Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 549428, 6 pages
http://dx.doi.org/10.1155/2014/549428

Research Article

Fixed Points of α-Admissible Mappings on Partial Metric Spaces

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Received 24 July 2014; Accepted 28 August 2014; Published 14 October 2014

Academic Editor: Wei-Shih Du

In this paper, a general class of α-admissible contractions on partial metric spaces is introduced. Fixed point theorems for these contractions on partial metric spaces and their consequences are stated and proved. Illustrative example is presented.

1. Introduction and Preliminaries

A rapid progress in the fixed point theory has been observed in the last few decades. This is a consequence of the fact that fixed point theory is a major tool in nonlinear analysis and has application in almost all branches of mathematics and natural sciences.

In 1992 Matthews ([1, 2]) introduced a new type of a metric called partial metric and a corresponding space called partial metric space (PMS), which have been defined due to a need in computer sciences. Partial metric spaces have been studied extensively since then; see [3–11] and references therein.

Improvement and generalization of the contractive conditions on the mappings are main concerns of most of the studies in fixed point theory. Such improvements and generalizations are usually done by means of auxiliary functions. Altering distance functions defined by Khan et al. [12] have been widely used for this reason both alone and combined with other functions.

In what follows, we employ two types of functions to define a class of contractions on partial metric spaces and investigate the existence and uniqueness of fixed points for these maps.

First, we introduce some basic concepts and notations to be used throughout the paper. We will denote by $\mathbb{N} = \{1, 2, 3, \ldots\}$ the set of natural numbers, denote by $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ the set of nonnegative integers, and denote by $\mathbb{R}^+ = [0, \infty)$ the set of nonnegative real numbers.

Definition 1 (see [12]). An altering distance function is a function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies the following.

(1) ψ is continuous and nondecreasing.

(2) $\psi(t) = 0 \Leftrightarrow t = 0$.

Partial metric space has been defined by Matthews as follows (See [1]).

Definition 2. Let X be a nonempty set and let $p : X \times X \to \mathbb{R}^+$ satisfy

$$\begin{aligned}
&(\mathrm{PM1}) \quad x = y \Longleftrightarrow p(x, x) = p(y, y) = p(x, y) \\
&(\mathrm{PM2}) \quad p(x, x) \le p(x, y) \\
&(\mathrm{PM3}) \quad p(x, y) = p(y, x) \\
&(\mathrm{PM4}) \quad p(x, y) \le p(x, z) + p(z, y) - p(z, z),
\end{aligned} \tag{1}$$

for all x, y, and $z \in X$. Then the pair (X, p) is called a partial metric space and p is called a partial metric on X.

One can easily see that the function $d_p : X \times X \to \mathbb{R}^+$, defined by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y), \tag{2}$$

is a metric on X. Moreover, every partial metric p on X generates a T_0 topology τ_p on X, whose base is a family of open p-balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$ where $B_p(x, \epsilon) = \{y \in X : p(x, y) \le p(x, x) + \epsilon\}$, for all $x \in X$ and $\epsilon > 0$.

Topological concepts such as convergence, Cauchy sequence, completeness, and continuity on PMS have also been defined in [1] as follows.

Definition 3. (1) A sequence $\{x_n\}$ in the PMS (X, p) converges to the limit x if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.

(2) A sequence $\{x_n\}$ in the PMS (X, p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.

(3) A PMS (X, p) is called complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

(4) A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $F(B_p(x_0, \delta)) \subseteq B_P(Fx_0, \epsilon)$.

Remark 4. The limit of a sequence $\{x_n\}$ in a partial metric space (X, p) may not be unique.

We give next some basic results in PMS.

Lemma 5 (see [1, 2, 6]). *(1) A sequence $\{x_n\}$ is a Cauchy sequence in the PMS (X, p) if and only if it is a Cauchy sequence in the metric space (X, d_p).*

(2) A PMS (X, p) is complete if and only if the metric space (X, d_p) is complete.

Moreover,

$$\lim_{n\to\infty} d_p(x, x_n) = 0 \Longleftrightarrow p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{3}$$

Lemma 6 (see [7, 9]). *Assume $x_n \to z$ as $n \to \infty$ in a PMS (X, p) such that $p(z, z) = 0$. Then $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for every $y \in X$.*

Lemma 7 (see [7, 9]). *Let (X, p) be a complete PMS.*

(A) *If $p(x, y) = 0$, then $x = y$;*

(B) *If $x \neq y$, then $p(x, y) > 0$.*

Admissible mappings have been defined recently by Samet et al. [13] and employed quite often in order to generalize the results on various contractions, see [14–17]. We state next the definitions of α-admissible mapping and triangular α-admissible mappings.

