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Research Article

The Center Conditions and Bifurcation of Limit Cycles at the Degenerate Singularity of a Three-Dimensional System

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We investigate multiple limit cycles bifurcation and center-focus problem of the degenerate equilibrium for a three-dimensional system. By applying the method of symbolic computation, we obtain the first four quasi-Lyapunov constants. It is proved that the system can generate 3 small limit cycles from nilpotent critical point on center manifold. Furthermore, the center conditions are found and as weak foci the highest order is proved to be the fourth; thus we obtain at most 3 small limit cycles from the origin via local bifurcation. To our knowledge, it is the first example of multiple limit cycles bifurcating from a nilpotent singularity for the flow of a high-dimensional system restricted to the center manifold.

1. Introduction

About dynamical behavior of the trajectories of three-dimensional system, bifurcation of limit cycles is one of major concerns; particularly, for Hopf bifurcation of a nondegenerate equilibrium with a pair of pure imaginary roots and a negative one, many investigations have been carried out in the past decades, for example, [1–4] for the three-dimensional chaotic systems, [5–8] for the three-dimensional Lotka-Volterra systems, and [9] for general three-dimensional systems.

However, up till now, study on bifurcation of limit cycles from the degenerate singularity for high-dimensional nonlinear dynamical systems is hardly seen in published references. In this paper, we investigate the following three-dimensional systems:

$$\begin{aligned}
\frac{dx}{dt} &= y + \mu x^2 + \sum_{k+2j+2l=3}^{\infty} A_{kjl} x^k y^j u^l = X(x, y, u),\\
\frac{dy}{dt} &= -2x^3 + 2\mu xy + \sum_{k+2j+2l=4}^{\infty} B_{kjl} x^k y^j u^l = Y(x, y, u), \qquad (1)\\
\frac{du}{dt} &= -du + \sum_{k+j+l=2}^{\infty} d_{kjl} x^k y^j u^l = U(x, y, u),
\end{aligned}$$

where $x, y, u, t, d, A_{kjl}, B_{kjl}, d_{kjl} \in \mathbb{R}$ $(d > 0, k, j, l \in \mathbb{N})$. Obviously system (1) has the Jacobian matrix at the origin as follows:

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & -d \end{pmatrix}, \quad \text{where } A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \qquad (2)$$

One notices that the origin has two zero eigenvalues and one negative eigenvalue and the block matrix A_1 is nilpotent. From the center manifold theorem, for the system (1), there exists the center manifold: $u = u(x, y)$ with $u(0,0) = 0$, $Du(0,0) = 0$, and more the flow on the center manifold is governed by a/a

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A_1 \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} X(x, y, u(x, y)) - y \\ Y(x, y, u(x, y)) \end{pmatrix}, \qquad (3)$$

namely,

$$\begin{aligned}
\frac{dx}{dt} &= y + \mu x^2 + \sum_{k+2j=3}^{\infty} a_{kj} x^k y^j = \widetilde{X}(x, y),\\
\frac{dy}{dt} &= -2x^3 + 2\mu xy + \sum_{k+2j=4}^{\infty} b_{kj} x^k y^j = \widetilde{Y}(x, y),
\end{aligned} \qquad (4)$$

where $x, y, t, a_{kj}, b_{kj} \in \mathbb{R}$ $(k, j \in \mathbb{N})$. It is usually called the reduced system and system (1) is topological equivalent to system (4) in the vicinity of the origin [10]. Thus by investigating system (4), we can solve Hopf bifurcation of the origin on center manifold of the three-dimensional system (1). At the same time, we also discuss the center-focus problem for the flow restricted to the center manifold, which closely relates to the maximum number of limit cycles bifurcating from the origin.

As far as investigation about limit cycles bifurcation of a nilpotent critical point in planar system is concerned, it is similar to nondegenerate case that detecting nilpotent center and calculating the focal value are needed [11]. There exist some available and classical ways, for instance, Poincare return map [12], Lyapunov function [13], and the normal form theory [14]. At the same time, some good results on the cyclicity were obtained [11, 12, 15, 16]. Recently in [17, 18], the authors gave an inverse integral factor method of calculating the quasi-Lyapunov constants of the three-order nilpotent critical point; it is convenient to compute the higher order focal values and solve the center-focus problem of the equilibrium. Here we extend this method's application to the three-dimensional system (1) and consider its specific example as follows:

$$\begin{aligned}\frac{dx}{dt} &= y + a_0 ux - 2xy = X,\\ \frac{dy}{dt} &= y^2 - 2x^3 + b_0 uy = Y,\\ \frac{du}{dt} &= -u + d_1 xy = U,\end{aligned} \tag{5}$$

where $a_0, b_0, d_1 \in \mathbb{R}$.

The rest of this paper is organized as follows. In Section 2, the corresponding quasi-Lyapunov constants are computed and the center conditions on the center manifold are determined. In Section 3, the multiple local bifurcations at the origin for system (5) are investigated, three limit cycles from the origin are obtained, and it is proved at most three small limit cycles from the origin via local bifurcation. In this work, the system and problem are all considered for the first time.

