

Research Article

Dynamics of a Predator-Prey System with Mixed Functional Responses

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A predator-prey system with two preys and one predator is considered. Especially, two different types of functional responses, Holling type and Beddington-DeAngelis type, are adopted. First, the boundedness of system is showed. Stabilities analysis of system is investigated via some properties about equilibrium points and stabilities of two subsystems without one of the preys of system. Also, persistence conditions of system are found out and some numerical examples are illustrated to substantiate our theoretical results.

1. Introduction

Ecological systems are mainly characterized by the interaction between species and their surrounding natural environment ([1]). Among them, two-species continuous time ecological models with one predator and one prey have been studied for several functional responses such as Holling-Tanner type ([2–4]), Beddington-DeAngelis type ([5–7]), and ratio-dependent type ([8, 9]). However, it has been recognized that such two-species ecological models are not sufficient to explain various phenomena observed in nature ([10–13]). For this reason, in recent years, ecological models with three and more species have been investigated by many authors in [14–18]. Particularly, in this paper, we will deal with a three-species ecological system with two different preys and one predator.

On the other hand, functional response between two species is known as the relationship between prey and predator. Most three-species systems in [5, 10, 11, 16, 18] have the same two functional responses. However, it is reasonable to consider two different functional responses since two preys in the system are different from each other. In fact, if one considers the handling time of the predator to capture the prey, one figures out that the predator has a Holling type-II functional response and if one thinks of the competitions

of predators with one another to catch the prey, Beddington-DeAngelis type functional response could be suitable. Thus, in this paper, we consider the following system with two preys and one predator with mixed two functional responses:

$$\begin{aligned}x_1'(t) &= x_1(t) \left(a_1 - b_1 x_1(t) - \frac{y(t)}{c + x_1(t)} \right), \\x_2'(t) &= x_2(t) \left(a_2 - b_2 x_2(t) - \frac{y(t)}{\alpha + x_2(t) + \beta y(t)} \right), \\y'(t) &= y(t) \left(-d + \frac{e_1 x_1(t)}{c + x_1(t)} + \frac{e_2 x_2(t)}{\alpha + x_2(t) + \beta y(t)} \right),\end{aligned}\quad (1)$$

where $x_1(t)$, $x_2(t)$, and $y(t)$ represent the population density of two preys and the predator at time t , respectively. The constants a_i ($i = 1, 2$) are called the intrinsic growth rates, b_i ($i = 1, 2$) are the coefficients of intraspecific competition, c and α are the half-saturation constants, e_i ($i = 1, 2$) are the per capita rate of predation of the predator, d is the death rate of the predator, and β scales the impact of the predator interference.

The main purpose of this paper is to look into dynamical properties of system (1). In Section 2, the boundedness of system (1), which means the solution of system (1) initiating in the nonnegative octant is bounded, is studied. Stabilities

analysis of system (1) is investigated via well-known properties about equilibrium points and the stabilities of two subsystems without one of the preys of system (1). Also, persistence conditions of the main system (1) are found out and some numerical examples are illustrated to substantiate our theoretical results in Section 4.

2. Boundedness of System (1)

First, let us consider the state space $\mathbb{R}_+^3 = \{(x, y, z) \mid x \geq 0, y \geq 0, z \geq 0\}$. It is easy to see that the functions in the right-hand side of system (1) are continuous and have continuous partial derivatives on \mathbb{R}_+^3 . Moreover elementary calculations yield the fact that they are Lipschitzian on \mathbb{R}_+^3 . Thus the solution of system (1) with nonnegative initial condition exists and is unique. In addition, the solution of system (1) initiating in the nonnegative octant is bounded as shown in the following theorem.

Theorem 1. *The solution of system (1) initiating in \mathbb{R}_+^3 is bounded for all $t \geq 0$.*

Proof. Since $dx_i(t)/dt \leq x_i(t)(a_i - b_i x(t))$, $i = 1, 2$, we have

$$\limsup_{t \rightarrow \infty} x_i(t) \leq \frac{a_i}{b_i}, \quad i = 1, 2. \quad (2)$$

Define $V(t) = e_1 x_1(t) + e_2 x_2(t) + y(t)$. Then

$$\frac{dV(t)}{dt} \leq \frac{a_1(a_1 + 1)e_1}{b_1} + \frac{a_2(a_2 + 1)e_2}{b_2} - mV(t), \quad (3)$$

where $m = \min\{1, d\}$. So, by comparison theorem, we obtain that $V(t) \leq (M/m) + ke^{-mt}$ for $t \geq 0$, where $M = (a_1(a_1 + 1)e_1/b_1) + (a_2(a_2 + 1)e_2/b_2)$ and k is a constant of integration. Thus $e_1 x_1(t) + e_2 x_2(t) + y(t) \leq M/m$ for sufficiently large t , which means that all species are uniformly bounded for any initial value in \mathbb{R}_+^3 . \square

It is easy to see that if $x_1(0) > 0$, then $x_1(t) > 0$ for all $t > 0$. The same is true for x_2 and y components. Therefore, we conclude clearly that the first octant \mathbb{R}_+^3 is an invariant domain of system (1).

