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*Research Article*

# On Certain Matrices of Bernoulli Numbers

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In this work we compute the determinant and inverse matrices for a certain symmetric matrix of Rayleigh sums. As a special case we also obtain the determinants and inverses for the matrices of the Bernoulli numbers and related numbers.

## 1. Introduction

The sequence of Bernoulli numbers $B_n$ is one of the most important sequences in mathematics. It has deep connections to number theory, for instance, the Bernoulli numbers are used to express the values of $\zeta(2n)$, where $\zeta(s)$ is the Riemann zeta function and $n$ is a positive integer [1, 2]. The Bernoulli numbers are also very important in analysis, for example, they appear in the Euler-Maclaurin formula [1], which is very important in mathematics and physics. The Bernoulli numbers are also very important in asymptotics of $q$-special functions; for example, in [3] we proved a complete asymptotic expansion of $q$-Gamma function $\Gamma_q(z)$ on the complex plane in terms of Bernoulli polynomials and Bernoulli polynomials. The applications of Bernoulli numbers in applied mathematics are just too many to list all of them; just to name a few, for example, see [4–6]. The Rayleigh sums $\sigma_\nu^{(n)}$ generalize $\zeta(2n)$ and it is known that $\sigma_{1/2}^{(n)}$ is a rational multiple of $B_{2n}$ [7]. In this work we first derive the inverse and determinant of a certain symmetric matrix defined by $\sigma_\nu^{(n)}$ and then specialize the result to the matrices defined by Bernoulli numbers $B_n$ and related numbers $S_n$.

But we have to emphasize that the present work demonstrated a method to compute inverses of certain Hankel matrices, not just determinants. In fact there are many known methods to compute determinants; for example, see [1, 8–11].

## 2. Preliminaries

For $\nu > -1$ the Bessel function of first kind is defined by [1, 7, 11, 12]:

$$J_\nu(z) = \frac{1}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(\nu+1)_k} \left(\frac{z}{2}\right)^{2k+\nu}, \tag{1}$$

where

$$\frac{1}{\Gamma(a)} = a \prod_{j=1}^{\infty} \left(1+\frac{a}{j}\right)\left(1+\frac{1}{j}\right)^{-a}, \qquad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n \in \mathbb{Z},\ a \in \mathbb{C}. \tag{2}$$

As a special case we have

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin z. \tag{3}$$

It is known that the even entire function $J_\nu(z) z^{-\nu}$ has infinitely many zeros, all of which are real. Let

$$0 < j_{\nu,1} < j_{\nu,2} < \cdots \tag{4}$$

be all its positive zeros; then the Rayleigh sum is defined by [7]

$$\sigma_\nu^{(n)} = \sum_{k=1}^{\infty} \frac{1}{j_{\nu,k}^{2n}}, \quad n \in \mathbb{N}. \tag{5}$$

Clearly [1],

$$\sigma_{1/2}^{(n)} = \sum_{n=1}^{\infty} \frac{1}{(n\pi)^{2k}} = \frac{(-1)^{n+1} 2^{2n-1} B_{2n}}{(2n)!}, \tag{6}$$

where the Bernoulli numbers $B_n$ are defined by [1, 2, 12]

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad |z| < 2\pi. \tag{7}$$

The related numbers $\{S_n\}_{n=1}^{\infty}$ are defined by [2, 13]

$$S_n = 2\left(\frac{2}{\pi}\right)^n \sum_{k=-\infty}^{\infty} \frac{1}{(4k+1)^n} \tag{8}$$

for $n = 2, 3, \ldots$ and $S_1 = 1$; it is known that

$$\frac{(-1)^{n+1} B_{2n}}{(2n)!} = \frac{S_{2n}}{2^{2n}\left(2^{2n} - 1\right)}, \quad n \in \mathbb{N}. \tag{9}$$