2. Quasi-Lyapunov Constants and Center Conditions

In this part, we firstly investigate the singular point quantities of the origin. For the center manifold of system (5), one can determine the formal expression $u = \tilde{u}(x, y)$ and obtain the same form as system (4):

$$\begin{aligned}\frac{dx}{dt} &= y + a_0 \tilde{u} x - 2xy = \widetilde{X},\\ \frac{dy}{dt} &= y^2 - 2x^3 + b_0 \tilde{u} y = \widetilde{Y}.\end{aligned} \tag{6}$$

Furthermore, we give the definition of quasi-Lyapunov constants for system (6) and the way of computing them; more details can be found in [17–19].

Lemma 1 (Theorems 8.7.1 and 8.7.2 in [17]). *For system* (6), *any positive integer s and a given number sequence* $\{c_{0\beta}\}$, $\beta \geqslant 3$, *we can derive successively and uniquely the terms of the following formal series with the coefficients* $c_{\alpha\beta}$ *satisfying* $\alpha \neq 0$,

$$M(x, y) = y^2 + \sum_{\alpha+\beta=3}^{\infty} c_{\alpha\beta} x^{\alpha} y^{\beta} \tag{7}$$

such that

$$\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) M - (s+1)\left(\frac{\partial M}{\partial x} X + \frac{\partial M}{\partial y} Y\right) = \sum_{m=6}^{\infty} \omega_m x^m. \tag{8}$$

And, if $\alpha \neq 0$, $c_{\alpha\beta}$ *is determined by the following recursive formula:*

$$\begin{aligned}c_{\alpha\beta} = \frac{1}{(s+1)\alpha} \sum_{k+j=2}^{\alpha+\beta-1} \{&[k - (s+1)(\alpha - k)] a_{kj} c_{\alpha-k,\beta-j+1}\\ &+ [j - (s+1)(\beta - j + 2)]\\ &\times b_{kj} c_{\alpha-k-1,\beta-j+2}\}\end{aligned} \tag{9}$$

and, for any positive integer $m \geqslant 6$, ω_m *is determined by the following recursive formula:*

$$\begin{aligned}\omega_m = \sum_{k+j=2}^{m-1} [&(k - (s+1)(\alpha - k + 1)) a_{kj} c_{m-k+1,-j}\\ &+ (j - (s+1)(1 - j)) b_{kj} c_{m-k,1-j}]\end{aligned} \tag{10}$$

and, when $\alpha < 0$ *or* $\beta < 0$, *we have let* $c_{\alpha\beta} = 0$.

Particularly, by choosing appropriate s and number sequence $\{c_{0\beta}\}$ $(\beta \geqslant 3)$, *we can make* $\omega_{2i+1}(s) = 0$ *and then let*

$$\omega_{2i+4}(s) = (2i - 4s - 1)\lambda_i. \tag{11}$$

Definition 2. The λ_i in (11) is called the ith quasi-Lyapunov constant of the origin of system (6), $i = 1, 2, \ldots$, and more if all the quasi-Lyapunov constants vanish, that is, $\lambda_m = 0$, $m = 1, 2, \ldots$, then the origin of system (5) is a center on the local center manifold at the origin.

Lemma 3. *For system* (6), *the mth focal value* $v_{2m}(-2\pi)$ *at the origin of system* (6) *is algebraic equivalent to* $\sigma_m \lambda_m$; *that is, for any positive integer* $m = 2, 3, \ldots$, *if* $v_2 = v_4 = \cdots = v_{2m-2} = 0$ *and* $\lambda_1 = \lambda_2 = \cdots = \lambda_{m-1} = 0$ *hold, then* $v_{2m} = \sigma_m \lambda_m$, *where* σ_m *is a defined constant (which is given in [17, 18]).*

Remark 4. According to Lemma 1, one sees that each ω_m in (10) is related to only the coefficients of the first $m-1$ degree terms of system (6). Here we determine the above $\tilde{u}(x, y)$ only up to a tenth degree polynomial with respect to x, y as follows:

$$\begin{aligned}\tilde{u}(x, y) = d_1 y (x - y) + d_1 y^2 (x + y)\\ + u_4 + u_5 + \cdots + u_{10},\end{aligned} \tag{12}$$

where $u_i, i = 4, \ldots, 10$ is a homogeneous polynomial in x, y of degree i, respectively, (which can be seen in the Appendix); thus $\widetilde{X}$ and $\widetilde{Y}$ in system (6) are two polynomials with degree 11. And more all a_{kj}, b_{kj} of $c_{\alpha\beta}$ in (9) are given definitely by the coefficients a_0, b_0, d_1 (the specific $c_{\alpha\beta}$ is available in Email address of the corresponding author); the ω_m in (10) is given by the following specific form:

$$\begin{aligned}
\omega_m = & -32d_1 (686 - 94a_0d_1 - 31b_0d_1) \\
& \times (21a_0 + b_0 - a_0m + 10a_0s - a_0ms) \\
& \times c[m-10, 0] \\
& - 2208d_1 (19a_0 + b_0 - a_0m + 9a_0s - a_0ms) \\
& \times c[m-9, 0] \\
& + 4d_1 (144 - 11a_0d_1 - 3b_0d_1) \\
& \times (17a_0 + b_0 - a_0m + 8a_0s - a_0ms) \\
& \times c[m-8, 0] \\
& + 40d_1 (15a_0 + b_0 - a_0m + 7a_0s - a_0ms) \\
& \times c[m-7, 0] \\
& - 24d_1 (13a_0 + b_0 - a_0m + 6a_0s - a_0ms) \\
& \times c[m-6, 0] \\
& + 2d_1 (9a_0 + b_0 - a_0m + 4a_0s - a_0ms) \\
& \times c[m-4, 0] + 2(1+s)c[m-3, 0].
\end{aligned} \tag{13}$$

Applying the powerful symbolic computation function of Mathematica system and the recursive formulas in Lemma 1, and from Remark 4, we obtain the first 7 quantities as follows:

$$\begin{aligned}
\omega_6 &= 0, \\
\omega_7 &= 3(1+s)c_{03}, \\
\omega_8 &= \frac{4}{5}d_1(2a_0 - b_0)(3-4s), \\
\omega_9 &= \frac{2}{3}a_0d_1(1-s), \\
\omega_{10} &= \frac{2}{7}a_0d_1(1-6a_0d_1), \\
\omega_{11} &= \frac{1}{112}\left(73a_0d_1 - 704a_0^2d_1^2 + 840c_{05}\right), \\
\omega_{12} &= -\frac{397}{105}a_0d_1.
\end{aligned} \tag{14}$$

In the above expression of each ω_k, $k = 7, \ldots, 12$, we have already let $\omega_6 = \cdots = \omega_{k-1} = 0$.

Particularly, in order to make $\omega_{2i+1} = 0$, $i = 1, \ldots, 5$, we let $s = 1$ and choose

$$c_{03} = 0, \qquad c_{05} = \frac{1}{840}\left(704a_0^2d_1^2 - 73a_0d_1\right). \tag{15}$$

Thus from the expression (11) in Lemma 1, we have the following.

Theorem 5. *For the flow on center manifold of system* (5), *we get the first 4 quasi-Lyapunov constants of the origin as follows:*

$$\begin{aligned}
\lambda_1 &= 0, \\
\lambda_2 &= \frac{4}{5}d_1(2a_0 - b_0), \\
\lambda_3 &= \frac{2}{7}a_0d_1(1-6a_0d_1), \\
\lambda_4 &= -\frac{397}{315}a_0d_1.
\end{aligned} \tag{16}$$

Theorem 6. *For system* (5), *the origin is a center on the local center manifold if and only if the following condition is satisfied:*

$$d_1 = 0 \quad or \quad a_0 = b_0 = 0. \tag{17}$$

Proof. The proof of the necessity is easy; then we omit it. Now we prove the sufficient condition; this technique derives from the method of Darboux (also see [20–23]). Obviously, if $d_1 = 0$ in the conditions (17) holds, then system (5) has the corresponding form as follows:

$$\begin{aligned}
\frac{dx}{dt} &= y + a_0ux - 2xy, \\
\frac{dy}{dt} &= y^2 - 2x^3 + b_0uy, \qquad \frac{du}{dt} = -u.
\end{aligned} \tag{18}$$

And more we can figure out easily one algebraic invariant surface for system (18): $F(x, y, u) = u$; in fact, there exists a polynomial $K(x, y, u) = -1$, the cofactor of $F(x, y, u)$, such that $dF/dt|_{(17)} = KF$. One can observe that $F(x, y, u) = 0$ is tangent to the center eigenspace, the (x, y)-plane, at the origin. Thus it forms a local center manifold in a neighborhood of the origin. From $F(x, y, u) = u = 0$, we substitute it into the first and second equations of the system defined by system (18) and we have the differential equations

$$\frac{dx}{dt} = y - 2xy, \qquad \frac{dy}{dt} = y^2 - 2x^3 \tag{19}$$

which is a Hamiltonian system with Hamiltonian function: $H(x, y) = x^4 - 2xy^2 + y^2$. Therefore the origin is a center for systems (18) or (19) as shown in Figure 1(a).

And if $a_0 = b_0 = 0$ in the conditions (17) holds, then system (5) has the corresponding form as follows:

$$\frac{dx}{dt} = y - 2xy, \qquad \frac{dy}{dt} = y^2 - 2x^3, \qquad \frac{du}{dt} = -u + d_1xy. \tag{20}$$

Clearly, the above $H = x^4 - 2xy^2 + y^2$ is also a first integral of system (20); thus the origin is a center for the flow of system (5) restricted to a center manifold. ☐

According to Lemma 3 and Theorem 6, then we have the following.

