\left(u_i^{k-j}\right)\right.$$
$$+ \frac{\tau}{2}\left[f'\left(U_i^{k-j}\right)\delta_t U_i^{k+(1/2)-j}\right. \tag{25}$$
$$\left.\left. - f'\left(u_i^{k-j}\right)\delta_t u_i^{k+(1/2)-j}\right]\right\}\delta_t e_i^{k+1/2},$$

$$I_2 = h\sum_{i=1}^{M-1} R_i^k \delta_t e_i^{k+1/2}.$$

The mathematical induction method will be used to prove Theorem 3. From (21), we have $\|e^k\|_\infty = 0$, for $-j \leq k \leq 0$. Suppose that (23) is true for $0 < k \leq l$, we will prove that (23) is also valid for $k = l + 1$.

From the inductive assumption, we have

$$\|e^k\|_\infty \leq C\left(\tau^2 + h^2\right), \quad 0 \leq k \leq l. \tag{26}$$

In the following, each term of (24) will be estimated. Consider

$$h\sum_{i=1}^{M-1} r_i^{k+1/2}\left(\delta_t e_i^{k+1/2}\right)^2 \geq c_0\left\|\delta_t e^{k+1/2}\right\|^2,$$
$$-dh\sum_{i=1}^{M-1}\delta_x^2 e_i^{k+1/2}\delta_t e_i^{k+1/2} = \frac{d}{2\tau}\left(\left|e^{k+1}\right|_1^2 - \left|e^k\right|_1^2\right). \tag{27}$$

From (H1) and (H2), we have

$$f\left(U_i^{k-j}\right) - f\left(u_i^{k-j}\right) + \frac{\tau}{2}\left[f'\left(U_i^{k-j}\right)\delta_t U_i^{k+(1/2)-j}\right.$$
$$\left. - f'\left(u_i^{k-j}\right)\delta_t u_i^{k+(1/2)-j}\right]$$
$$\leq c_2\left|e_i^{k-j}\right| + \frac{\tau}{2}\left[f'\left(U_i^{k-j}\right)\delta_t e_i^{k+(1/2)-j}\right. \tag{28}$$
$$+ \left(f'\left(U_i^{k-j}\right) - f'\left(u_i^{k-j}\right)\right)$$
$$\times\left.\left(\delta_t U_i^{k+(1/2)-j} - \delta_t e_i^{k+(1/2)-j}\right)\right]$$
$$\leq C\left(\left|e_i^{k-j}\right| + \left|e_i^{k+1-j}\right|\right).$$

Using the above inequality, we have

$$I_1 \leq Ch\sum_{i=1}^{M-1}\left(\left|e_i^{k-j}\right| + \left|e_i^{k+1-j}\right|\right)\left|\delta_t e_i^{k+1/2}\right|$$
$$\leq \varepsilon h\sum_{i=1}^{M-1}\left(\delta_t e_i^{k+1/2}\right)^2 + Ch\left(\sum_{i=1}^{M-1}\left(e_i^{k-j}\right)^2 + \sum_{i=1}^{M-1}\left(e_i^{k+1-j}\right)^2\right)$$
$$= \varepsilon\left\|\delta_t e^{k+1/2}\right\|^2 + C\left(\left\|e^{k-j}\right\|^2 + \left\|e^{k+1-j}\right\|^2\right),$$
$$I_2 \leq \varepsilon h\sum_{i=1}^{M-1}\left(\delta_t e_i^{k+1/2}\right)^2 + Ch\sum_{i=1}^{M-1}\left(R_i^k\right)^2$$
$$\leq \varepsilon\left\|\delta_t e^{k+1/2}\right\|^2 + C\left(\tau^2 + h^2\right)^2. \tag{29}$$

Inserting (27)–(29) into (24), we obtain

$$c_0\left\|\delta_t e^{k+1/2}\right\|^2 + \frac{d}{2\tau}\left(\left|e^{k+1}\right|_1^2 - \left|e^k\right|_1^2\right)$$
$$\leq 2\varepsilon\left\|\delta_t e^{k+1/2}\right\|^2 + C\left(\left\|e^{k-j}\right\|^2 + \left\|e^{k+1-j}\right\|^2\right) \tag{30}$$
$$+ C\left(\tau^2 + h^2\right)^2, \quad 0 \leq k \leq l.$$

Taking $\varepsilon = c_0/2$, we have

$$\frac{d}{2\tau}\left(\left|e^{k+1}\right|_1^2 - \left|e^k\right|_1^2\right) \leq C\left(\left\|e^{k-j}\right\|^2 + \left\|e^{k+1-j}\right\|^2\right)$$
$$+ C\left(\tau^2 + h^2\right)^2, \quad 0 \leq k \leq l. \tag{31}$$

