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Research Article

Dynamics of the Zeros of Analytic Continued (h, q)-Euler Polynomials

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In this paper, we study that the (h, q)-Euler numbers $E_{n,q}^{(h)}$ and (h, q)-Euler polynomials $E_{n,q}^{(h)}(x)$ are analytic continued to $E_q^{(h)}(s)$ and $E_q^{(h)}(s, w)$. We investigate the new concept of dynamics of the zeros of analytic continued polynomials related to solution of Bernoulli equation. Finally, we observe an interesting phenomenon of "scattering" of the zeros of $E_q^{(h)}(s, w)$.

1. Introduction

By using software, many mathematicians can explore concepts much more easily than in the past. The ability to create and manipulate figures on the computer screen enables mathematicians to quickly visualize and produce many problems, examine properties of the figures, look for patterns, and make conjectures. This capability is especially exciting because these steps are essential for most mathematicians to truly understand even basic concept. Recently, the computing environment would make more and more rapid progress and there has been increasing interest in solving mathematical problems with the aid of computers. Mathematicians have studied different kinds of the Euler, Bernoulli, Tangent, and Genocchi numbers and polynomials. Numerical experiments of Bernoulli polynomials, Euler polynomials, Genocchi polynomials, and Tangent polynomials have been the subject of extensive study in recent year and much progress has been made both mathematically and computationally (see [1–18]). Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers. Let q be a complex number with $|q| < 1$ and $h \in \mathbb{Z}$. Bernoulli equation is one of the well known nonlinear differential equations of the first order. It is written as

$$\frac{dy}{dt} + p(x)\, y = g(x)\, y^m \quad (m \text{ any real number}), \tag{1}$$

where $p(x)$ and $g(x)$ are continuous functions. For $m = 0$ and $m = 1$ the equation is linear, and otherwise it is nonlinear. When $m = 2$, the Bernoulli equation has the solution which is the function of exponential generating function of the Euler numbers. Simsek [18] introduced the (h, q)-Euler numbers $E_{n,q}^{(h)}$ and polynomials $E_{n,q}^{(h)}(x)$. He gave recurrence identities (h, q)-Euler polynomials and the alternating sums of powers of consecutive (h, q)-integers. In [13], we described the beautiful zeros of the (h, q)-Euler polynomials $E_{n,q}^{(h)}(x)$ using a numerical investigation. Also we investigated distribution and structure of the zeros of the (h, q)-Euler polynomials $E_{n,q}^{(h)}(x)$ by using computer.

Let us define the (h, q)-Euler numbers $E_{n,q}^{(h)}$ and polynomials $E_{n,q}^{(h)}(x)$ as follows:

$$F_q^{(h)}(t) = \frac{2}{q^h e^t + 1} = \sum_{n=0}^{\infty} E_{n,q}^{(h)} \frac{t^n}{n!}, \tag{2}$$

$$F_q^{(h)}(x, t) = \frac{2}{q^h e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q}^{(h)}(x) \frac{t^n}{n!}. \tag{3}$$

Observe that if $q \to 1$, then $E_{n,q}^{(h)}(x) = E_n(x)$ and $E_{n,q}^{(h)} = E_n$, where $E_n(x)$ and E_n denote the Euler polynomials and the numbers, respectively (see [2, 5, 8, 16, 17]).

Thus (h,q)-Euler numbers $E_{n,q}^{(h)}$ are defined by means of the generating function

$$F_q^{(h)}(t) = \sum_{n=0}^{\infty} E_{n,q}^{(h)} \frac{t^n}{n!} = 2\sum_{m=0}^{\infty} (-1)^m q^{hm} e^{mt}. \tag{4}$$

As is well known, when $m = 2$ a special Bernoulli equation

$$\frac{dy}{dt} + y = \frac{1}{2}y^2 \tag{5}$$

has the solution

$$y = \frac{2}{q^h e^t + 1} = \sum_{n=0}^{\infty} E_{n,q,}^{(h)} \frac{t^n}{n!}. \tag{6}$$

That is, the Bernoulli equation has the solution which is the function of exponential generating function of the (h,q)-Euler numbers. Thus, a realistic study for the analytic continued polynomials $E_q^{(h)}(s,w)$ is very interesting by using computer. It is the aim of this paper to observe an interesting phenomenon of "scattering" of the zeros of the analytic continued polynomials $E_q^{(h)}(s,w)$ in complex plane.

By using computer, the (h,q)-Euler numbers $E_{n,q,}^{(h)}$ can be determined explicitly. A few of them are

$$\begin{aligned} E_{0,q}^{(h)} &= \frac{2}{1+q^h}, \\ E_{1,q}^{(h)} &= -\frac{2q^h}{(1+q^h)^2}, \\ E_{2,q}^{(h)} &= -\frac{2q^h}{(1+q^h)^2} + \frac{4q^{2h}}{(1+q^h)^3}, \\ E_{3,q}^{(h)} &= -\frac{2q^h}{(1+q^h)^2} + \frac{12q^{2h}}{(1+q^h)^3} - \frac{12q^{3h}}{(1+q^h)^4}. \end{aligned} \tag{7}$$

Theorem 1. *For $n \in \mathbb{N}_0$, we have*

$$E_{n,q,}^{(h)}(x) = \sum_{k=0}^{n} \binom{n}{k} E_{k,q}^{(h)} x^{n-k}. \tag{8}$$