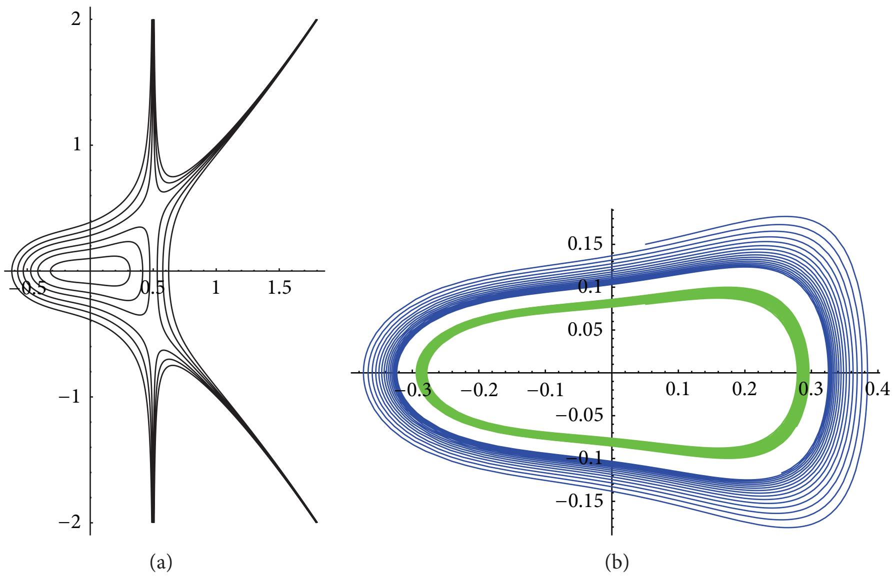

FIGURE 1: (a) The phase portrait of system (19) on the (x, y)-plane which shows that it is integrable at the origin; (b) the stable limit cycle bifurcated from the origin of system (24) with $\delta = 0.001$, $d_1 = 1$, $a_0 = 1/6$, $b_0 = 1/3 + 0.001$.

Theorem 7. *For the origin of system* (5) *as a weak focus on center manifold, the highest is the fourth order and its first 4 focal values are as follows:*

$$v_2(-2\pi) = \sigma_1\lambda_1, \qquad v_4(-2\pi) = \sigma_2\lambda_2,$$
$$v_6(-2\pi) = \sigma_3\lambda_3, \qquad v_8(-2\pi) = \sigma_4\lambda_4, \tag{21}$$

where, for the expression of v_4, *we have let* $v_2 = 0$, *and* $v_2 = v_4 = 0$ *for* v_6, $v_2 = v_4 = v_6 = 0$ *for* v_8.

3. Multiple Limit Cycle Bifurcation of the System

In this section, we apply the focal values obtained in last section to discuss multiple bifurcation of the equilibrium and demonstrate there exist 3 and at most 3 limit cycles.

From Theorems 5 and 7, we have

Theorem 8. *For the flow on center manifold of* (5), *the origin is a 4th-order fine focus; that is,* $\lambda_1 = \lambda_2 = \lambda_3 = 0$, $\lambda_4 \neq 0$ *if and only if*

$$b_0 = 2a_0, \qquad 6a_0d_1 = 1$$
$$\text{or} \quad b_0 = \frac{1}{3d_1}, \qquad a_0 = \frac{1}{6d_1} \tag{22}$$
$$(d_1 \neq 0).$$

Proof. Firstly, we prove the necessity, from the expressions (16) in Theorem 5; we let $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and consider $\lambda_4 \neq 0$ and then only $b_0 = 2a_0$ and $6a_0d_1 = 1$ are obtained at the same time. Then for the proof of the sufficiency, by substituting the above conditions into $\lambda_i, i = 1, 2, 3, 4$, one can get the conclusion easily. Furthermore we can transform the above conditions into the following form: $b_0 = 1/3d_1$, $a_0 = (1/6d_1)$ $(d_1 \neq 0)$. Thus the proof is completed. □

Remark 9. For the coefficients of system (5) or (6), when $d_1 \neq 0$, there always exists a group of critical values: $a_0 = 1/6d_1$, $b_0 = 1/3d_1$ such that the conditions (22) hold.

From Theorem 5, one calculates easily the Jacobian determinant with respect to the functions λ_2, λ_3 and variables a_0, b_0,

$$J = \frac{\partial(\lambda_2, \lambda_3)}{\partial(a_0 b_0)} = \frac{8}{35}d_1^2(1 - 12a_0d_1). \tag{23}$$

Considering the conditions (22) and when $d_1 \neq 0$ holds, we obtain $J = -(8/35)d_1^2 \neq 0$. At the same time, one can get that $\lambda_4 = -(397/1890) \neq 0$ holds.