The above inequality has the following form:

$$\left|e^{k+1}\right|_1^2 \leq \left|e^k\right|_1^2 + C\tau\left(\left\|e^{k-j}\right\|^2 + \left\|e^{k+1-j}\right\|^2\right)$$
$$+ C\tau\left(\tau^2 + h^2\right)^2, \quad 0 \leq k \leq l. \tag{32}$$

Summing up (32) for k, noticing (21), and exploiting (17), we have

$$\left|e^{k+1}\right|_1^2 \leq \left|e^0\right|_1^2 + C\tau \sum_{m=0}^{k}\left(\left\|e^{m-j}\right\|^2 + \left\|e^{m+1-j}\right\|^2\right)$$
$$+ C\tau \sum_{k=0}^{l}\left(\tau^2 + h^2\right)^2$$
$$\leq C\tau \sum_{m=1}^{k+1-j}\left\|e^m\right\|^2 + C\left(\tau^2 + h^2\right)^2$$
$$\leq C\tau \sum_{m=1}^{k}\left|e^m\right|_1^2 + C\left(\tau^2 + h^2\right)^2, \quad 0 \leq k \leq l. \tag{33}$$

By Lemma 1, we have

$$\left|e^{l+1}\right|_1^2 \leq C\left(\tau^2 + h^2\right)^2, \tag{34}$$

where C is a constant which depends on c_0, c_1, c_2, d, and T. From (16), we obtain

$$\left\|e^{l+1}\right\|_\infty \leq C\left(\tau^2 + h^2\right). \tag{35}$$

By the inductive principle, this completes the proof. □

Remark 4. Theorem 3 shows that the convergence order of the variable coefficient delay partial differential equations (1) is $o(t^2 + h^2)$. However, for the constant coefficient delay partial differential equations ($r(x,t) = 1$ in (1)), a Crank-Nicolson scheme with $o(t^2 + h^2)$ convergence is constructed in [5], and a new difference scheme with $o(t^2 + h^4)$ convergence is constructed in [9].

To discuss the stability of the difference scheme (13), we consider the following problem:

$$r(x,t)v_t - dv_{xx} = f(v(x,t-s)), \quad (x,t) \in (0,1) \times (0,T],$$
$$v(x,t) = \phi(x,t) + \psi(x,t), \quad x \in [0,1], \ t \in [-s,0],$$
$$v(0,t) = \alpha(t), \quad v(1,t) = \beta(t), \quad t \in (0,T]. \tag{36}$$

The following difference scheme solving for (36) can be obtained:

$$r_i^{k+1/2}\delta_t v_i^{k+1/2} - d\delta_x^2 v_i^{k+1/2}$$
$$= f\left(v_i^{k-j}\right) + \frac{\tau}{2}f'\left(v_i^{k-j}\right)\delta_t v_i^{k+(1/2)-j}, \tag{37}$$
$$v_i^k = \phi(x_i, t_k) + \psi_i^k, \quad 0 \leq i \leq M, \ -j \leq k \leq 0,$$
$$v_0^k = \alpha(t_k), \quad v_M^k = \beta(t_k), \quad 1 \leq k \leq N,$$

where ψ_i^k is a perturbation of $\phi(x_i, t_k)$.

Similar to the proof of Theorem 3, the following stability result can be obtained.

Theorem 5. *Denote*

$$\eta_i^k = v_i^k - u_i^k, \quad 0 \leq i \leq M, \ -j \leq k \leq N. \tag{38}$$

Then, there exist constants c_5 and c_6 such that

$$\left\|\eta^k\right\|_\infty \leq c_5 \sqrt{\tau h \sum_{m=-j}^{0}\sum_{i=1}^{M-1}\left(\psi_i^k\right)^2} \tag{39}$$

under the condition that h and τ are small enough and $\max_{-j \leq k \leq 0, 0 \leq i \leq M}|\psi_i^k| \leq c_6$.

Remark 6. Under the condition of assumptions (H1) and (H2) and $\max_{-j \leq k \leq 0, 0 \leq i \leq M}|\psi_i^k| \leq c_6$, for small h and τ, we can get the stability results of Theorem 5 (which can be referred to in [8, 10, 11]), where the difficulty is that $r_i^{k+1/2} \neq 1$; the proof can be referred to in the proof of Theorem 3.