## 3. Main Results

**Theorem 1.** *Given a nonnegative integer n, one has*

$$\begin{aligned}
&\det\left(\sigma_\nu^{(j+k+1)}\right)_{j,k=1}^n = \frac{2^{(n+1)(2n+1)}}{((\nu+1)/2)_{n+1} \prod_{k=0}^n (\nu+1)_{2k}^2}, \\
&\left\{\left(\sigma_\nu^{(j+k+1)}\right)_{j,k=1}^n\right\}^{-1} \\
&= \Bigg(\sum_{m=0}^n \Big((-1)^{j+k}(2m+\nu+1)(m+j)!(m+k)! \\
&\qquad \times (\nu+1)_{m+j}(\nu+1)_{m+k} \\
&\qquad \times \big(4^{j+k+1}(2j)!(2k)!(m-j)!(m-k)! \\
&\qquad \times (\nu+1)_{m-j}(\nu+1)_{m-k}\big)^{-1}\Big)\Bigg)_{j,k=0}^n,
\end{aligned} \tag{10}$$

*for* $\nu > -1$.

**Corollary 2.** *For any nonnegative integer n, one has*

$$\begin{aligned}
&\det\left(\frac{B_{2j+2k+2}}{(2j+2k+2)!}\right)_{j,k=1}^n = \frac{1}{(3/4)_{n+1} \prod_{k=0}^n (3/2)_{2k}^2}, \\
&\left\{\left(\frac{B_{2(j+k+1)}}{(2j+2k+2)!}\right)_{j,k=0}^n\right\}^{-1} \\
&= \Bigg(\sum_{m=0}^n \Big(\left(m+\frac{3}{4}\right)(m+j)!(m+k)!\left(\frac{3}{2}\right)_{m+j}\left(\frac{3}{2}\right)_{m+k} \\
&\qquad \times \Big((2j)!(2k)!(m-j)!(m-k)!\left(\frac{3}{2}\right)_{m-j} \\
&\qquad \times \left(\frac{3}{2}\right)_{m-k}\Big)^{-1}\Big)\Bigg)_{j,k=0}^n,
\end{aligned} \tag{11}$$

*or, equivalently,*

$$\det\left(\frac{S_{2j+2k+2}}{4^{j+k+1} - 1}\right)_{j,k=0}^n = \frac{4^{(n+1)^2}}{(3/4)_{n+1} \prod_{k=0}^n (3/2)_{2k}^2}, \tag{12}$$

$$\begin{aligned}
&\left\{\left(\frac{S_{2j+2k+2}}{4^{j+k+1} - 1}\right)_{j,k=0}^n\right\}^{-1} \\
&= \Bigg(\sum_{m=0}^n \Big(\left(m+\frac{3}{4}\right)(m+j)!(m+k)!\left(\frac{3}{2}\right)_{m+j}\left(\frac{3}{2}\right)_{m+k} \\
&\qquad \times \Big((-4)^{j+k}(2j)!(2k)!(m-j)!(m-k)! \\
&\qquad \times \left(\frac{3}{2}\right)_{m-j}\left(\frac{3}{2}\right)_{m-k}\Big)^{-1}\Big)\Bigg)_{j,k=0}^n.
\end{aligned} \tag{13}$$

## 4. Proofs

Given a probability measure $d\mu(x)$ on $\mathbb{R}$ such that $\int_{\mathbb{R}} x^{2n} d\mu(x) < \infty$ for all $n \in \mathbb{R}$, we define the inner product for $d\mu(x)$ square integrable functions $f(x)$ and $g(x)$ by

$$(f, g) = \int_{-\infty}^{\infty} f(x)\, g(x)\, d\mu(x). \tag{14}$$

For each $n \in \mathbb{N} \cup \{0\}$, let $G_n = (m_{j,k})_{j,k=0}^n$ with $m_{j,k} = (u_j, u_k)$ for $j, k = 0, 1, \ldots, n$ where $\{u_k(x)\}_{k=0}^{\infty}$ is a sequence of polynomials with $u_0(x) = 1$ such that, for each $n$, $\{u_k(x)\}_{k=0}^n$ are linearly independent. Then there is a unique orthonormal system $\{p_k(x)\}_{k=0}^{\infty}$ [1, 10, 11]:

$$\begin{aligned}
p_n(x) &= \frac{1}{\sqrt{\det G_n \det G_{n-1}}} \\
&\quad \times \det\begin{pmatrix}
m_{0,0} & m_{0,1} & m_{0,2} & \cdots & m_{0,n} \\
m_{1,0} & m_{1,1} & m_{1,2} & \cdots & m_{1,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
m_{n-1,0} & m_{n-1,1} & m_{n-1,2} & \cdots & m_{n-1,n} \\
u_0(x) & u_1(x) & u_2(x) & \cdots & u_n(x)
\end{pmatrix},
\end{aligned} \tag{15}$$

with positive leading coefficient in $u_n(x)$. Clearly we have $p_n(x) = \sum_{j=0}^n a_{n,j} u_j(x)$ for some real numbers $a_{j,k}$ for $j, k = 0, 1, \ldots, n$ and $a_{j,k} = 0$ for $k > j$.