Thus we take some appropriate perturbations for the coefficients of system (6) and get the perturbed system with the following form:

$$\frac{dx}{dt} = \delta x + y + a_0\tilde{u}x - 2xy,$$
$$\frac{dy}{dt} = 2\delta y + y^2 - 2x^3 + b_0\tilde{u}y, \tag{24}$$

where $0 < \delta \ll 1$. According to Theorem 2.2 in [18], system (6) has 3 limit cycles in the neighborhood of the origin; that is, system (5) has 3 limit cycles on the center manifold necessarily. One can refer to [9, 17, 18] for more details about constructing of limit cycles. And using numerical method, we get an example of one stable limit cycle for system (24) as shown in Figure 1(b).

Therefore, from the above discussion and Theorem 7, we have the following.

Theorem 10. *For system* (5), *3 and at most 3 small amplitude limit cycles can be bifurcated from the origin on the center manifold.*

4. Conclusions

In summary, based on precise symbolic computation, we have investigated deeply multiple limit cycles bifurcation in the vicinity of the degenerate equilibrium for a three-dimensional system. Firstly by computing its quasi-Lyapunov constants, we solved its center-focus problem on center manifold; thus the center conditions are found and as weak foci the highest order is proved to be the fourth. Then we obtained that the system can generate 3 small limit cycles from nilpotent critical point and at most 3 small limit cycles from the origin via local bifurcation. For the flow of a high-dimensional system restricted to the center manifold, only one interesting example of multiple limit cycles bifurcating from its nilpotent singularity was given here, but we believe that more outcomes on this problem will be shown in the near future.

Appendix

Consider

$$\begin{aligned}
u_4 = -d_1 \big(&-2x^4 + 12x^3 y - 36x^2 y^2 + a_0 d_1 x^2 y^2 \\
&+ b_0 d_1 x^2 y^2 + 72xy^3 - 3a_0 d_1 xy^3 \\
&-5b_0 d_1 xy^3 - 69y^4 + 3a_0 d_1 y^4 + 7b_0 d_1 y^4\big),
\end{aligned}$$

$$\begin{aligned}
u_5 = d_1 y \big(&20x^4 - 134x^3 y + 474x^2 y^2 - 4a_0 d_1 x^2 y^2 \\
&- 5b_0 d_1 x^2 y^2 - 876xy^3 + 5a_0 d_1 xy^3 \\
&+ 5b_0 d_1 xy^3 + 600y^4 + 7a_0 d_1 y^4 + 28b_0 d_1 y^4\big),
\end{aligned}$$

$$\begin{aligned}
u_6 = -d_1 \big(&24x^6 - 288x^5 y + 14a_0 d_1 x^5 y + 6b_0 d_1 x^5 y \\
&+ 1732x^4 y^2 - 144a_0 d_1 x^4 y^2 - 88b_0 d_1 x^4 y^2 \\
&- 6944x^3 y^3 + 744a_0 d_1 x^3 y^3 + 552b_0 d_1 x^3 y^3 \\
&- 3a_0^2 d_1^2 x^3 y^3 - 6a_0 b_0 d_1^2 x^3 y^3 - 3b_0^2 d_1^2 x^3 y^3 \\
&+ 20358x^2 y^4 - 2443a_0 d_1 x^2 y^4 - 2081b_0 d_1 x^2 y^4 \\
&+ 17a_0^2 d_1^2 x^2 y^4 + 46a_0 b_0 d_1^2 x^2 y^4 + 33b_0^2 d_1^2 x^2 y^4 \\
&- 42468xy^5 + 5038a_0 d_1 xy^5 + 4882b_0 d_1 xy^5 \\
&- 40a_0^2 d_1^2 xy^5 - 134a_0 b_0 d_1^2 xy^5 - 126b_0^2 d_1^2 xy^5 \\
&+ 45468y^6 - 5003a_0 d_1 y^6 - 5153b_0 d_1 y^6 \\
&+40a_0^2 d_1^2 y^6 + 152a_0 b_0 d_1^2 y^6 + 168b_0^2 d_1^2 y^6\big),
\end{aligned}$$

$$\begin{aligned}
u_7 = d_1 \big(&40x^7 - 1104x^6 y + 12060x^5 y^2 \\
&- 260a_0 d_1 x^5 y^2 - 128b_0 d_1 x^5 y^2 - 77700x^4 y^3 \\
&+ 2860a_0 d_1 x^4 y^3 + 1700b_0 d_1 x^4 y^3 + 337632x^3 y^4 \\
&- 15498a_0 d_1 x^3 y^4 - 10704b_0 d_1 x^3 y^4 + 24a_0^2 d_1^2 x^3 y^4 \\
&+ 56a_0 b_0 d_1^2 x^3 y^4 + 33b_0^2 d_1^2 x^3 y^4 - 1012896x^2 y^5 \\
&+ 49266a_0 d_1 x^2 y^5 + 39042b_0 d_1 x^2 y^5 - 94a_0^2 d_1^2 x^2 y^5 \\
&- 251a_0 b_0 d_1^2 x^2 y^5 - 169b_0^2 d_1^2 x^2 y^5 + 1898388xy^6 \\
&- 84891a_0 d_1 xy^6 - 72276b_0 d_1 xy^6 + 66a_0^2 d_1^2 xy^6 \\
&+ 67a_0 b_0 d_1^2 xy^6 - 166b_0^2 d_1^2 xy^6 - 1625580y^7 \\
&+ 54873a_0 d_1 y^7 + 45075b_0 d_1 y^7 + 174a_0^2 d_1^2 y^7 \\
&+915a_0 b_0 d_1^2 y^7 + 1419b_0^2 d_1^2 y^7\big),
\end{aligned}$$