4. Numerical Test

In this section, a numerical example is considered to validate the algorithm provided in this paper, and the numerical solutions u_i^k of the example are obtained by exploiting scheme (13). Define

$$E_\infty(h,\tau) = \max_{0 \leq i \leq M, 0 \leq k \leq N}\left|u(x_i, t_k) - u_i^k\right|. \tag{40}$$

Consider the following problem:

$$r(x,t)u_t - u_{xx} = u(x,t-0.1), \quad x \in (0,1), \ t \in (0,1],$$
$$u(x,t) = e^{-x}(1+t), \quad x \in (0,1), \ t \in [-0.1,0],$$
$$u(0,t) = 1+t, \quad u(1,t) = e^{-1}(1+t), \quad t \in (0,1], \tag{41}$$

where $r(x,t) = 2(t + 0.95)$. The exact solution of (41) is $u(x,t) = e^{-x}(1+t)$.

Table 1 provides some numerical results of difference scheme (13) solving for (41) with step size $h = \tau = 0.01$. Table 2 gives the maximum absolute errors between numerical solutions and exact solutions with different step sizes. From Table 2, we can see that when both the space step

TABLE 1: Numerical results of (41) when $h = \tau = 1/100$.

(x, t)	Numerical solution	Exact solution	$\|u(x_i, t_k) - u_i^k\|$
$(0.5, 0.1)$	0.667184	0.667184	$2.501e - 007$
$(0.5, 0.2)$	0.727837	0.727837	$4.324e - 007$
$(0.5, 0.3)$	0.788490	0.788490	$5.702e - 007$
$(0.5, 0.4)$	0.849144	0.849143	$6.812e - 007$
$(0.5, 0.5)$	0.909797	0.909796	$7.753e - 007$
$(0.5, 0.6)$	0.970450	0.970449	$8.586e - 007$
$(0.5, 0.7)$	1.031103	1.031102	$9.347e - 007$
$(0.5, 0.8)$	1.091756	1.091755	$1.006e - 006$
$(0.5, 0.9)$	1.152409	1.152408	$1.074e - 006$
$(0.5, 1.0)$	1.213062	1.213061	$1.140e - 006$

TABLE 2: Maximum norm errors of (41) with different step sizes.

h	τ	$E_\infty(h, \tau)$	$E_\infty(h, \tau)/E_\infty(h/2, \tau/2)$
1/10	1/10	$1.138e - 004$	$*$
1/20	1/20	$2.872e - 005$	3.962
1/40	1/40	$7.182e - 006$	3.999
1/80	1/80	$1.796e - 006$	4.000
1/160	1/160	$4.489e - 007$	4.000

FIGURE 1: Error curves of difference scheme (13) solving for problem (41) with different step sizes, when $t = 1$.
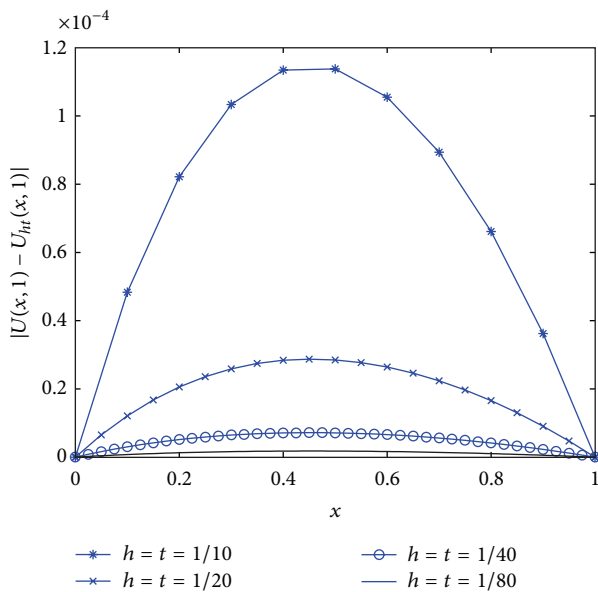

FIGURE 2: Error surface maps of difference scheme (13) solving for problem (41) with step size $h = \tau = 1/80$.

size and the time step size are reduced by a factor of 1/2, then the maximum absolute errors are reduced by a factor of approximately 1/4.

Figure 1 provides us with the error curves of numerical solutions for (41) at $t = 1$ by using scheme (13). Figures 2 and 3 give the error surface of the numerical solutions with step sizes $h = \tau = 1/80$ and $h = \tau = 1/160$, respectively.