**Lemma 3.** *For each nonnegative integer n, let* $G_n = (m_{j,k})_{j,k=0}^n$ *and* $A_n = (a_{j,k})_{j,k=0}^n$. *Then*

$$\det G_n = \prod_{j=0}^n a_{j,j}^{-2}, \qquad G_n^{-1} = A_n^T A_n. \tag{16}$$

*Proof.* From (15) and $p_n(x) = \sum_{j=0}^{n} a_{n,j} u_j(x)$ it is clear that

$$a_{n,n} = \sqrt{\frac{\det G_{n-1}}{\det G_n}}, \qquad \det G_n = \prod_{j=0}^{n} a_{j,j}^{-2}. \tag{17}$$

For each $n$, since both $\{p_k(x)\}_{k=0}^{n}$ and $\{u_k(x)\}_{k=0}^{n}$ are a basis for the same set of polynomials, $A_n$ must be invertible for each $n \in \mathbb{N} \cup \{0\}$. We denote $A_n^{-1} = (s_{j,k})_{j,k=0}^{n}$; then $u_j(x) = \sum_{\ell=0}^{n} s_{j,\ell} p_\ell(x)$ for $j = 0, 1, \ldots, n$. Clearly, $s_{j,\ell} = 0$ for $\ell > j$. Thus,

$$m_{j,k} = \left(u_j(x), u_k(x)\right) = \sum_{m=0}^{n} s_{j,m} s_{k,m}, \tag{18}$$

for $j, k = 0, 1, \ldots, n$, which is

$$G_n = A_n^{-1}\left(A_n^{-1}\right)^T = A_n^{-1}\left(A_n^T\right)^{-1}, \tag{19}$$

and hence $G_n^{-1} = A_n^T A_n$. □

*4.1. Proof of Theorem 1.* The normalized even order Lommel polynomials are defined by [11]

$$\begin{aligned} h_n(x) &= \frac{\sqrt{2n+\nu+1}}{2} h_{2n,\nu+1}(x) \\ &= \sum_{k=0}^{n} \frac{(-1)^{n-k}\sqrt{2n+\nu+1}\,(n+k)!(\nu+1)_{n+k}}{2^{2k+1}(2k)!\,(n-k)!(\nu+1)_{n-k}}\left(x^2\right)^k, \end{aligned} \tag{20}$$

for $n \in \mathbb{N}$ and $h_0(x) = \sqrt{\nu+1}/2$. They satisfy the orthogonal relation

$$\sum_{k=1}^{\infty} \frac{1}{j_{\nu,k}^2} h_m\left(\frac{1}{j_{\nu,k}}\right) h_n\left(\frac{1}{j_{\nu,k}}\right) = \delta_{m,n}. \tag{21}$$

For $n = 0, 1, \ldots$, it is clear that the $n$th moment with respect to the measure of orthogonality is

$$m_n = \sum_{k=1}^{\infty} \frac{1}{j_{\nu,k}^{2+2n}} = \sigma_\nu^{(n+1)}. \tag{22}$$

Let $u_n(x) = x^{2n}$ for $n = 0, 1, \ldots$; then

$$a_{n,k} = \frac{(-1)^{n-k}\sqrt{2n+\nu+1}\,(n+k)!(\nu+1)_{n+k}}{2^{2k+1}(2k)!\,(n-k)!(\nu+1)_{n-k}}. \tag{23}$$