Now, we will discuss conditions that render certain species extinct. According to system (1), even if one of the preys is extinct, predator species could survive since the predator has two preys. However, the higher the death rate of the predator is, the higher the possibility of predator extinction is. Thus the following theorem indicates that if the death rate of the predator is less than a certain value depending on the growth rate of two preys, then the predator will not face extinction.

Theorem 2. *A necessary condition for the predator species y to survive is*

$$d < \frac{a_1 e_1}{a_1 + b_1 c} + \frac{a_2 e_2}{a_2 + \alpha b_2}. \quad (4)$$

Proof. From the third equation of system (1), we get

$$\begin{aligned} \frac{dy(t)}{dt} &= y(t) \left(-d + \frac{e_1 x_1(t)}{c + x_1(t)} + \frac{e_2 x_2(t)}{\alpha + x_2(t) + \beta y(t)} \right) \\ &\leq y(t) \left(-d + \frac{e_1 x_1(t)}{c + x_1(t)} + \frac{e_2 x_2(t)}{\alpha + x_2(t)} \right). \end{aligned} \quad (5)$$

In the proof of Theorem 1, $\limsup_{t \rightarrow \infty} x_i(t) \leq a_i/b_i$, $i = 1, 2$, is shown. Then

$$\frac{dy(t)}{dt} \leq y(t) \left(-d + \frac{a_1 e_1}{a_1 + b_1 c} + \frac{a_2 e_2}{a_2 + \alpha b_2} \right) \quad (6)$$

and hence $y(t) \leq y(0)e^{At}$, where $A = -d + (a_1 e_1/(a_1 + b_1 c)) + (a_2 e_2/(a_2 + \alpha b_2))$. Thus if $A < 0$, that is $d > (a_1 e_1/(a_1 + b_1 c)) + (a_2 e_2/(a_2 + \alpha b_2))$, then $\lim_{t \rightarrow \infty} y(t) = 0$. Therefore $d < (a_1 e_1/(a_1 + b_1 c)) + (a_2 e_2/(a_2 + \alpha b_2))$ is a necessary condition for the predator species y to survive. \square

3. Stability Analysis of System (1)

In order to study stabilities of equilibria of system (1), we first take into account a subsystem of system (1) when the second prey (x_2) is absent as follows:

$$x_1'(t) = x_1(t) (a_1 - b_1 x_1(t)) - \frac{x_1(t) y(t)}{c + x_1(t)}, \quad (7)$$

$$y'(t) = y(t) \left(-d + \frac{e_1 x_1(t)}{c + x_1(t)} \right).$$

Kolmogorov's theorem in [19] assumes the existence of either a stable equilibrium point or a stable limit cycle behavior in the positive quadrant of phase space of a two-dimensional (2D) dynamical system, provided certain conditions are satisfied.

In fact, it is easy to see that the subsystem (7) is a Kolmogorov system under the following condition:

$$0 < \frac{cd}{e_1 - d} < \frac{a_1}{b_1}. \quad (8)$$

For this reason, from now on, we assume that subsystem (7) satisfies condition (8). By applying the local stability analysis ([20]) to Kolmogorov system (7) we have the following results.

- (1) The equilibrium point $E_{10} = (0, 0)$ always exists and is a saddle point.
- (2) The equilibrium point $E_{11} = (a_1/b_1, 0)$ always exists and is a saddle point under condition (8).
- (3) The positive equilibrium point $E_{12} = (\hat{x}, \hat{y})$ exists, where

$$\hat{x} = \frac{cd}{e_1 - d}, \quad \hat{y} = (a_1 - b_1 \hat{x})(c + \hat{x}), \quad (9)$$

and it is a locally asymptotically stable point if the following condition holds:

$$d > \frac{e_1(a_1 - b_1 c)}{a_1 + b_1 c}. \quad (10)$$

Moreover, if the condition $d < e_1(a_1 - b_1c)/(a_1 + b_1c)$ holds, the solution of subsystem (7) approaches to a stable limit cycle even though the system is not a Kolmogorov system.

Secondly, we focus on another subsystem of system (1) when the first prey (x_1) is absent as follows:

$$\begin{aligned} x_2'(t) &= x_2(t) \left(a_2 - b_2x_2(t) - \frac{x_2(t)y(t)}{\alpha + x_2(t) + \beta y(t)} \right), \\ y'(t) &= y(t) \left(-d + \frac{e_2x_2(t)}{\alpha + x_2(t) + \beta y(t)} \right). \end{aligned} \tag{11}$$

Subsystem (11) is a Kolmogorov system if the following condition is satisfied:

$$0 < \frac{d\alpha}{e_2 - d} < \frac{a_2}{b_2}. \tag{12}$$

Simple calculation yields that there exist at most three nonnegative equilibrium points of subsystem (11). Moreover, the stability of such equilibrium points can be studied by applying the local stability analysis to subsystem (11) as the previous case. Thus we summarize results about local stability as follows.

- (1) The equilibrium point $E_{20} = (0, 0)$ always exists and is a saddle point.
- (2) The equilibrium point $E_{21} = (a_2/b_2, 0)$ always exists and is also a saddle point under condition (12).
- (3) The positive equilibrium point $E_{22} = (\bar{x}, \bar{y})$ exists, where

$$\begin{aligned} \bar{x} &= \frac{(a_2\beta + d - e_2) + \sqrt{(\beta a_2 + d - e_2)^2 + 4b_2\alpha d\beta}}{2b_2\beta}, \\ \bar{y} &= \frac{(e_2 - d)\bar{x} - d\alpha}{d\beta}. \end{aligned} \tag{13}$$

In [15], the authors have investigated the local stability of the equilibrium point E_{22} .