Generally speaking, from the results of the tables and the figures provided, we can see that the numerical results are coincident with the theoretical results.
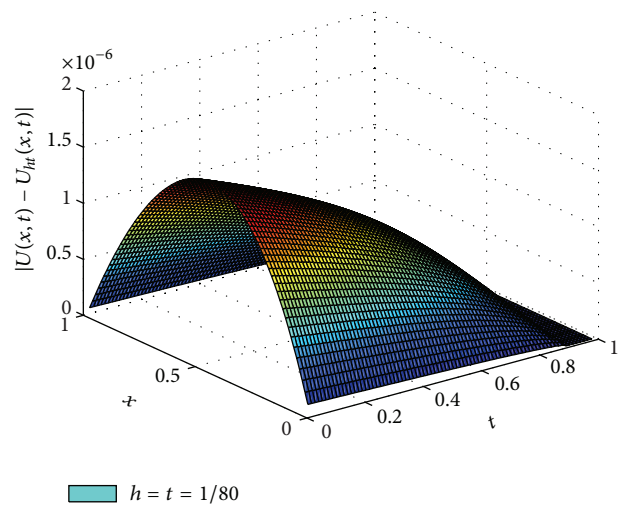
5. Conclusion

In this paper, a type of variable coefficient delay partial differential equations is considered. A linearized Crank-Nicolson scheme is constructed and is proved to be unconditionally stable and convergent. Finally, a numerical test is provided to illustrate the theoretical results.

Conflict of Interests

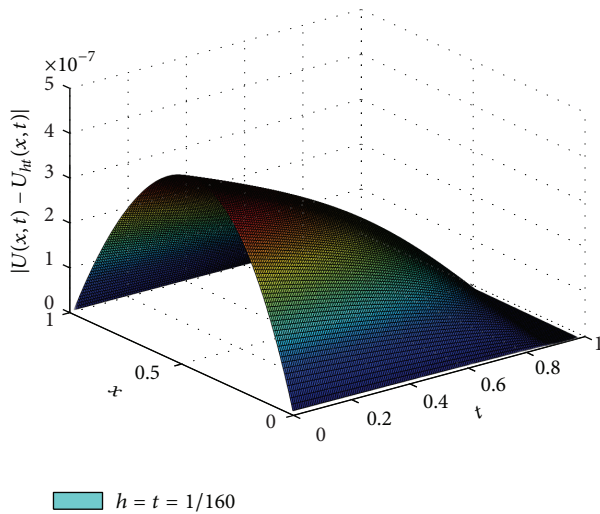The authors declare that there is no conflict of interests regarding the publication of this paper.

FIGURE 3: Error surface maps of difference scheme (13) solving for problem (41) with step size $h = \tau = 1/160$.

Acknowledgments

References

[1] D. F. Li and C. J. Zhang, "Nonlinear stability of discontinuous Galerkin methods for delay differential equations," *Applied Mathematics Letters*, vol. 23, no. 4, pp. 457–461, 2010.

[2] N. V. Azbelev, V. P. Maksimov, and L. F. Rakhmatullina, *Introduction to the Theory of Functional Differential Equations*, Nauka, Moscow, Russia, 1991.

[3] J. K. Hale, *Theory of Functional Differential Equations*, Springer, New York, NY, USA, 1993.

[4] O. Diekmann, S. A. van Gils, S. V. Lunel, and H.-O. Walther, *Delay Equations: Functional-, Complex-, and Nonlinear Analysis*, Springer, New York, NY, USA, 1995.

[5] Z.-B. Zhang and Z.-Z. Sun, "A Crank-Nicolson scheme for a class of delay nonlinear parabolic differential equations," *Journal on Numerical Methods and Computer Applications*, vol. 31, no. 2, pp. 131–140, 2010.

[6] Z.-Z. Sun and Z.-B. Zhang, "A linearized compact difference scheme for a class of nonlinear delay partial differential equations," *Applied Mathematical Modelling*, vol. 37, no. 3, pp. 742–752, 2013.

[7] Q. F. Zhang and C. J. Zhang, "A new linearized compact multisplitting scheme for the nonlinear convection-reaction-diffusion equations with delay," *Communications in Nonlinear Science and Numerical Simulation*, vol. 18, no. 12, pp. 3278–3288, 2013.

[8] Z.-Z. Sun, *The Numerical Methods for Partial Equations*, Science Press, Beijing, China, 2005. (Chinese).

[9] Y. F. Jin, J. X. Jiang, C. M. Hou, and D. H. Guan, "New difference scheme for general delay parabolic equations," *Journal of Information and Computational Science*, vol. 9, article 18, pp. 5579–5586, 2012.

[10] C. M. Huang and S. Vandewalle, "An analysis of delay-dependent stability for ordinary and partial differential equations with fixed and distributed delays," *SIAM Journal on Scientific Computing*, vol. 25, no. 5, pp. 1608–1632, 2004.

[11] C. M. Huang and S. Vandewalle, "Unconditionally stable difference methods for delay partial differential equations," *Numerische Mathematik*, vol. 122, no. 3, pp. 579–601, 2012.