By Lemma 3, the matrix $(\sigma_\nu^{(j+k+1)})_{j,k=0}^{n}$ has determinant

$$\prod_{k=0}^{n} \frac{2^{4k+2}}{(\nu+1+2k)(\nu+1)_{2k}^2} = \frac{2^{(n+1)(2n+1)}}{((\nu+1)/2)_{n+1}\prod_{k=0}^{n}(\nu+1)_{2k}^2} \tag{24}$$

and its inverse $(\gamma_{j,k})_{j,k=0}^{n}$ has elements

$$\begin{aligned} \gamma_{j,k} = \sum_{m=0}^{n} \Big( &(-1)^{j+k}(2m+\nu+1)(m+j)!\,(m+k)! \\ &\times (\nu+1)_{m+j}(\nu+1)_{m+k} \\ &\times \left(4^{j+k+1}(2j)!\,(2k)!\,(m-j)!\,(m-k)! \right. \\ &\left.\times (\nu+1)_{m-j}(\nu+1)_{m-k}\right)^{-1}\Big). \end{aligned} \tag{25}$$

*4.2. Proof of Corollary 2.* From (24), (25), and (6), we get

$$\begin{aligned} &\det\left(\frac{(-1)^{j+k}2^{2j+2k+1}B_{2(j+k+1)}}{(2j+2k+2)!}\right)_{j,k=0}^{n} \\ &\quad = \frac{2^{(n+1)(2n+1)}}{(3/4)_{n+1}\prod_{k=0}^{n}(3/2)_{2k}^2}, \\ &\left\{\left(\frac{(-1)^{j+k}2^{2j+2k+1}B_{2(j+k+1)}}{(2j+2k+2)!}\right)_{j,k=0}^{n}\right\}^{-1} \\ &\quad = \left(\sum_{m=0}^{n}\left((-1)^{j+k}\left(2m+\frac{3}{2}\right)(m+j)!\,(m+k)!\right.\right. \\ &\qquad \times \left(\frac{3}{2}\right)_{m+j}\left(\frac{3}{2}\right)_{m+k} \\ &\qquad \times \left(4^{j+k+1}(2j)!\,(2k)!\,(m-j)!\,(m-k)!\left(\frac{3}{2}\right)_{m-j}\right. \\ &\qquad \left.\left.\left.\times \left(\frac{3}{2}\right)_{m-k}\right)^{-1}\right)\right)_{j,k=0}^{n}. \end{aligned} \tag{26}$$

They are simplified to

$$\begin{aligned} &\det\left(\frac{B_{2(j+k+1)}}{(2j+2k+2)!}\right) = \frac{1}{(3/4)_{n+1}\prod_{k=0}^{n}(3/2)_{2k}^2}, \\ &\left\{\left(\frac{B_{2(j+k+1)}}{(2j+2k+2)!}\right)_{j,k=0}^{n}\right\}^{-1} \\ &\quad = \left(\sum_{m=0}^{n}\left(\left(m+\frac{3}{4}\right)(m+j)!\,(m+k)!\left(\frac{3}{2}\right)_{m+j}\left(\frac{3}{2}\right)_{m+k}\right.\right. \\ &\qquad \times \left((2j)!\,(2k)!\,(m-j)!\,(m-k)!\left(\frac{3}{2}\right)_{m-j}\right. \\ &\qquad \left.\left.\left.\times \left(\frac{3}{2}\right)_{m-k}\right)^{-1}\right)\right)_{j,k=0}^{n}. \end{aligned} \tag{27}$$

By (9) we get

$$\det\left(\frac{S_{2j+2k+2}}{2\left(4^{j+k+1}-1\right)}\right)_{j,k=0}^{n}=\frac{2^{(n+1)(2n+1)}}{(3/4)_{n+1}\prod_{k=0}^{n}(3/2)_{2k}^{2}},$$

$$\left\{\left(\frac{S_{2j+2k+2}}{2\left(4^{j+k+1}-1\right)}\right)_{j,k=0}^{n}\right\}^{-1}$$
$$=\left(\sum_{m=0}^{n}\left((-1)^{j+k}\left(2m+\frac{3}{2}\right)(m+j)!\,(m+k)!\left(\frac{3}{2}\right)_{m+j}\times\left(\frac{3}{2}\right)_{m+k}\times\left(4^{j+k+1}(2j)!\,(2k)!\,(m-j)!\,(m-k)!\left(\frac{3}{2}\right)_{m-j}\times\left(\frac{3}{2}\right)_{m-k}\right)^{-1}\right)\right)_{j,k=0}^{n},\quad(28)$$

which are simplified to (12) and (13), respectively.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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