By Theorem 1, after some elementary calculations, we have

$$\begin{aligned} \int_a^b E_{n,q}^{(h)}(x)\,dx &= \sum_{l=0}^{n} \binom{n}{l} E_{l,q}^{(h)} \int_a^b x^{n-l}dx \\ &= \frac{1}{n+1}\sum_{l=0}^{n+1} \binom{n+1}{l} E_{l,q}^{(h)}\, x^{n-l+1}\Big|_a^b \\ &= \frac{E_{n+1,q}^{(h)}(b) - E_{n+1,q}^{(h)}(a)}{n+1}. \end{aligned} \tag{9}$$

Since $E_{n,q,}^{(h)}(0) = E_{n,q}^{(h)}$, by (9), we obtain

$$E_{n,q}^{(h)}(x) = E_{n,q}^{(h)} + n\int_0^x E_{n-1,q}^{(h)}(t)\,dt \quad \text{for } n \in \mathbb{N}. \tag{10}$$

Then, it is easy to deduce that $E_{n,q}^{(h)}(x)$ are polynomials of degree n. Here is the list of the first (h,q)-Euler's polynomials:

$$\begin{aligned} E_{0,q}^{(h)}(x) &= \frac{2}{1+q^h}, \\ E_{1,q}^{(h)}(x) &= -\frac{2q^h}{(1+q^h)^2} + \frac{2x}{1+q^h}, \\ E_{2,q}^{(h)}(x) &= -\frac{2q^h}{(1+q^h)^2} + \frac{4q^{2h}}{(1+q^h)^3} - \frac{4q^h x}{(1+q^h)^2} + \frac{2x^2}{1+q^h}, \\ E_{3,q}^{(h)}(x) &= -\frac{2q^h}{(1+q^h)^2} + \frac{12q^{2h}}{(1+q^h)^3} - \frac{12q^{3h}}{(1+q^h)^4} \\ &\quad + \frac{12q^{2h}x}{(1+q^h)^3} - \frac{6q^h x}{(1+q^h)^2} - \frac{6q^h x^2}{(1+q^h)^2} + \frac{2x^3}{1+q^h}. \end{aligned} \tag{11}$$

2. Analytic Continuation of (h,q)-Euler Numbers $E_{n,q}^{(h)}$

In this section, we introduce the (h,q)-Euler zeta function and Hurwitz (h,q)-Euler zeta function. By (h,q)-Euler zeta function, we consider the function $E_q^{(h)}(s)$ as the analytic continuation of (h,q)-Euler numbers. For more studies in this subject, you may see [2–5, 7–9, 12, 13, 18].

From (4), we note that

$$\begin{aligned} \frac{d^k}{dt^k} F_q^{(h)}(t)\Big|_{t=0} &= 2\sum_{m=0}^{\infty} (-1)^m q^{hm} m^k \\ &= E_{k,q}^{(h)}, \quad (k \in \mathbb{N}). \end{aligned} \tag{12}$$

By using the above equation, we are now ready to define (h,q)-Euler zeta functions.

Definition 2. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$. Consider

$$\zeta_{E,q}^{(h)}(s) = 2\sum_{n=1}^{\infty} \frac{(-1)^n q^{hn}}{n^s}. \tag{13}$$

Observe that $\zeta_{E,q}^{(h)}(s)$ is a meromorphic function on $\mathbb{C}$. Clearly, $\lim_{q\to 1}\zeta_{E,q}^{(h)}(s) = \zeta_E(s)$ (see [3, 4, 7–9, 12, 13, 18]). Notice that the (h,q)-Euler zeta function can be analytically continued to the whole complex plane, and these zeta functions have the values of the (h,q)-Euler numbers at negative integers.

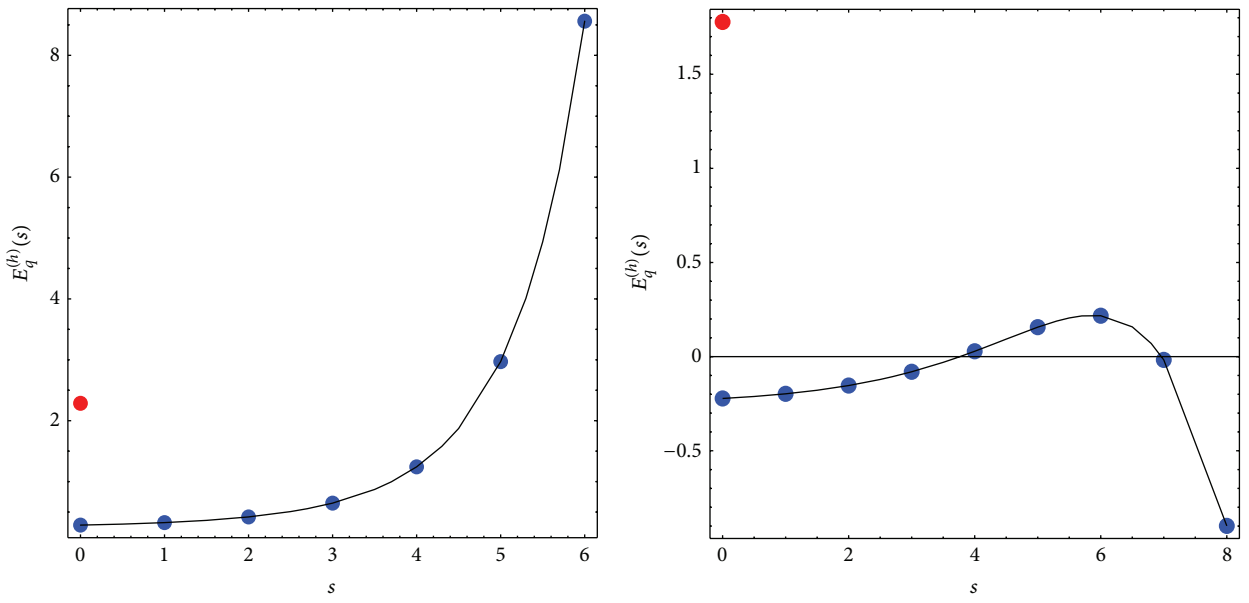

FIGURE 1: The curve $E_q^{(h)}(s)$ runs through the points of all $E_{n,q}^{(h)}$ except $E_{0,q}^{(h)}$.