$$\begin{aligned}
u_8 = -d_1 \big(&-576x^8 + 44a_0 d_1 x^8 + 12b_0 d_1 x^8 \\
&+ 10976x^7 y - 1264a_0 d_1 x^7 y - 496b_0 d_1 x^7 y \\
&- 106352x^6 y^2 + 16048a_0 d_1 x^6 y^2 \\
&+ 7696b_0 d_1 x^6 y^2 \\
&- 114a_0^2 d_1^2 x^6 y^2 - 116a_0 b_0 d_1^2 x^6 y^2 \\
&- 34b_0^2 d_1^2 x^6 y^2 + 704496x^5 y^3 \\
&- 126632a_0 d_1 x^5 y^3 - 69528b_0 d_1 x^5 y^3 \\
&+ 1700a_0^2 d_1^2 x^5 y^3 + 2172a_0 b_0 d_1^2 x^5 y^3 \\
&+ 816b_0^2 d_1^2 x^5 y^3 - 3558660x^4 y^4 \\
&+ 710088a_0 d_1 x^4 y^4 + 428808b_0 d_1 x^4 y^4 \\
&- 12764a_0^2 d_1^2 x^4 y^4 - 19064a_0 b_0 d_1^2 x^4 y^4 \\
&- 8340b_0^2 d_1^2 x^4 y^4 + 14a_0^3 d_1^3 x^4 y^4 \\
&+ 42a_0^2 b_0 d_1^3 x^4 y^4 + 42a_0 b_0^2 d_1^3 x^4 y^4 \\
&+ 14b_0^3 d_1^3 x^4 y^4 + 14104992x^3 y^5 \\
&- 2960538a_0 d_1 x^3 y^5 - 1907570b_0 d_1 x^3 y^5 \\
&+ 61685a_0^2 d_1^2 x^3 y^5 + 103855a_0 b_0 d_1^2 x^3 y^5 \\
&+ 50130b_0^2 d_1^2 x^3 y^5 - 125a_0^3 d_1^3 x^3 y^5 \\
&- 463a_0^2 b_0 d_1^3 x^3 y^5 - 591a_0 b_0^2 d_1^3 x^3 y^5 \\
&- 261b_0^3 d_1^3 x^3 y^5 - 43327872x^2 y^6 \\
&+ 9065880a_0 d_1 x^2 y^6 + 6166752b_0 d_1 x^2 y^6
\end{aligned}$$

$$- 200895a_0^2d_1^2x^2y^6 - 374342a_0b_0d_1^2x^2y^6$$
$$- 196423b_0^2d_1^2x^2y^6 + 504a_0^3d_1^3x^2y^6$$
$$+ 2186a_0^2b_0d_1^3x^2y^6 + 3360a_0b_0^2d_1^3x^2y^6$$
$$+ 1854b_0^3d_1^3x^2y^6 + 94249296xy^7$$
$$- 18564504a_0d_1xy^7 - 13274868b_0d_1xy^7$$
$$+ 412530a_0^2d_1^2xy^7 + 833201a_0b_0d_1^2xy^7$$
$$+ 468601b_0^2d_1^2xy^7 - 1097a_0^3d_1^3xy^7$$
$$- 5317a_0^2b_0d_1^3xy^7 - 9303a_0b_0^2d_1^3xy^7$$
$$- 6011b_0^3d_1^3xy^7 - 105628356y^8$$
$$+ 18948615a_0d_1y^8 + 13977981b_0d_1y^8$$
$$- 411312a_0^2d_1^2y^8 - 868420a_0b_0d_1^2y^8$$
$$- 503532b_0^2d_1^2y^8 + 1097a_0^3d_1^3y^8$$
$$+ 5673a_0^2b_0d_1^3y^8 + 10687a_0b_0^2d_1^3y^8$$
$$+7551b_0^3d_1^3y^8\big),$$