Theorem 3 (see [15]). *The positive equilibrium point $E_{22} = (\bar{x}, \bar{y})$ of Kolmogorov system (11) is locally asymptotically stable if one of the following sets of conditions is satisfied:*

- (1) $e_2\beta \geq 1$,
- (2) $e_2\beta < 1$ and $\Delta_1^2 - 4\Delta_2 \leq 0$,
- (3) $e_2\beta < 1$ and $\Delta_1^2 - 4\Delta_2 > 0$, with $0 < \bar{x} \leq R_1$ or $R_2 \leq \bar{x} < 1$.

However, the solution of subsystem (11) approaches to a stable limit cycle for $R_1 < \bar{x} < R_2$. Here $\Delta_1 = d(1 - e_2\beta)(d - e_2)/b_2e_2^2\beta$, $\Delta_2 = \alpha d^2(1 - e_2\beta)/b_2e_2^2\beta$, $R_1 = (1/2)(-\Delta_1 - \sqrt{\Delta_1^2 - 4\Delta_2})$, and $R_2 = (1/2)(-\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_2})$.

Now, we turn our concerns on system (1) to investigate the existence and local stability of the equilibrium points of the system. In fact, there are at most seven nonnegative equilibrium points of system (1). The existence conditions of them are mentioned in the following lemma.

Lemma 4. (1) *The trivial equilibrium point $E_0 = (0, 0, 0)$ and one-prey equilibrium points $E_1 = (a_1/b_1, 0, 0)$ and $E_2 = (0, a_2/b_2, 0)$ always exist.*

(2) *Two-species equilibrium points $E_3 = (a_1/b_1, a_2/b_2, 0)$, $E_4 = (\hat{x}, 0, \hat{y})$, and $E_5 = (0, \bar{x}, \bar{y})$ exist in the interior of positive quadrant of x_1x_2 , x_1y , and x_2y planes, respectively, if the Kolmogorov conditions $0 < cd/(e_1 - d) < a_1/b_1$ and $0 < d\alpha/(e_2 - d) < a_2/b_2$ hold, where \hat{x} , \hat{y} and \bar{x} , \bar{y} are given in (9) and (13), respectively.*

(3) *The positive equilibrium point $E_6 = (x_1^*, x_2^*, y^*)$ exists in the interior of the first octant if*

$$e_2 < d < e_1, \quad \frac{c(d - e_2)}{e_1 + e_2 - d} < x_1^* < \min \left\{ \frac{cd}{e_1 - d}, \frac{a_1}{b_1} \right\}, \tag{14}$$

where

$$\begin{aligned} x_2^* &= \frac{((d - e_1)x_1^* + cd)(\alpha + \beta(a_1 - b_1x_1^*)(c + x_1^*))}{(e_1 + e_2 - d)x_1^* + c(e_2 - d)}, \\ y^* &= (a_1 - b_1x_1^*)(c + x_1^*) \end{aligned} \tag{15}$$

and x_1^* satisfies the following equation:

$$A_5x_1^{*5} + A_4x_1^{*4} + A_3x_1^{*3} + A_2x_1^{*2} + A_1x_1^* + A_0 = 0. \tag{16}$$

Here,

$$\begin{aligned} A_0 &= b_2c(\alpha + a_1\beta c)^2 de_2 - a_2c(\alpha + a_1\beta c)e_2(-d + e_2) \\ &\quad + a_1c^2(-d + e_2)^2, \\ A_1 &= 2b_2\beta c(a_1 - b_1c)(\alpha + a_1\beta c)de_2 \\ &\quad + b_2(\alpha + a_1\beta c)^2(d - e_1)e_2 \\ &\quad - a_2\beta c(a_1 - b_1c)e_2(-d + e_2) - b_1c^2(-d + e_2)^2 \\ &\quad - a_2(\alpha + a_1\beta c)e_2(-d + e_1 + e_2) \\ &\quad + 2a_1c(-d + e_2)(-d + e_1 + e_2), \\ A_2 &= b_2\beta^2c(a_1 - b_1c)^2de_2 - 2b_1b_2\beta c(\alpha + a_1\beta c)de_2 \\ &\quad + 2b_2\beta(a_1 - b_1c)(\alpha + a_1\beta c)(d - e_1)e_2 \\ &\quad + a_2b_1\beta ce_2(-d + e_2) \end{aligned}$$

$$\begin{aligned}
 & -a_2\beta(a_1 - b_1c)e_2(-d + e_1 + e_2) \\
 & -2b_1c(-d + e_2)(-d + e_1 + e_2) + a_1(-d + e_1 + e_2)^2, \\
 A_3 = & -2b_1b_2\beta^2c(a_1 - b_1c)de_2 \\
 & + b_2\beta^2(a_1 - b_1c)^2(d - e_1)e_2 \\
 & - 2b_1b_2\beta(\alpha + a_1\beta c)(d - e_1)e_2 \\
 & + (-d + e_1 + e_2)(a_2b_1\beta e_2 - b_1(-d + e_1 + e_2)), \\
 A_4 = & b_1b_2(-2a_1d + 3b_1cd + 2a_1e_1 - 2b_1ce_1)e_2\beta^2, \\
 A_5 = & b_1^2b_2(d - e_1)e_2\beta^2.
 \end{aligned} \tag{17}$$