Theorem 3. *For $k \in \mathbb{N}$, we have*

$$\zeta_{E,q}^{(h)}(-k) = E_{k,q}^{(h)}. \tag{14}$$

Observe that $\zeta_{E,q}^{(h)}(s)$ function interpolates $E_{k,q}^{(h)}$ numbers at nonnegative integers.

By using (3), we note that

$$\begin{aligned} \frac{d^k}{dt^k} F_q^{(h)}(x,t)\Big|_{t=0} &= 2\sum_{m=0}^{\infty} (-1)^m q^{hm} (x+m)^k \\ &= E_{k,q}^{(h)}(x), \quad (k \in \mathbb{N}), \end{aligned} \tag{15}$$

$$\left(\frac{d}{dt}\right)^k \left(\sum_{n=0}^{\infty} E_{n,q,}^{(h)}(x)\frac{t^n}{n!}\right)\Big|_{t=0} = E_{k,q}^{(h)}(x), \quad \text{for } k \in \mathbb{N}. \tag{16}$$

By (16), we are now ready to define the Hurwitz-type (h,q)-Euler zeta functions.

Definition 4. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$. Consider

$$\zeta_{E,q}^{(h)}(s,x) = 2\sum_{n=0}^{\infty} \frac{(-1)^n q^{hn}}{(n+x)^s}. \tag{17}$$

Note that $\zeta_{E,q}^{(h)}(s,x)$ is a meromorphic function on $\mathbb{C}$ (see [3, 4, 7–9, 12, 13, 18]). Relation between $\zeta_{E,q}^{(h)}(s,x)$ and $E_{k,q}^{(h)}(x)$ is given by the following theorem.

Theorem 5. *For $k \in \mathbb{N}$, we have*

$$\zeta_{E,q}^{(h)}(-k,x) = E_{k,q}^{(h)}(x). \tag{18}$$

We now consider the function $E_q^{(h)}(s)$ as the analytic continuation of (h,q)-Euler numbers. From the above analytic continuation of (h,q)-Euler numbers, we consider

$$\begin{aligned} E_{n,q}^{(h)} &\longmapsto E_q^{(h)}(s), \\ \zeta_{E,q}^{(h)}(-n) = E_{n,q}^{(h)} &\longmapsto \zeta_{E,q}^{(h)}(-s) = E_q^{(h)}(s). \end{aligned} \tag{19}$$

In Figure 1(a), we choose $q = -1/2$ and $h = 3$. In Figure 1(b), we choose $q = 1/2$ and $h = 3$.

All the (h,q)-Euler numbers $E_{n,q}^{(h)}$ agree with $E_q^{(h)}(n)$, the analytic continuation of (h,q)-Euler numbers evaluated at n (see Figure 1),

$$\begin{aligned} &E_{n,q}^{(h)} = E_q^{(h)}(n) \quad \text{for } n \geq 1 \\ &\text{except } E_q^{(h)}(0) = \frac{-2q^h}{1+q^h}, \qquad \text{but } E_{0,q}^{(h)} = \frac{2}{1+q^h}. \end{aligned} \tag{20}$$

In fact, we can express $E_q^{(h)'}(s)$ in terms of $\zeta_{E,q}^{(h)'}(s)$, the derivative of $\zeta_E^{(h)}(s)$, as follows:

$$\begin{aligned} E_q^{(h)}(s) &= \zeta_{E,q}^{(h)}(-s), \\ E_q^{(h)'}(s) &= -\zeta_{E,q}^{(h)'}(-s), \\ E_q^{(h)'}(2n+1) &= -\zeta_{E,q}^{(h)'}(-2n-1) \quad \text{for } n \in \mathbb{N}_0. \end{aligned} \tag{21}$$

From the relation (21), we can define the other analytic continued half of (h,q)-Euler numbers

$$\begin{aligned} &E_q^{(h)}(s) = \zeta_{E,q}^{(h)}(-s), \qquad E_q^{(h)}(-s) = \zeta_{E,q}^{(h)}(s) \\ &\Longrightarrow E_q^{(h)}(-n) = \zeta_{E,q}^{(h)}(n), \quad n \in \mathbb{N}. \end{aligned} \tag{22}$$