$$u_9 = d_1\big(-2208x^9 + 77328x^8y - 2384a_0d_1x^8y$$
$$- 784b_0d_1x^8y - 1227512x^7y^2$$
$$+ 64396a_0d_1x^7y^2 + 26272b_0d_1x^7y^2$$
$$+ 12357160x^6y^3 - 835460a_0d_1x^6y^3$$
$$- 396608b_0d_1x^6y^3 + 3144a_0^2d_1^2x^6y^3$$
$$+ 3498a_0b_0d_1^2x^6y^3 + 1096b_0^2d_1^2x^6y^3$$
$$- 89203392x^5y^4 + 6916384a_0d_1x^5y^4$$
$$+ 3676396b_0d_1x^5y^4 - 49214a_0^2d_1^2x^5y^4$$
$$- 63780a_0b_0d_1^2x^5y^4 - 22834b_0^2d_1^2x^5y^4$$
$$+ 483032256x^4y^5 - 40282356a_0d_1x^4y^5$$
$$- 23266264b_0d_1x^4y^5 + 378254a_0^2d_1^2x^4y^5$$
$$+ 553724a_0b_0d_1^2x^4y^5 + 217218b_0^2d_1^2x^4y^5$$
$$- 180a_0^3d_1^3x^4y^5 - 607a_0^2b_0d_1^3x^4y^5$$
$$- 687a_0b_0^2d_1^3x^4y^5 - 261b_0^3d_1^3x^4y^5$$
$$- 1968992136x^3y^6 + 169196340a_0d_1x^3y^6$$
$$+ 104279988b_0d_1x^3y^6 - 1780906a_0^2d_1^2x^3y^6$$
$$- 2890212a_0b_0d_1^2x^3y^6 - 1233114b_0^2d_1^2x^3y^6$$
$$+ 1292a_0^3d_1^3x^3y^6 + 4961a_0^2b_0d_1^3x^3y^6$$
$$+ 6477a_0b_0^2d_1^3x^3y^6 + 2877b_0^3d_1^3x^3y^6$$
$$+ 5820320664x^2y^7 - 495558804a_0d_1x^2y^7$$
$$- 321861864b_0d_1x^2y^7 + 5224500a_0^2d_1^2x^2y^7$$
$$+ 9158682a_0b_0d_1^2x^2y^7 + 4161294b_0^2d_1^2x^2y^7$$
$$- 3274a_0^3d_1^3x^2y^7 - 12840a_0^2b_0d_1^3x^2y^7$$
$$- 16145a_0b_0^2d_1^3x^2y^7 - 5819b_0^3d_1^3x^2y^7$$
$$- 11169394848xy^8 + 901925700a_0d_1xy^8$$
$$+ 606394284b_0d_1xy^8 - 8530348a_0^2d_1^2xy^8$$
$$- 15428426a_0b_0d_1^2xy^8 - 6980247b_0^2d_1^2xy^8$$
$$+ 949a_0^3d_1^3xy^8 - 2716a_0^2b_0d_1^3xy^8$$
$$- 23065a_0b_0^2d_1^3xy^8 - 31601b_0^3d_1^3xy^8$$
$$+ 10324368000y^9 - 750336780a_0d_1y^9$$
$$- 509164044b_0d_1y^9 + 5239852a_0^2d_1^2y^9$$
$$+ 8941749a_0b_0d_1^2y^9 + 3331701b_0^2d_1^2y^9$$
$$+ 7827a_0^3d_1^3y^9 + 50215a_0^2b_0d_1^3y^9$$
$$+119361a_0b_0^2d_1^3y^9 + 108056b_0^3d_1^3y^9\big),$$

$$u_{10} = -d_1\big(21952x^{10} - 3008a_0d_1x^{10} - 992b_0d_1x^{10}$$
$$- 605184x^9y + 107232a_0d_1x^9y + 42144b_0d_1x^9y$$
$$- 1104a_0^2d_1^2x^9y - 752a_0b_0d_1^2x^9y - 160b_0^2d_1^2x^9y$$
$$+ 8513712x^8y^2 - 1841888a_0d_1x^8y^2$$
$$- 822896b_0d_1x^8y^2 + 34440a_0^2d_1^2x^8y^2$$
$$+ 29616a_0b_0d_1^2x^8y^2 + 7896b_0^2d_1^2x^8y^2$$
$$- 81848832x^7y^3 + 20765520a_0d_1x^7y^3$$
$$+ 10179424b_0d_1x^7y^3 - 527376a_0^2d_1^2x^7y^3$$
$$- 530448a_0b_0d_1^2x^7y^3 - 162816b_0^2d_1^2x^7y^3$$
$$+ 1050a_0^3d_1^3x^7y^3 + 1774a_0^2b_0d_1^3x^7y^3$$
$$+ 1102a_0b_0^2d_1^3x^7y^3 + 250b_0^3d_1^3x^7y^3$$
$$+ 602777304x^6y^4 - 173386212a_0d_1x^6y^4$$
$$- 90723084b_0d_1x^6y^4 + 5270270a_0^2d_1^2x^6y^4$$
$$+ 5948592a_0b_0d_1^2x^6y^4 + 2010036b_0^2d_1^2x^6y^4$$