Proof. We only consider the existence of the positive equilibrium point E_6 . It is easy to see that the equilibrium point $E_6 = (x_1^*, x_2^*, y^*)$ exists in the interior of the first octant if and only if there exists a positive solution to the following algebraic nonlinear simultaneous equations:

$$\begin{aligned}
 f_1(x, y, z) &= a_1 - b_1x_1(t) - \frac{y(t)}{c + x_1(t)} = 0, \\
 f_2(x, y, z) &= a_2 - b_2x_2(t) - \frac{y(t)}{\alpha + x_2(t) + \beta y(t)} = 0, \\
 f_3(x, y, z) &= -d + \frac{e_1x_1(t)}{c + x_1(t)} + \frac{e_2x_2(t)}{\alpha + x_2(t) + \beta y(t)} = 0.
 \end{aligned} \tag{18}$$

From the first and third equations in (18) we can have

$$\begin{aligned}
 x_2^* &= \frac{((d - c)x_1^* + dc)(\alpha + \beta(a_1 - b_1x_1^*)(c + x_1^*))}{(e_1 + e_2 - d)x_1^* + c(e_2 - d)}, \\
 y^* &= (a_1 - b_1x_1^*)(c + x_1^*).
 \end{aligned} \tag{19}$$

By using (19) in the second equation of (18) and by elementary calculation we can obtain the following equation:

$$A_5x_1^{*5} + A_4x_1^{*4} + A_3x_1^{*3} + A_2x_1^{*2} + A_1x_1^* + A_0 = 0. \tag{20}$$

Since the degree of equation (20) is 5, it has at least one real root x_1^* . Moreover if condition (14) is satisfied then all values of x_1^* , x_2^* , and y^* are positive. \square

It is worth noting that since predator dies out in the absence of all preys the equilibrium point $(0, 0, z_c)$ with $z_c > 0$ does not exist.

In order to investigate stabilities of the equilibrium points, we need to consider the variational matrix $V(x_1, x_2, y)$ of system (1). Thus we get the following matrix:

$$V(x_1, x_2, y) = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}, \tag{21}$$

where

$$\begin{aligned}
 v_{11} &= a_1 - 2b_1x_1 - \frac{cy}{(c + x_1)^2}, \\
 v_{12} &= 0, \quad v_{13} = -\frac{x_1}{c + x_1}, \\
 v_{21} &= 0, \quad v_{22} = a_2 - 2b_2x_2 - \frac{y(\alpha + \beta y)}{(\alpha + x_2 + \beta y)^2}, \\
 v_{23} &= -\frac{x_2(\alpha + x_2)}{(\alpha + x_2 + \beta y)^2}, \\
 v_{31} &= \frac{ce_1y}{(c + x_1)^2}, \quad v_{32} = \frac{e_2y(\alpha + \beta y)}{(\alpha + x_2 + \beta y)^2}, \\
 v_{33} &= -d + \frac{e_1x_1}{c + x_1} + \frac{e_2x_2(\alpha + x_2)}{(\alpha + x_2 + \beta y)^2}.
 \end{aligned} \tag{22}$$

By using the variational matrix (21), local stabilities of system (1) near the equilibrium points are studied in the following theorems.

Theorem 5. (1) The trivial equilibrium point $E_0 = (0, 0, 0)$ is a hyperbolic saddle point. In fact, near E_0 both prey populations are increasing while the predator population is decreasing. And the equilibrium points $E_1 = (a_1/b_1, 0, 0)$ and $E_2 = (0, a_2/b_2, 0)$ are also hyperbolic saddle points.

(2) The equilibrium point $E_3 = (a_1/b_1, a_2/b_2, 0)$ is always unstable; actually, a saddle point with locally stable manifold in the x_1x_2 plane and with local unstable manifold in the y declines if Kolmogorov conditions (8) and (12) hold.

(3) The equilibrium point $E_4 = (\bar{x}, 0, \bar{y})$ is stable if $(1 - a_2\beta)\bar{y} > a_2\alpha$ and is unstable if $(1 - a_2\beta)\bar{y} < a_2\alpha$.

(4) Assume that hypotheses of Theorem 3 hold; then the equilibrium point $E_5 = (0, \bar{x}, \bar{y})$ is stable if $\bar{y} > a_1c$ and is unstable if $\bar{y} < a_1c$.

Proof. (1) The eigenvalues of the matrix $V(0, 0, 0)$ are a_1, a_2 , and $-d$ and their eigenvectors are $(1, 0, 0), (0, 1, 0)$, and $(0, 0, -1)$. Furthermore, the eigenvalues of the matrices $V(a_1/b_1, 0, 0)$ and $V(0, a_2/b_2, 0)$ are $-a_1, a_2, (a_1e_1/(b_1c + a_1)) - d$ and $a_1, -a_2, (a_2e_2/(b_2\alpha + a_2)) - d$, respectively. Thus the equilibrium points E_0, E_1 , and E_2 are hyperbolic saddle.