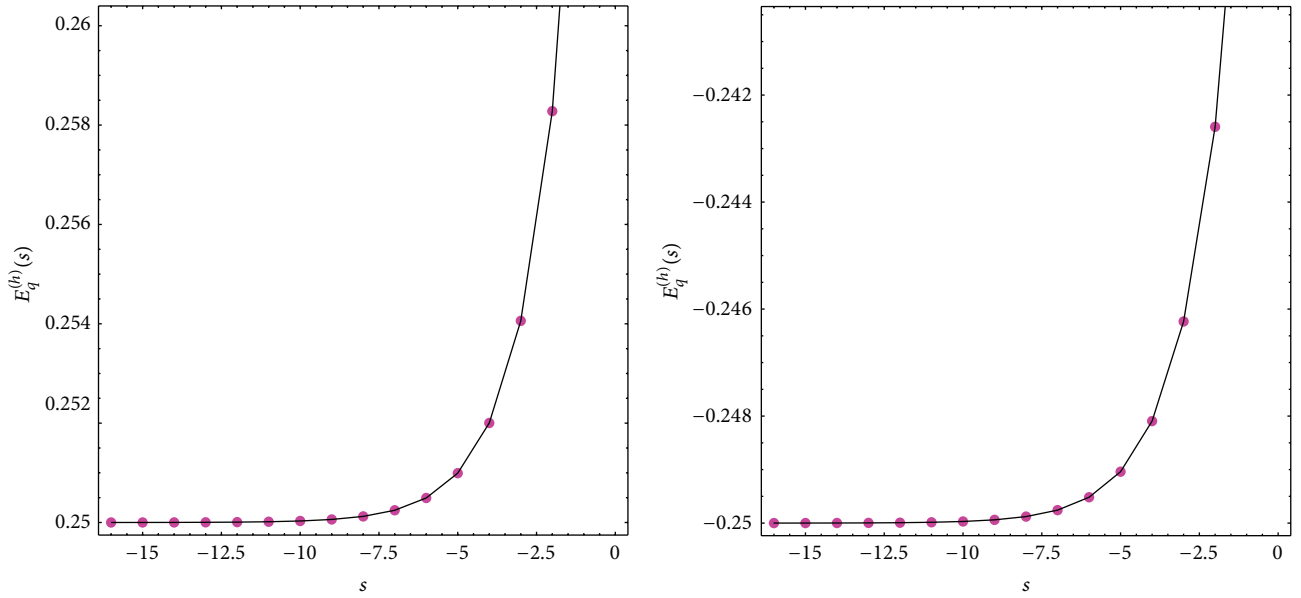

FIGURE 2: The curve $E_q^{(h)}(s)$ runs through the points $E_{-n,q}^{(h)}$.

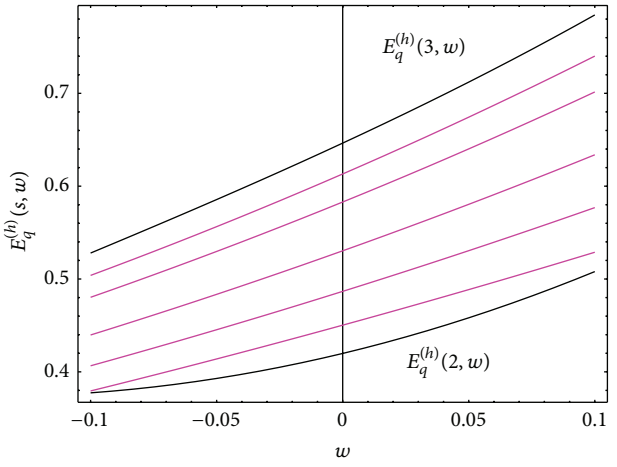

FIGURE 3: The curve of $E_q^{(h)}(s, w)$, $2 \le s \le 3$, $-0.1 \le w \le 0.1$.

By (22), we have

$$\lim_{n \to \infty} E_{-n,q}^{(h)} = \zeta_{E,q}^{(h)}(n) = -2q^h. \tag{23}$$

The curve $E_q^{(h)}(s)$ runs through the points $E_{-n,q}^{(h)} = E_q^{(h)}(-n)$ and grows $\sim -2q^h$ asymptotically as $-n \to \infty$ (see Figure 2).

In Figure 2(a), we choose $q = -1/2$ and $h = 3$. In Figure 2(b), we choose $q = 1/2$ and $h = 3$.

3. Analytic Continuation of Euler Polynomials $E_{n,q}^{(h)}(x)$

In this section, we observe the analytic continued (h, q)-Euler polynomials. Looking back at (13) and (22), for consistency with the definition of $E_{n,q}^{(h)}(x) = E_q^{(h)}(n, x)$, (h, q)-Euler polynomials should be analogously redefined as

$$\begin{aligned} E_q^{(h)}(0, x) &= -q^h E_{0,q}^{(h)}(x), \\ E_q^{(h)}(n, x) &= \sum_{l=0}^{n} \binom{n}{l} E_{l,q}^{(h)} x^{n-l}. \end{aligned} \tag{24}$$

Let $\Gamma(s)$ be the gamma function. The analytic continuation can be then obtained as

$$\begin{aligned} & n \longmapsto s \in \mathbb{R}, \qquad x \longmapsto w \in \mathbb{C}, \\ & E_{0,q}^{(h)} \longmapsto E_q^{(h)}(0) = -\frac{1}{q^h} \zeta_{E,q}^{(h)}(0), \\ & E_{k,q}^{(h)} \longmapsto E_q^{(h)}(k + s - [s]) = \zeta_{E,q}^{(h)}(-(k + (s - [s]))), \\ & \binom{n}{k} \longmapsto \frac{\Gamma(1 + s)}{\Gamma(1 + k + (s - [s]))\,\Gamma(1 + [s] - k)} \\ & \Longrightarrow E_{n,q}^{(h)}(w) \longmapsto E_q^{(h)}(s, w) \\ & = \sum_{k=-1}^{[s]} \frac{\Gamma(1 + s)\, E_q^{(h)}(k + s - [s])\, w^{[s]-k}}{\Gamma(1 + k + (s - [s]))\,\Gamma(1 + [s] - k)} \\ & = \sum_{k=0}^{[s]+1} \frac{\Gamma(1 + s)\, E_q^{(h)}((k - 1) + s - [s])\, w^{[s]+1-k}}{\Gamma(k + (s - [s]))\,\Gamma(2 + [s] - k)}, \end{aligned} \tag{25}$$

where $[s]$ gives the integer part of s, and so $s - [s]$ gives the fractional part.