$$
\begin{aligned}
&- 21020a_0^3d_1^3x^6y^4 - 41960a_0^2b_0d_1^3x^6y^4\\
&- 31132a_0b_0^2d_1^3x^6y^4 - 8640b_0^3d_1^3x^6y^4\\
&- 3601377936x^5y^5 + 1131810888a_0d_1x^5y^5\\
&+ 620482512b_0d_1x^5y^5 - 38357066a_0^2d_1^2x^5y^5\\
&- 47295826a_0b_0d_1^2x^5y^5 - 17155148b_0^2d_1^2x^5y^5\\
&+ 212538a_0^3d_1^3x^5y^5 + 481702a_0^2b_0d_1^3x^5y^5\\
&+ 405182a_0b_0^2d_1^3x^5y^5 + 128818b_0^3d_1^3x^5y^5\\
&- 85a_0^4d_1^4x^5y^5 - 340a_0^3b_0d_1^4x^5y^5\\
&- 510a_0^2b_0^2d_1^4x^5y^5 - 340a_0b_0^3d_1^4x^5y^5\\
&- 85b_0^4d_1^4x^5y^5\\
&+ 17877283056x^4y^6 - 5874570084a_0d_1x^4y^6\\
&- 3333151404b_0d_1x^4y^6 + 212121436a_0^2d_1^2x^4y^6\\
&+ 280352816a_0b_0d_1^2x^4y^6 + 107174808b_0^2d_1^2x^4y^6\\
&- 1401156a_0^3d_1^3x^4y^6 - 3518659a_0^2b_0d_1^3x^4y^6\\
&- 3250715a_0b_0^2d_1^3x^4y^6 - 1130262b_0^3d_1^3x^4y^6\\
&+ 1085a_0^4d_1^4x^4y^6 + 5094a_0^3b_0d_1^4x^4y^6\\
&+ 9160a_0^2b_0^2d_1^4x^4y^6 + 7498a_0b_0^3d_1^4x^4y^6\\
&+ 2363b_0^4d_1^4x^4y^6 - 73199185920x^3y^7\\
&+ 23982302232a_0d_1x^3y^7\\
&+ 13978222776b_0d_1x^3y^7\\
&- 892808908a_0^2d_1^2x^3y^7 - 1249355132a_0b_0d_1^2x^3y^7\\
&- 497385264b_0^2d_1^2x^3y^7 + 6463496a_0^3d_1^3x^3y^7\\
&+ 17699203a_0^2b_0d_1^3x^3y^7 + 17662897a_0b_0^2d_1^3x^3y^7\\
&+ 6564610b_0^3d_1^3x^3y^7 - 6527a_0^4d_1^4x^3y^7\\
&- 34734a_0^3b_0d_1^4x^3y^7 - 72008a_0^2b_0^2d_1^4x^3y^7\\
&- 69410a_0b_0^3d_1^4x^3y^7 - 26441b_0^4d_1^4x^3y^7\\
&+ 237058519752x^2y^8 - 73717828776a_0d_1x^2y^8\\
&- 44037236592b_0d_1x^2y^8 + 2751173532a_0^2d_1^2x^2y^8\\
&+ 4044825060a_0b_0d_1^2x^2y^8\\
&+ 1668601218b_0^2d_1^2x^2y^8\\
&- 20693125a_0^3d_1^3x^2y^8 - 60906347a_0^2b_0d_1^3x^2y^8\\
&- 64886985a_0b_0^2d_1^3x^2y^8 - 25527395b_0^3d_1^3x^2y^8\\
&+ 23205a_0^4d_1^4x^2y^8 + 135806a_0^3b_0d_1^4x^2y^8\\
&+ 313360a_0^2b_0^2d_1^4x^2y^8 + 341634a_0b_0^3d_1^4x^2y^8\\
&+ 150475b_0^4d_1^4x^2y^8 - 541133408592xy^9\\
&+ 153053040600a_0d_1xy^9 + 93565298520b_0d_1xy^9\\
&- 5592047067a_0^2d_1^2xy^9 - 8557415979a_0b_0d_1^2xy^9\\
&- 3635890530b_0^2d_1^2xy^9 + 42251824a_0^3d_1^3xy^9\\
&+ 131348672a_0^2b_0d_1^3xy^9 + 147010021a_0b_0^2d_1^3xy^9\\
&+ 60353479b_0^3d_1^3xy^9 - 48844a_0^4d_1^4xy^9\\
&- 305846a_0^3b_0d_1^4xy^9 - 760242a_0^2b_0^2d_1^4xy^9\\
&- 901682a_0b_0^3d_1^4xy^9 - 438506b_0^4d_1^4xy^9\\
&+ 634052720592y^{10} - 159806071620a_0d_1y^{10}\\
&- 99249887196b_0d_1y^{10} + 5639205735a_0^2d_1^2y^{10}\\
&+ 8832784710a_0b_0d_1^2y^{10} + 3813012729b_0^2d_1^2y^{10}\\
&- 42181381a_0^3d_1^3y^{10} - 135185562a_0^2b_0d_1^3y^{10}\\
&- 155218542a_0b_0^2d_1^3y^{10} - 64874815b_0^3d_1^3y^{10}\\
&+ 48844a_0^4d_1^4y^{10} + 318016a_0^3b_0d_1^4y^{10}\\
&+ 824332a_0^2b_0^2d_1^4y^{10} + 1024232a_0b_0^3d_1^4y^{10}\\
&+525776b_0^4d_1^4y^{10}\Big).
\end{aligned}
\tag{A.1}
$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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