Definition 8. A mapping $T : X \to X$ is called α-admissible if for all $x, y \in X$ we have

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1, \tag{4}$$

where $\alpha : X \times X \to \mathbb{R}^+$ is a given function.

Definition 9. A mapping $T : X \to X$ is called triangular α-admissible if it is α-admissible and satisfies

$$\begin{matrix} \alpha(x, y) \geq 1 \\ \alpha(y, z) \geq 1 \end{matrix} \Longrightarrow \alpha(x, z) \geq 1, \tag{5}$$

where $x, y, z \in X$ and $\alpha : X \times X \to \mathbb{R}^+$ is a given function.

In [16], Alsulami et al. defined the following weaker condition which is sufficient in the proof of existence and uniqueness theorems.

Definition 10. A mapping $T : X \to X$ is said to be weak triangular α-admissible if it is α-admissible and satisfies

$$\alpha(x, Tx) \geq 1 \Longrightarrow \alpha(x, T^2x) \geq 1, \tag{6}$$

where $x \in X$ and $\alpha : X \times X \to \mathbb{R}^+$ is a given function.

Weak triangular α-admissible mappings satisfy a property stated in the following Lemma the proof of which easily follows from the definition and can be found in [15].

Lemma 11 (see [15]). *Let $T : X \to X$ be a weak triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. If $x_n = T^n x_0$, then $\alpha(x_m, x_n) \geq 1$ for all $m, n \in \mathbb{N}_0$ with $m < n$.*

2. Fixed Point Theorems on Complete Partial Metric Spaces

Our main results include theorems on existence and uniqueness of fixed points for a class of weak triangular α-admissible mappings defined on partial metric spaces. Inspired by a recent study of Alsulami et al. [16] and Yan et al. [18], we define a class of α-admissible contractions on a PMS via auxiliary functions and discuss the existence and uniqueness of their fixed points.

Our main theorem is stated below.

Theorem 12. *Let (X, p) be a complete partial metric space. Let $T : X \to X$ be a continuous, weak triangular α-admissible mapping such that*

$$\alpha(x, y)\psi(p(Tx, Ty)) \leq \phi(M(x, y)), \quad \forall x, y \in X, \tag{7}$$

where ψ is an altering distance functions, $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying $\psi(t) > \phi(t)$, for all $t > 0$, and

$$M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty)\}. \tag{8}$$

If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has a fixed point.

Proof. Take $x_0 \in X$ which satisfies $\alpha(x_0, Tx_0) \geq 1$ and define the sequence $\{x_n\}$ as $x_{n+1} = Tx_n$ for $n \in \mathbb{N}_0$.

If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}_0$, then obviously, $x_{n_0} = Tx_{n_0}$ a fixed point of T. Suppose that $p(x_n, x_{n+1}) > 0$, for all $n \in \mathbb{N}_0$.

Since T is α-admissible and $\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1$, we deduce

$$\begin{aligned} &\alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1, \quad \text{therefore} \\ &\alpha(Tx_1, Tx_2) = \alpha(x_2, x_3) \geq 1, \end{aligned} \tag{9}$$

and continuing in this way, we get

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \forall n \in \mathbb{N}_0. \tag{10}$$

Due to (10) we can put $x = x_n$ and $y = x_{n-1}$ in (7) which gives

$$\begin{aligned} &\psi(p(x_{n+1}, x_n)) \\ &\quad \leq \alpha(x_n, x_{n-1})\psi(p(x_{n+1}, x_n)) \\ &\quad = \alpha(x_n, x_{n-1})\psi(p(Tx_n, Tx_{n-1})) \leq \phi(M(x_n, x_{n-1})), \end{aligned} \tag{11}$$

where

$$\begin{aligned} M(x_n, x_{n-1}) &= \max\{p(x_n, x_{n-1}), p(x_n, Tx_n), p(x_{n-1}, Tx_{n-1})\} \\ &= \max\{p(x_n, x_{n-1}), p(x_n, x_{n+1})\}. \end{aligned} \tag{12}$$