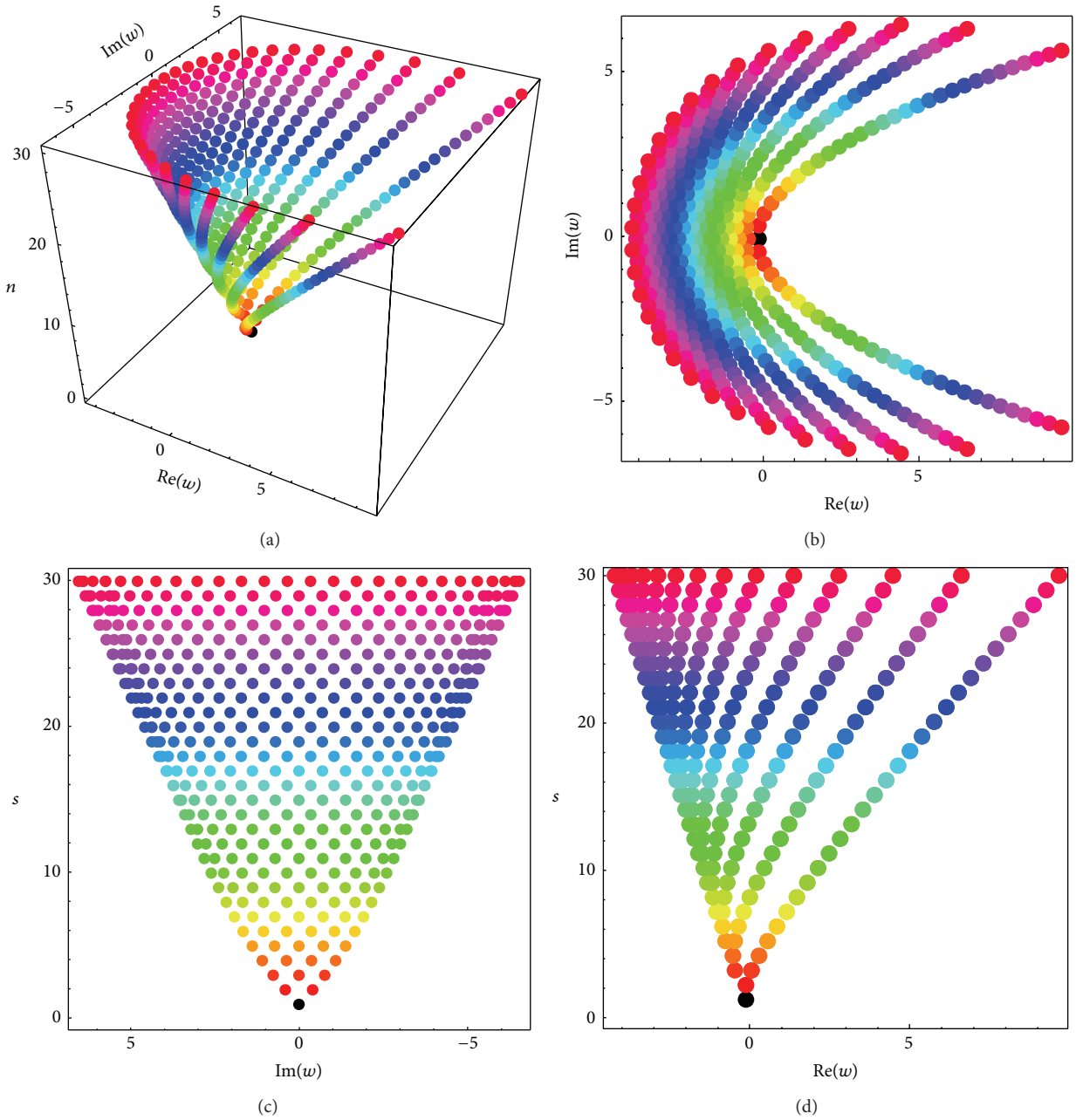

FIGURE 4: Stacks of zeros of $E_q^{(h)}(n, w)$ for $1 \le n \le 30$.

By (25), we obtain analytic continuation of (h, q)-Euler polynomials for $q = -1/2$ and $h = 3$ as follows:

$$
\begin{aligned}
E_q^{(h)}(0, w) &\approx 2.28571,\\
E_q^{(h)}(1, w) &\approx 0.326531 + 2.28571w,\\
E_q^{(h)}(2, w) &\approx 0.41982 + 0.65306w + 0.28571w^2,\\
E_q^{(h)}(2.2, w) &\approx 0.45027 + 0.74682w + 0.38478w^2\\
&\quad + 0.02373w^3,\\
E_q^{(h)}(2.4, w) &\approx 0.48668 + 0.85139w + 0.50111w^2\\
&\quad + 0.06112w^3,\\
E_q^{(h)}(2.6, w) &\approx 0.53037 + 0.96937w + 0.63694w^2\\
&\quad + 0.11503w^3,\\
E_q^{(h)}(2.8, w) &\approx 0.58293 + 1.10403w + 0.79519w^2\\
&\quad + 0.18867w^3,\\
E_q^{(h)}(2.9, w) &\approx 0.61316 + 1.178859w + 0.88385w^2\\
&\quad + 0.23401w^3,\\
E_q^{(h)}(3, w) &\approx 0.64639 + 1.25947w + 0.97959w^2\\
&\quad + 2.28571w^3.
\end{aligned}
\tag{26}
$$