(2) The eigenvalues of the matrix $V(a_1/b_1, a_2/b_2, 0)$ are $-a_1, -a_2$, and $(a_1e_1/(b_1c + a_1)) + (a_2e_2/(b_2\alpha + a_2)) - d$. Therefore, since Kolmogorov conditions (8) and (12) are satisfied, the sign of $(a_1e_1/(b_1c + a_1)) + (a_2e_2/(b_2\alpha + a_2)) - d$ is always positive. Thus the point E_3 is unstable.

(3) Now, consider the equilibrium point E_4 . The point $E_4 = (\bar{x}, 0, \bar{y})$ has the same stability as E_{12} in the interior of positive coordinate plane x_1y . Furthermore, since the equilibrium point E_{12} is always stable under condition (8), the local stability of the point E_4 depends on the sign of the eigenvalue $a_2 - (\bar{y}/(\alpha + \beta\bar{y}))$ of the x_2 -direction.

(4) Similar to the case of the point E_4 , the point $E_5 = (0, \bar{x}, \bar{y})$ has the same stability behavior as E_{22} in the interior of positive coordinate plane x_2y . Thus, since the point E_{22} is

locally stable, if one of the conditions of Theorem 3 is satisfied, then the point E_5 is locally stable or unstable along the x_1 -direction according to the sign of the eigenvalue $a_1 - (\bar{y}/c)$ of the x_1 -direction. \square

Example 6. In this example we simulate system (1) numerically by using Runge-Kutta method of order 4 to substantiate Theorem 5 when the parameters are as follows:

$$\begin{aligned} a_1 = 0.8, \quad b_1 = 1, \quad a_2 = 0.7, \quad b_2 = 1, \\ c = 0.5, \quad d = 0.2, \quad e_1 = 0.8, \quad (23) \\ e_2 = 0.9, \quad \alpha = 1.3, \quad \beta = 0.5. \end{aligned}$$

Then it follows from Theorem 5 (4) that $E_5 = (0, \bar{x}, \bar{y}) = (0, 0.4516, 0.5609)$ is stable since $\bar{y} > a_1c = 0.4$. Figure 1 illustrates the phase portrait of system (1) and time series for $x_1(t)$, $x_2(t)$, and $y(t)$ when initial condition is $(0.2, 0.2, 0.2)$.

In order to discuss the stability of the equilibrium point $E_6 = (x_1^*, x_2^*, y^*)$, let $V^* = (v_{ij}^*)$ be the variational matrix at E_6 . Then it follows from (21) that V^* can be written as follows:

$$V^* = \begin{pmatrix} -b_1x_1^* + \frac{x_1^*y^*}{(c+x_1^*)^2} & 0 & -\frac{x_1^*}{c+x_1^*} \\ 0 & -b_2x_2^* + \frac{x_2^*y^*}{(\alpha+x_2^*+\beta y^*)^2} & -\frac{x_2^*(\alpha+x_2^*)}{(\alpha+x_2^*+\beta y^*)^2} \\ \frac{ce_1y^*}{(c+x_1^*)^2} & \frac{e_2y^*(\alpha+\beta y^*)}{(\alpha+x_2^*+\beta y^*)^2} & -\frac{e_2\beta x_2^*y^*}{(\alpha+x_2^*+\beta y^*)^2} \end{pmatrix}. \quad (24)$$

Thus the characteristic equation of the matrix V^* is obtained as $\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0$, where $B_1 = -(v_{11}^* + v_{22}^* + v_{33}^*)$, $B_2 = v_{11}^*v_{22}^* + v_{11}^*v_{33}^* + v_{22}^*v_{33}^* - v_{13}^*v_{31}^* - v_{23}^*v_{32}^*$, $B_3 = (v_{23}^*v_{32}^* - v_{22}^*v_{33}^*)v_{11}^* + v_{22}^*v_{13}^*v_{31}^*$, $v_{11}^* = -b_1x_1^* + (x_1^*y^*/(c+x_1^*)^2)$, $v_{12}^* = 0$, $v_{13}^* = -x_1^*/(c+x_1^*)$, $v_{21}^* = 0$, $v_{22}^* = -b_2x_2^* + (x_2^*y^*/(\alpha+x_2^*+\beta y^*)^2)$, $v_{23}^* = -x_2^*(\alpha+x_2^*)/(\alpha+x_2^*+\beta y^*)^2$, $v_{31}^* = ce_1y^*/(c+x_1^*)^2$, $v_{32}^* = e_2y^*(\alpha+\beta y^*)/(\alpha+x_2^*+\beta y^*)^2$, and $v_{33}^* = -e_2\beta x_2^*y^*/(\alpha+x_2^*+\beta y^*)^2$.

From the Routh-Hurwitz criterion ([20]), we know that $E_6 = (x_1^*, x_2^*, y^*)$ is locally asymptotically stable if and only if B_1, B_3 , and $B_1B_2 - B_3$ are positive. It is not easy to find the conditions $B_1 > 0$ and $B_3 > 0$ and $B_1B_2 - B_3 > 0$. However, we give a sufficient condition to guarantee the local stability of the equilibrium point $E_6 = (x_1^*, x_2^*, y^*)$ in the following theorem.