If $M(x_n, x_{n-1}) = p(x_n, x_{n+1})$ for some n, then the inequality (11) becomes

$$0 < \psi(p(x_{n+1}, x_n)) \le \phi(p(x_n, x_{n+1})), \tag{13}$$

which is not possible since $\psi(t) > \phi(t)$ for $t > 0$. Then, we should have $M(x_n, x_{n-1}) = p(x_n, x_{n-1})$ for all $n \ge 1$ and, thus,

$$0 < \psi(p(x_{n+1}, x_n)) \le \phi(p(x_n, x_{n-1})) < \psi(p(x_n, x_{n-1})), \tag{14}$$

which gives

$$p(x_{n+1}, x_n) < p(x_n, x_{n-1}), \tag{15}$$

since ψ is a nondecreasing function. Therefore, the sequence $\{p(x_{n+1}, x_n)\}$ is a decreasing sequence bounded below by 0 and hence converges to a limit; say $r \ge 0$. Taking limit as $n \to \infty$ in (11), we get

$$\psi(r) \le \phi(r). \tag{16}$$

However, since by definition of ψ and ϕ we have $\psi(t) > \phi(t)$ for $t > 0$, the above inequality is possible only for $r = 0$, that is,

$$r = \lim_{n\to\infty} p(x_{n+1}, x_n) = 0. \tag{17}$$

On the other hand, by (PM2), we have

$$p(x_n, x_n) \le p(x_{n+1}, x_n), \tag{18}$$

or upon letting $n \to \infty$,

$$0 \le \lim_{n\to\infty} p(x_n, x_n) \le \lim_{n\to\infty} p(x_{n+1}, x_n) = 0, \tag{19}$$

that is,

$$\lim_{n\to\infty} p(x_n, x_n) = 0. \tag{20}$$

We prove next that $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_p), where d_p is the metric defined in (2) associated with the partial metric p. Assume that $\{x_n\}$ is not Cauchy. Then, for some $\varepsilon > 0$ there exist subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with

$$n_k > m_k > k, \qquad d_p(x_{n_k}, x_{m_k}) \ge \varepsilon, \tag{21}$$

for all $k \ge 1$, where corresponding to each m_k, we choose n_k to be smallest integer for which (21) holds. Then

$$d_p(x_{n_k-1}, x_{m_k}) < \varepsilon. \tag{22}$$

Note that from

$$d_p(x_n, x_{n+1}) = 2p(x_n, x_{n+1}) - p(x_n, x_n) - p(x_{n+1}, x_{n+1}), \tag{23}$$

we have

$$\lim_{n\to\infty} d_p(x_n, x_{n+1}) = \lim_{n\to\infty} [2p(x_n, x_{n+1}) - p(x_n, x_n) - p(x_{n+1}, x_{n+1})] = 0. \tag{24}$$

Using triangle inequality and regarding (21) and (22), we obtain

$$\begin{aligned} \varepsilon &\le d_p(x_{n_k}, x_{m_k}) \le d_p(x_{n_k}, x_{n_k-1}) + d_p(x_{n_k-1}, x_{m_k}) \\ &\le d_p(x_{n_k}, x_{n_{k-1}}) + \varepsilon. \end{aligned} \tag{25}$$

Letting $k \to \infty$ in the above inequality and using (24), we get

$$\lim_{k\to\infty} d_p(x_{n_k}, x_{m_k}) = \varepsilon. \tag{26}$$

On the other hand, we also have

$$\begin{aligned} d_p(x_{n_k-1}, x_{m_k-1}) &\le d_p(x_{n_k-1}, x_{m_k}) + d_p(x_{m_k}, x_{m_k-1}) \\ &\le \varepsilon + d_p(x_{m_k}, x_{m_k-1}). \end{aligned} \tag{27}$$

Again by letting $k \to \infty$ and using (24) and (26), we get

$$\lim_{k\to\infty} d_p(x_{n_{k-1}}, x_{m_{k-1}}) = \varepsilon. \tag{28}$$

From (26) and (28) and using (20) it is easy to see that

$$\begin{aligned} \varepsilon &= \lim_{k\to\infty} d_p(x_{n_k}, x_{m_k}) \\ &= \lim_{k\to\infty} [2p(x_{n_k}, x_{m_k}) - p(x_{n_k}, x_{n_k}) - p(x_{m_k}, x_{m_k})] \\ &= \lim_{k\to\infty} 2p(x_{n_k}, x_{m_k}) \\ \varepsilon &= \lim_{k\to\infty} d_p(x_{n_k-1}, x_{m_k-1}) \\ &= \lim_{k\to\infty} [2p(x_{n_k-1}, x_{m_k-1}) - p(x_{n_k-1}, x_{n_k-1}) \\ &\qquad - p(x_{m_k-1}, x_{m_k-1})] \\ &= \lim_{k\to\infty} 2p(x_{n_k-1}, x_{m_k-1}), \end{aligned} \tag{29}$$