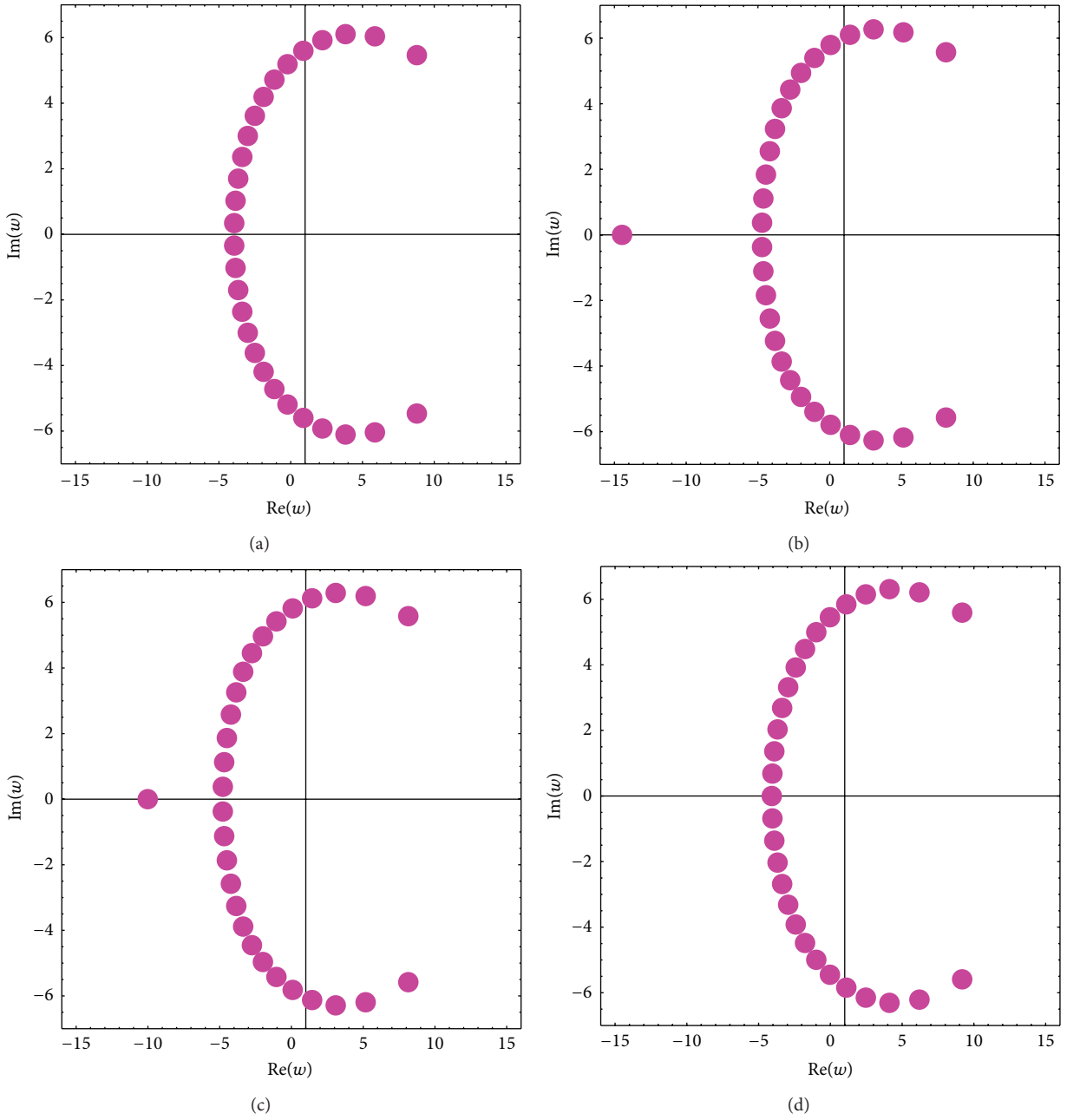

Figure 5: Zeros of $E_q^{(h)}(s,w)$ for $s = 28, 28.8, 28.9, 29$.

By using (26), we plot the deformation of the curve $E_q^{(h)}(2,w)$ into the curve of $E_q^{(h)}(3,w)$ via the real analytic continuation $E_q^{(h)}(s,w)$, $2 \le s \le 3$, $w \in \mathbb{R}$ (see Figure 3).

Next, we investigate the beautiful zeros of the $E_q^{(h)}(n,w)$ by using a computer. We plot the zeros of $E_q^{(h)}(n,w)$ for $n \in \mathbb{N}$, $q = 1/2$, $h = 3$, and $w \in \mathbb{C}$ (Figure 4). In Figure 4(b), we draw x and y axes but no z axis in three dimensions. In Figure 4(c), we draw y and z axes but no x axis in three dimensions. In Figure 4(d), we draw x and z axes but no y axis in three dimensions.

In Figure 4, we observe that $E_q^{(h)}(n,w)$, $w \in \mathbb{C}$, has $\mathrm{Im}(w) = 0$ reflection symmetry analytic complex functions (Figure 4). The obvious corollary is that the zeros of $E_q^{(h)}(n,w)$ will also inherit these symmetries:

$$\text{if } E_q^{(h)}(n,w_0) = 0, \quad \text{then } E_q^{(h)}(n,w_0^*) = 0, \tag{27}$$

where $*$ denotes complex conjugation.

Finally, we investigate the beautiful zeros of the $E_q^{(h)}(s,w)$ by using a computer. We plot the zeros of $E_q^{(h)}(s,w)$ for $s = 28, 20.8, 20.9, 29$, $q = -1/2$, $h = 3$, and $w \in \mathbb{C}$ (Figure 5).