Theorem 7. Suppose that the positive equilibrium point E_6 exists in the interior of the positive octant. Then E_6 is locally asymptotically stable if

$$y^* < b_1(c+x_1^*)^2, \quad (25)$$

$$y^* < b_2(\alpha+x_1^*+\beta y^*)^2. \quad (26)$$

Proof. It is from elementary calculation that $B_1 > 0$, $B_3 > 0$, and $B_1B_2 - B_3 > 0$ under conditions (25) and (26). \square

Example 8. In order to substantiate Theorem 7, we set the parameters as follows:

$$\begin{aligned} a_1 = 0.8, \quad b_1 = 1, \quad a_2 = 0.7, \\ b_2 = 1, \quad c = 0.8, \quad d = 0.5, \quad (27) \\ e_1 = 0.9, \quad e_2 = 0.3, \quad \alpha = 1.3, \quad \beta = 0.5. \end{aligned}$$

The point $E_6 = (x_1^*, x_2^*, y^*) = (0.667992, 0.603106, 0.193787)$ is locally stable since $y^* < b_1(c+x_1^*)^2 = 2.155$ and $y^* < b_2(\alpha+x_1^*+\beta y^*)^2 = 4.2638$. The phase portrait of system (1) and time series for $x_1(t)$, $x_2(t)$, and $y(t)$ are shown in Figure 2.

On the other hand, if one of the conditions (25) and (26) is not satisfied, the positive equilibrium point E_6 could not be stable. In order to illustrate an example, we take the parameters as follows:

$$\begin{aligned} a_1 = 0.8, \quad b_1 = 0.5, \quad a_2 = 0.7, \\ b_2 = 1, \quad c = 0.2, \quad d = 0.5, \quad (28) \\ e_1 = 0.9, \quad e_2 = 0.3, \quad \alpha = 1.3, \quad \beta = 0.5. \end{aligned}$$

Then we have the point $E_6 = (0.171044, 0.567449, 0.265103)$. Since $y^* > b_1(c+x_1^*)^2 = 0.068836$ and $y^* < b_2(\alpha+x_1^*+\beta y^*)^2 = 2.57152$, the point E_6 does not satisfy condition (25) and moreover Figure 3 exhibits numerically that E_6 is unstable. As shown in Figure 3 even if the positive point becomes an unstable point a stable limit cycle could occur.

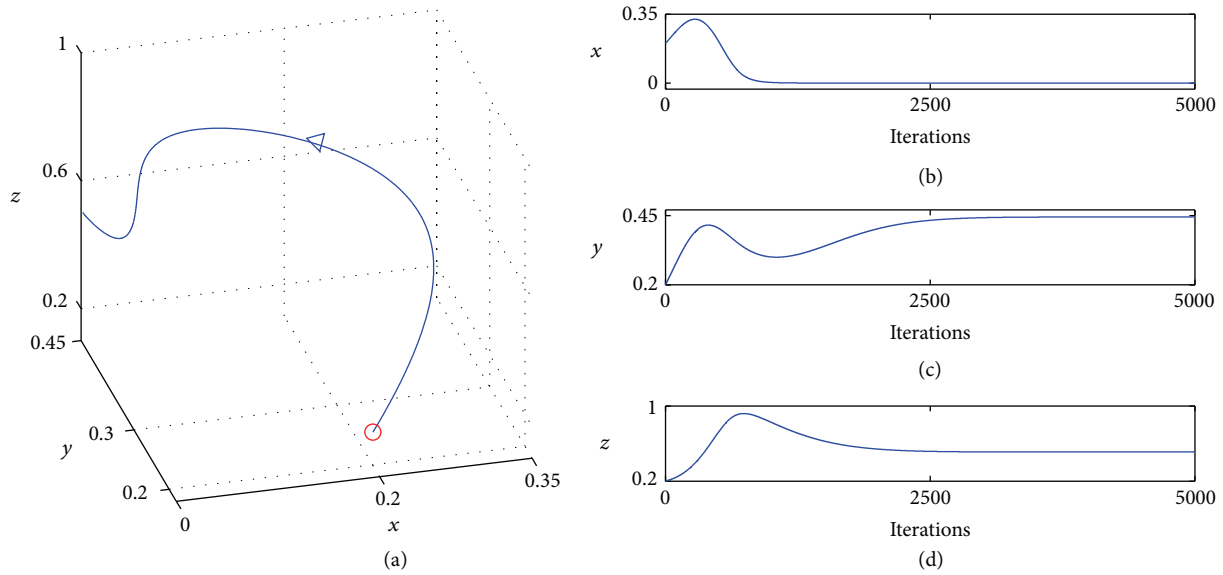


FIGURE 1: (a) A phase portrait of system (1) with initial condition (0.2, 0.2, 0.2) and (b)–(d) time series for $x_1(t)$, $x_2(t)$, and $y(t)$, respectively, when $a_1 = 0.8$, $b_1 = 1$, $a_2 = 0.7$, $b_2 = 1$, $c = 0.5$, $d = 0.2$, $e_1 = 0.8$, $e_2 = 0.9$, $\alpha = 1.3$, and $\beta = 0.5$.