or

$$\lim_{k\to\infty} p(x_{n_k}, x_{m_k}) = \lim_{k\to\infty} p(x_{n_{k-1}}, x_{m_{k-1}}) = \frac{\varepsilon}{2}. \tag{30}$$

Thus, the limit of

$$\begin{aligned} M(x_{n_{k-1}}, x_{m_{k-1}}) &= \max\{p(x_{n_{k-1}}, x_{m_{k-1}}), p(x_{n_{k-1}}, Tx_{n_{k-1}}), \\ &\qquad p(x_{m_{k-1}}, Tx_{m_{k-1}})\}, \end{aligned} \tag{31}$$

as $k \to \infty$, is calculated as

$$\lim_{k\to\infty} M\left(x_{n_{k-1}}, x_{m_{k-1}}\right) = \lim_{k\to\infty} \max\left\{\frac{\varepsilon}{2}, 0, 0\right\} = \frac{\varepsilon}{2}, \quad (32)$$

due to (17) and (30). Recall that T is weak triangular α-admissible. Then, from Lemma 11 we have $\alpha(x_{n_{k-1}}, x_{m_{k-1}}) \geq 1$. Therefore, we can apply condition (7) with $x_{n_{k-1}}$ and $x_{m_{k-1}}$ to obtain

$$\begin{aligned} 0 < \psi\left(p\left(x_{n_k}, x_{m_k}\right)\right) &\leq \alpha\left(x_{n_k-1}, x_{m_k-1}\right)\psi\left(p\left(x_{n_k}, x_{m_k}\right)\right) \\ &\leq \phi\left(M\left(x_{n_{k-1}}, x_{m_{k-1}}\right)\right). \end{aligned} \quad (33)$$

Letting $k \to \infty$ and taking into account (30) and (32), we have

$$0 < \psi\left(\frac{\varepsilon}{2}\right) \leq \phi\left(\frac{\varepsilon}{2}\right). \quad (34)$$

Note however that the condition $\psi(t) > \phi(t)$, for $t > 0$ implies that the above inequality holds only if $\varepsilon/2 = 0$, or, equivalently, $\varepsilon = 0$ which contradicts the assumption that $\{x_n\}$ is not a Cauchy sequence. Thus, $\{x_n\}$ must be a Cauchy sequence in the metric space (X, d_p). By Lemma 5, the sequence $\{x_n\}$ is also a Cauchy sequence in the PMS (X, p) which is a complete PMS. Again by Lemma 5, (X, d_p) is a complete metric space. Therefore, there exists $u \in X$ such that

$$\lim_{n\to\infty} d_p\left(x_n, u\right) = 0. \quad (35)$$

Notice that from Lemma 5 we also have

$$p(u, u) = \lim_{n\to\infty} p\left(x_n, u\right). \quad (36)$$

Finally, the continuity of T gives

$$u = \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = Tu, \quad (37)$$

that is, u is a fixed point of T, which completes the proof. □

The continuity condition on α-admissible mappings is not required for the existence of a fixed point if the space under consideration has the following property.

(I) If $\{x_n\}$ is a sequence in X such that

$$x_n \longrightarrow x, \quad \alpha\left(x_n, x_{n+1}\right) \geq 1 \quad \forall n \in \mathbb{N}_0, \quad (38)$$

then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ for which

$$\alpha\left(x_{n_k}, x\right) \geq 1 \quad \forall k \in \mathbb{N}_0. \quad (39)$$

Under this condition, we can state another existence theorem as follows.

Theorem 13. *Let (X, p) be a complete PMS on which the condition (I) holds. Let $T : X \to X$ be a weak triangular α-admissible mapping such that*

$$\alpha(x, y)\psi(p(Tx, Ty)) \leq \phi(M(x, y)), \quad \forall x, y \in X, \quad (40)$$

where ψ is an altering distance functions, $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying $\psi(t) > \phi(t)$ for all $t > 0$ and

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}. \quad (41)$$

If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has a fixed point.