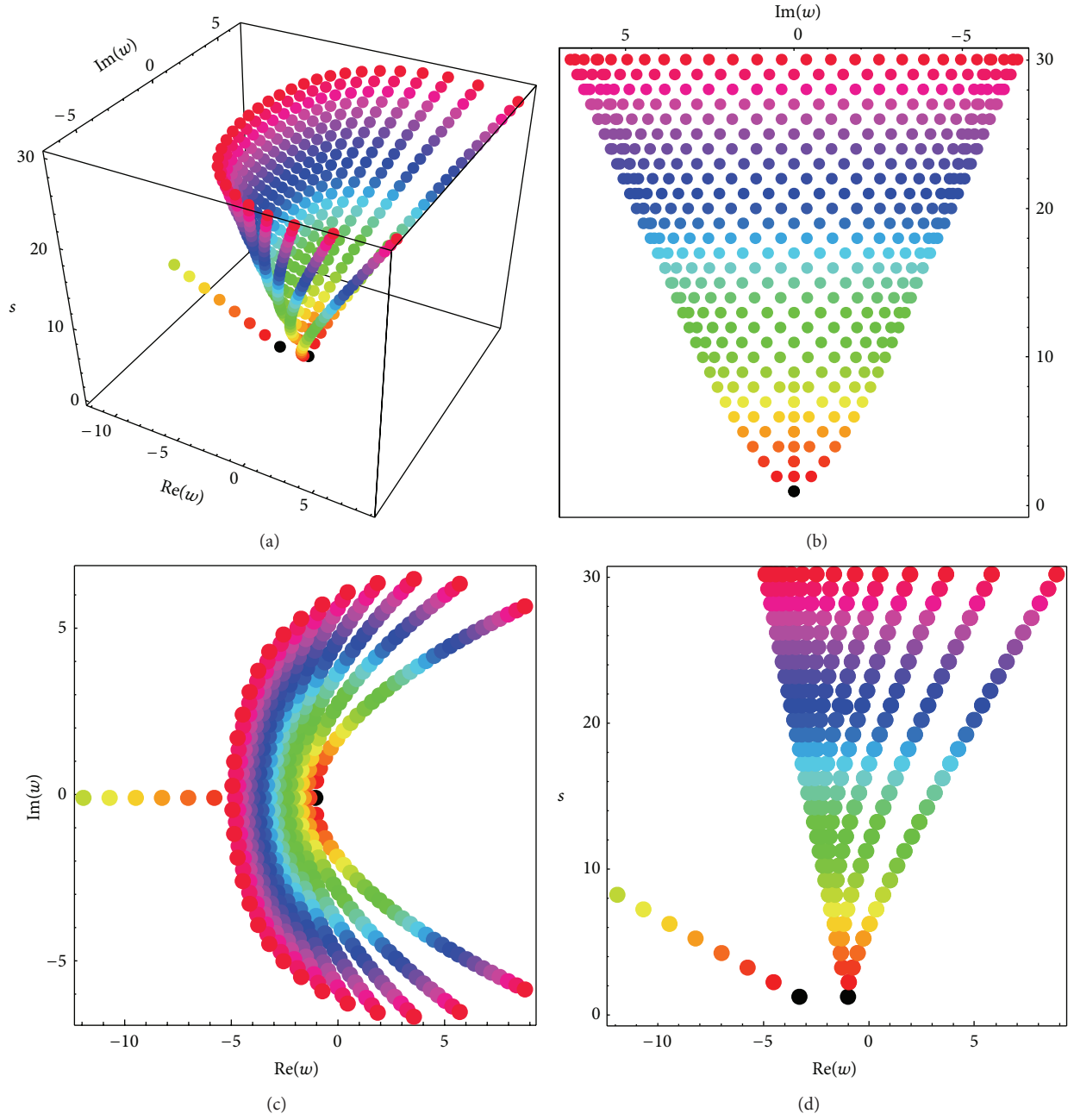

FIGURE 6: Stacks of zeros of $E_q^{(h)}(s, w)$ for $1 \le n \le 30$.

In Figure 5(a), we choose $s = 28$. In Figure 5(b), we choose $s = 28.8$. In Figure 5(c), we choose $s = 28.9$. In Figure 5(d), we choose $s = 29$.

Since

$$\sum_{n=0}^{\infty} E_{n,q^{-1}}^{(h)}(1-x)\frac{(-1)^n t^n}{n!} = \frac{2}{q^{-h}e^{-t}+1}e^{(1-x)(-t)} = q^h\left(\frac{2}{q^h e^t+1}\right)e^{xt} = \sum_{n=0}^{\infty} q^h E_{n,q}^{(h)}(x)\frac{t^n}{n!}, \tag{28}$$

we obtain

$$q^h E_{n,q}^{(h)}(x) = (-1)^n E_{n,q^{-1}}^{(h)}(1-x). \tag{29}$$

Observe that $E_n(x)$, $x \in \mathbb{C}$, has $\mathrm{Re}(x) = 1/2$ reflection symmetry in addition to the usual $\mathrm{Im}(x) = 0$ reflection symmetry analytic complex functions (see [14]). The question is, what happens with the reflexive symmetry (29), when one considers (h, q)-Euler polynomials? Prove that $E_q^{(h)}(n, w)$, $w \in \mathbb{C}$, has no $\mathrm{Re}(w) = 1/2$ reflection symmetry analytic complex functions (Figure 4). However, we observe that $E_q^{(h)}(s, w)$,