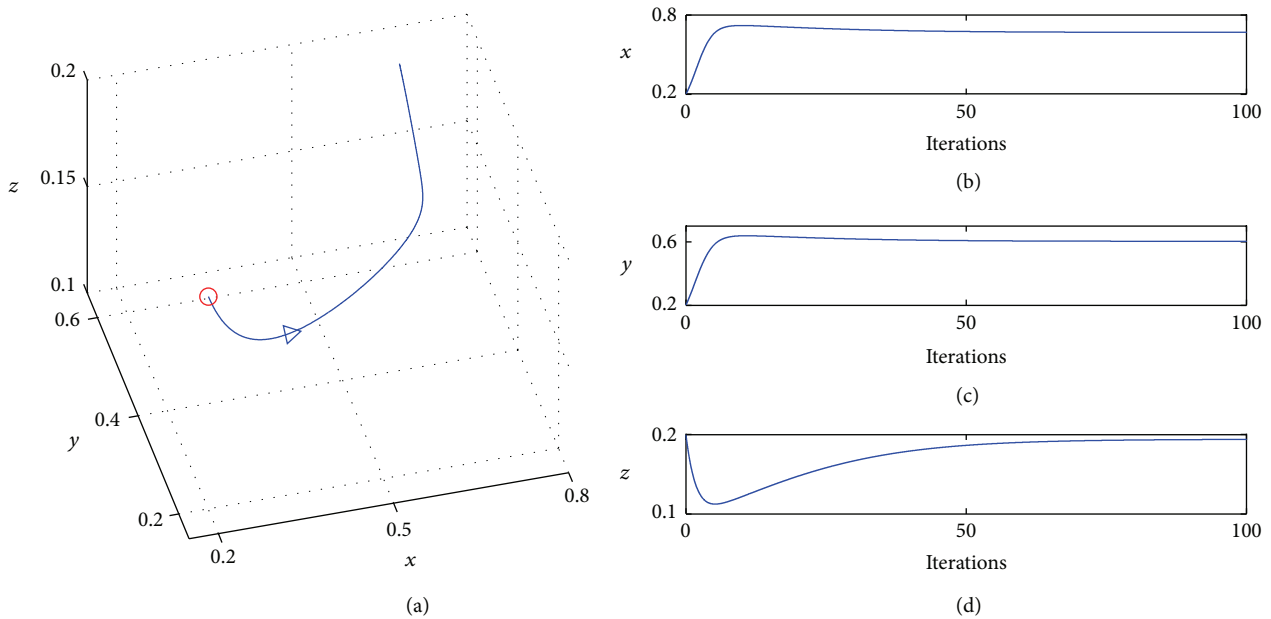


FIGURE 2: (a) A phase portrait of system (1) with initial condition (0.2, 0.2, 0.2) and (b)–(d) time series for $x_1(t)$, $x_2(t)$, and $y(t)$, respectively, when $a_1 = 0.8$, $b_1 = 1$, $a_2 = 0.7$, $b_2 = 1$, $c = 0.8$, $d = 0.5$, $e_1 = 0.9$, $e_2 = 0.3$, $\alpha = 1.3$, and $\beta = 0.5$.

4. Persistence of System (1)

The term persistence is given to systems in which strict solutions do not approach the boundary of the nonnegative cones as time goes to infinity. Therefore, for the continuous biological system, survival of all interacting species and the persistence are equivalent. In the following theorem, we find out some persistence conditions of system (1).

Theorem 9. Suppose that system (1) has no nontrivial periodic solutions in the boundary planes and satisfies the hypothesis

of Theorem 3 and condition $e_1(a_1 - b_1c)/(a_1 + b_1c) < d < a_1e_1/(a_1 + b_1c)$ holds. Then the necessary conditions for the persistence of system (1) are

$$\bar{\lambda}_1 = a_1 - \frac{\bar{y}}{c} \geq 0, \tag{29}$$

$$\hat{\lambda}_2 = a_2 - \frac{\hat{y}}{\alpha + \beta\hat{y}} \geq 0, \tag{30}$$