Proof. As in the proof of Theorem 12, we take $x_0 \in X$ which satisfies $\alpha(x_0, Tx_0) \geq 1$ and define the sequence $\{x_n\}$ as $x_{n+1} = Tx_n$ for $n \in \mathbb{N}_0$. The proof of convergence of this sequence to a limit $u \in X$ is exactly the same as the proof of Theorem 12. Since $\lim_{n\to\infty} x_n = u$, then the condition (I) implies $\alpha(x_{n_k}, u) \geq 1$, for all $k \in \mathbb{N}_0$. Applying the inequality (40) with $x = x_{n_k}$ and $y = u$ we get

$$\begin{aligned} \psi\left(p\left(x_{n_k+1}, Tu\right)\right) &\leq \alpha\left(x_{n_k}, u\right)\psi\left(p\left(x_{n_k+1}, Tu\right)\right) \\ &= \alpha\left(x_{n_k}, u\right)\psi\left(p\left(Tx_{n_k}, Tu\right)\right) \\ &\leq \phi\left(M\left(x_{n_k}, u\right)\right), \end{aligned} \quad (42)$$

where

$$M\left(x_{n_k}, u\right) = \max\left\{p\left(x_{n_k}, u\right), p\left(x_{n_k}, x_{n_k+1}\right), p(u, Tu)\right\}. \quad (43)$$

Taking limit as $k \to \infty$ and regarding the continuity of ψ and ϕ, we get

$$\begin{aligned} \psi(p(u, Tu)) &\leq \phi(\max\{p(u, u), 0, p(u, Tu)\}) \\ &= \phi(p(u, Tu)). \end{aligned} \quad (44)$$

Again, using the fact that $\psi(t) > \phi(t)$, for $t > 0$, we conclude that $p(u, Tu) = 0$ and hence, from Lemma 7, $Tu = u$, which completes the proof. □

For the uniqueness of fixed points of α-admissible contractions we need an extra condition. This condition reads as follows:

$$\begin{aligned} &\text{(II)} \ \ \forall x, y \in X, \ \text{there exists } z \in X \text{ such that} \\ &\qquad \alpha(x, z) \geq 1, \qquad \alpha(y, z) \geq 1. \end{aligned} \quad (45)$$

We prove the uniqueness of a fixed point for a subclass of contractions defined in Theorems 12 and 13. The reason for this is that the condition (I) is not sufficient for the uniqueness of fixed points of maps defined in these two theorems.

Theorem 14. *Let (X, p) be a complete partial metric space satisfying the condition (II). Let $T : X \to X$ be a weak triangular α-admissible mapping such that*

$$\alpha(x, y)\psi(p(Tx, Ty)) \leq \phi(p(x, y)), \quad \forall x, y \in X, \quad (46)$$

where ψ is an altering distance functions, $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying $\psi(t) > \phi(t)$ for all $t > 0$. Assume that either T is continuous or X satisfies the condition (I). If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has a unique fixed point.

Proof. The existence proof is similar to that of Theorem 12 (resp., Theorem 13) and hence we omit the details. To show the uniqueness, we assume that T has two different fixed points; say $x, y \in X$. From the condition (II), there exists $z \in X$, such that

$$\alpha(x, z) \geq 1, \quad \alpha(y, z) \geq 1. \tag{47}$$

Then, since T is α-admissible, we have from (47)

$$\begin{aligned} \alpha(T^n x, T^n z) &= \alpha(x, T^n z) \geq 1, \\ \alpha(T^n y, T^n z) &= \alpha(y, T^n z) \geq 1, \end{aligned} \tag{48}$$

for all $n \in \mathbb{N}_0$. Define the sequence $\{z_n\} \in X$ as $z_n = T^n z$. If $x = z_n$ for some $n \in \mathbb{N}_0$, then, $Tz_n = Tx = x$, that is, the sequence $\{z_n\}$ converges to the fixed point x. Assume that $x \neq z_n$ for all $n \in \mathbb{N}_0$. Applying (46) with $x = x$ and $y = z_n$ we get

$$\begin{aligned} 0 < \psi(p(x, z_n)) &\leq \alpha(x, z_{n-1}) \psi(p(Tx, Tz_{n-1})) \\ &\leq \phi(p(x, z_{n-1})) < \psi(p(x, z_{n-1})). \end{aligned} \tag{49}$$