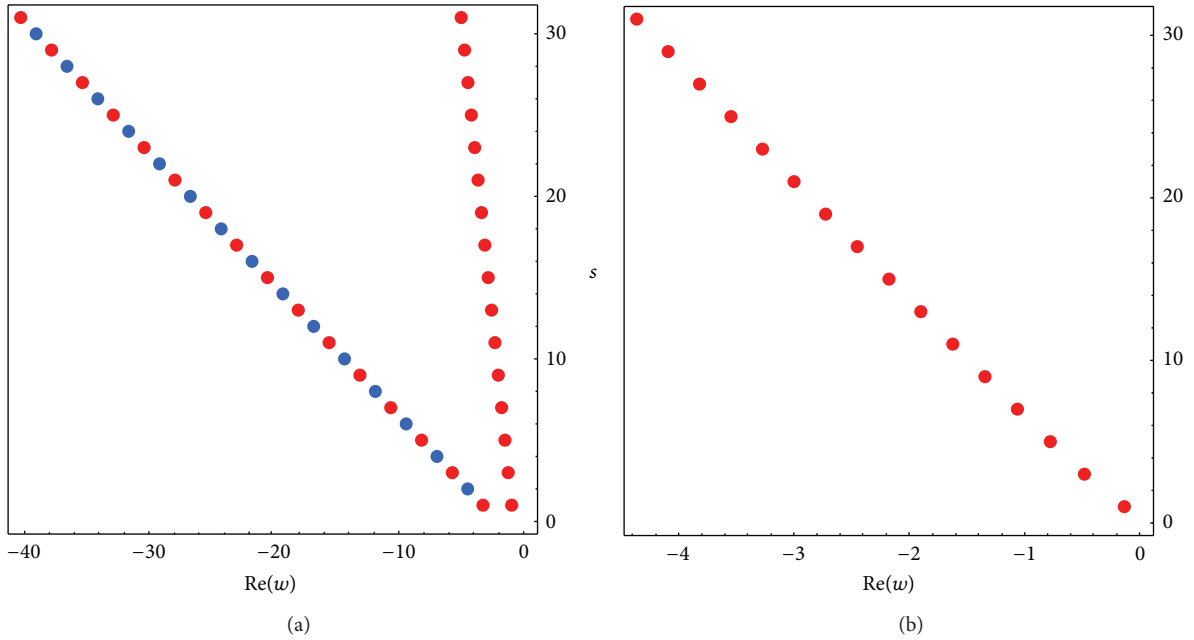

Figure 7: Real zeros of $E_q^{(h)}(s, w)$.

$w \in \mathbb{C}$, has $\mathrm{Im}(x) = 0$ reflection symmetry analytic complex functions (Figure 5).

Stacks of zeros of $E_q^{(h)}(s, w)$ for $s = n + 1/2$, $h = 3$, $1 \le n \le 30$ from a 3-D structure are presented (Figure 6).

In Figure 6(b), we draw y and z axes but no x axis in three dimensions. In Figure 6(c), we draw x and y axes but no z axis in three dimensions. In Figure 6(d), we draw x and z axes but no y axis in three dimensions.

Our numerical results for approximate solutions of real zeros of $E_q^{(h)}(s, w)$, $q = -1/2$, $h = 3$ are displayed. We observe a remarkably regular structure of the complex roots of (h, q)-Euler polynomials. We hope to verify a remarkably regular structure of the complex roots of (h, q)-Euler polynomials (Table 1).

Next, we calculated an approximate solution satisfying $E_q^{(h)}(s, w)$, $q = -1/2$, $h = 3$, $w \in \mathbb{R}$. The results are given in Table 2.

In Figure 7, we plot the real zeros of the (h, q)-Euler polynomials $E_q^{(h)}(s, w)$ for $s = n + 1/2$, $q = -1/2$, $h = 3$, and $w \in \mathbb{C}$ (Figure 7). In Figure 7(a), we choose $s = n + 1/2$. In Figure 7(b), we choose $s = n$. We want to find a formula that best fits a given set of data points. The least squares method is used to fit polynomials or a set of functions to a given set of data points. Using the least squares method, we can find a and b such that $x = a + bn$ is the least squares fit to the data given in Table 2. The graph of the data points is shown in Figure 7. We obtain $x = -0.0818486 - 0.138376n$ for $n = 1, 3, 5, \ldots$. We also obtain $x = -1.48453 - 1.22923n$ for $n = 1, 3, 5, \ldots$ and $x = -0.850454 - 0.134332n$ for $n = 1.5, 2.5, 3.5, 4.5, \ldots$. The real zero $x \sim -\infty$ asymptotically as $n \to \infty$.

The (h, q)-Euler polynomials $E_q^{(h)}(n, w)$ are polynomials of degree n. Thus, $E_q^{(h)}(n, w)$ has n zeros and $E_q^{(h)}(n + 1, w)$ has $n + 1$ zeros. When discrete n is analytic continued to continuous parameter s, it naturally leads to the following question.

Table 1: Numbers of real and complex zeros of $E_q^{(h)}(s, w)$.

s	Real zeros	Complex zeros
1.5	2	0
2.5	1	2
3.5	2	2
4.5	1	4
5.5	2	4
6.5	1	6
7.5	2	6
8	0	8
8.5	1	8
9	1	8
9.5	2	8
10	0	10
10.5	1	10
11	1	10
11.5	2	10

How does $E_q^{(h)}(s, w)$, the analytic continuation of $E_q^{(h)}(n, w)$, pick up an additional zero as s increases continuously by one?

This introduces the exciting concept of the dynamics of the zeros of analytic continued Euler polynomials, the idea of looking at how the zeros move about in the w complex plane as we vary the parameter s.

Table 2: Approximate solutions of $E_q^{(h)}(s,\ w) = 0$, $h = 3$, $w \in \mathbb{R}$.

s	w
6	×
6.5	−9.47111
7	−1.06648
7.5	−10.6998, −1.8568
8	×
8.5	−11.9289
9	−1.34687
9.5	−13.1581, −2.12315
10	×
10.5	−14.3874
11	−1.62471
11.5	−15.6169, −2.3908

To have a physical picture of the motion of the zeros in the complex w plane, imagine that each time as s increases gradually and continuously by one, an additional real zero flies in from positive infinity along the real positive axis, gradually slowing down as if "it is flying through a viscous medium."

For more studies and results in this subject, you may see [6, 11–15].

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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