Since ψ is nondecreasing, then $p(x, z_n) \leq p(x, z_{n-1})$ for all $n \in \mathbb{N}$. Thus, the sequence $\{p(x, z_n)\}$ is a positive non increasing sequence and hence, converges to a limit say $L \geq 0$. Taking limit as $n \to \infty$ in (49), and regarding continuity of ψ and ϕ, we deduce

$$0 \leq \psi(L) \leq \phi(L), \tag{50}$$

which is possible only if $L = 0$. Hence, we conclude that

$$\lim_{n \to \infty} p(x, z_n) = 0. \tag{51}$$

In a similar way, we obtain

$$\lim_{n \to \infty} p(y, z_n) = 0. \tag{52}$$

By Lemma 6 and (51) and (52), it follows that

$$p(x, x) = \lim_{n \to \infty} (x, z_n) = \lim_{n \to \infty} p(y, z_n) = p(y, x) = 0, \tag{53}$$

and using the fact that $p(x, x) = p(y, y) = 0$, the condition (PM1) implies $x = y$, which completes the proof of uniqueness. □

3. Consequences and an Example

The class of contractions defined in Theorems 12 and 13 is quite general and many particular results can be concluded from these theorems. Some of these conclusions are stated below.

Corollary 15. *Let (X, p) be a complete PMS. Let $T : X \to X$ be weak triangular α-admissible mapping such that*

$$\alpha(x, y) p(Tx, Ty) \leq kM(x, y), \quad \forall x, y \in X, \tag{54}$$

where $0 < k < 1$ and

$$M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty)\}. \tag{55}$$

Assume that either T is continuous or X satisfies the condition (I). If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has a fixed point.

Proof. Proof is obvious by choosing $\psi(t) = t$ and $\phi(t) = kt$ in Theorem 12. □

Corollary 16. *Let (X, p) be a complete PMS. Let $T : X \to X$ be a weak triangular α-admissible mapping such that*

$$\alpha(x, y) p(Tx, Ty) \leq ap(x, y) + bp(x, Tx) + cp(y, Ty), \tag{56}$$

for all $x, y \in X$, where $0 < a + b + c < 1$. Assume that either T is continuous or X satisfies the condition (I). If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has a fixed point.

Proof. Due to the fact that

$$ad(x, y) + bd(x, Tx) + cd(y, Ty) \leq kM(x, y), \tag{57}$$

proof follows from Corollary 15. □

Last, we give the following example to illustrate our results.

Example 1. Let $X = \mathbb{R}^+ = [0, \infty)$, and define $p(x, y)$ and $\alpha(x, y)$ on X as

$$\begin{aligned} p(x, y) &= \max\{x, y\}, \\ \alpha(x, y) &= \begin{cases} 2 & \text{if } x, y \in [0, 1] \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{58}$$

respectively. Let $\psi(t) = 2t$ and $\phi(t) = t$ and T be defined as

$$Tx = \begin{cases} x - \dfrac{4}{5} & \text{if } x > 1 \\ \dfrac{x}{5} & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } x \leq 0. \end{cases} \tag{59}$$

Clearly, T is continuous. Then for $x, y \in [0, 1]$ with $x \leq y$ we have

$$\begin{aligned} &\alpha(x, y) \psi(p(Tx, Ty)) \\ &\quad = \frac{4y}{5} \leq \phi(\max\{p(x, y),\ p(x, Tx),\ p(y, Ty)\}) \\ &\quad = \max\{y, x, y\} = y, \end{aligned} \tag{60}$$

and similarly, for $x, y \in [0, 1]$ with $y \leq x$,

$$\begin{aligned} &\alpha(x, y) \psi(p(Tx, Ty)) \\ &\quad = \frac{4x}{5} \leq \phi(\max\{p(x, y),\ p(x, Tx),\ p(y, Ty)\}) \\ &\quad = \max\{x, x, y\} = x. \end{aligned} \tag{61}$$

For $x \notin [0,1]$, $y \notin [0,1]$, or $x, y \notin [0,1]$ the contractive condition of Theorem 12 is already satisfied since in this case $\alpha(x, y) = 0$. In addition, for $x = 1/2$ we have

$$\alpha\left(\frac{1}{2}, T\frac{1}{2}\right) = \alpha\left(\frac{1}{2}, \frac{1}{10}\right) = 2 \geq 1. \qquad (62)$$

Since all conditions of Theorem 12 hold, then T has a fixed point which clearly is $x = 0$.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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