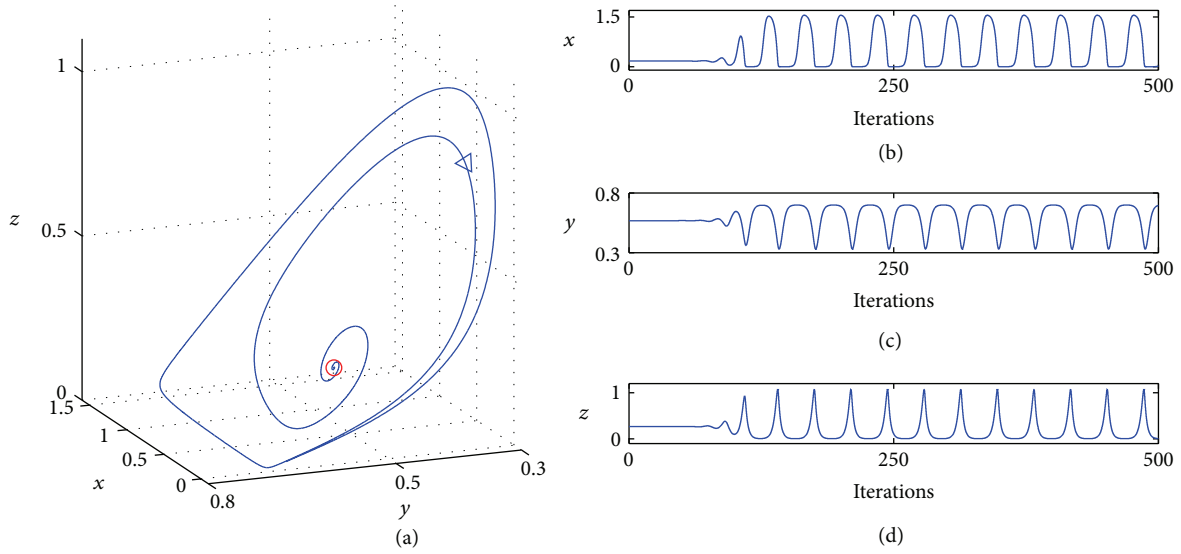


FIGURE 3: (a) A phase portrait of system (1) with initial condition (0.171, 0.567, 0.265) and (b)–(d) time series for $x_1(t)$, $x_2(t)$, and $y(t)$, respectively, when $a_1 = 0.8$, $b_1 = 0.5$, $a_2 = 0.7$, $b_2 = 1$, $c = 0.2$, $d = 0.5$, $e_1 = 0.9$, $e_2 = 0.3$, $\alpha = 1.3$, and $\beta = 0.5$.

and the sufficient conditions for the persistence of system (1) are

$$\bar{\lambda}_1 = a_1 - \frac{\bar{y}}{c} > 0, \tag{31}$$

$$\hat{\lambda}_2 = a_2 - \frac{\hat{y}}{\alpha + \beta \hat{y}} > 0. \tag{32}$$

Proof. Note that the boundedness of system (1) is shown in Theorem 1 and E_{12} is locally stable under Kolmogorov condition (12). Since E_{12} and E_{22} are locally stable by the assumptions, the signs of the eigenvalues $\hat{\lambda}_2$ and $\bar{\lambda}_1$ determine the stability of the equilibrium points $E_4 = (\hat{x}, 0, \hat{y})$ and $E_5 = (0, \bar{x}, \bar{y})$. In fact, if there are no nontrivial periodic solutions in the x_2y plane and (29) does not hold (i.e., $\bar{\lambda}_1 < 0$) then there is an orbit in the positive cone, which approaches to E_5 . Hence, condition (29) is one of the necessary conditions for the persistence. Similarly, we obtain the other necessary condition (30) for the persistence of system (1) by applying the same method as mentioned above to the equilibrium point E_4 .

Now, we will use the abstract theorem of Freedman and Waltman [19] to figure out sufficient conditions for the persistence of system (1). In order to do this, consider the growth functions f_1 , f_2 , and f_3 in (18) of system (1). Then it is shown that the following four conditions are satisfied.

- (1) Clearly, we have $\partial f_i / \partial y < 0$, $\partial f_3 / \partial x_i > 0$, $i = 1, 2$.
- (2) Each prey population grows up to its carrying capacity in the absence of predators; that is, $f_1(0, 0, 0) = a_1 > 0$ and $f_2(0, 0, 0) = a_2 > 0$ and $(\partial f_i / \partial x_i)(x_1, x_2, 0) = b_i < 0$ ($i = 1, 2$) and $f_1(a_1/b_1, 0, 0) = 0 = f_2(0, a_2/b_2, 0)$. Furthermore, the

predator population dies out in the absence of preys; that is, consider $f_3(0, 0, 0) = -d < 0$.

- (3) $\partial f_1 / \partial x_2 = 0$ and $\partial f_2 / \partial x_1 = 0$. There exists exactly one point $E_3 = (b_1/a_1, b_2/a_2, 0)$ satisfying $f_i(b_1/a_1, b_2/a_2, 0) = 0$, $i = 1, 2$.
- (4) In the absence of each prey species the predator can survive on the other prey. This is always true under the Kolmogorov conditions (8) and (12). There exist uniquely $E_4 = (\hat{x}, 0, \hat{y})$ and $E_5 = (0, \bar{x}, \bar{y})$ satisfying $f_1(\hat{x}, 0, \hat{y}) = f_3(\hat{x}, 0, \hat{y}) = f_2(0, \bar{x}, \bar{y}) = f_3(0, \bar{x}, \bar{y}) = 0$. According to Kolmogorov conditions (8) and (12), we can get that $f_3(a_1/b_1, 0, 0) > 0$ and $f_3(0, a_2/b_2, 0) > 0$, respectively.
- (5) It follows from (8), (12), (31), and (32) that the inequalities $f_3(a_1/b_1, a_2/b_2, 0) > 0$, $f_1(0, \bar{x}, \bar{y}) > 0$, and $f_2(\hat{x}, 0, \hat{y}) > 0$ hold.

Therefore, by Freedman and Waltman theorem ([19]), system (1) persists under the hypotheses. \square

Example 10. Now let the parameters be as follows:

$$\begin{aligned} a_1 &= 0.8, & b_1 &= 0.5, & a_2 &= 1.0, \\ b_2 &= 1.0, & c &= 0.8, & d &= 0.199, \\ e_1 &= 0.3, & e_2 &= 0.5, & \alpha &= 1.0, & \beta &= 0.7. \end{aligned} \tag{33}$$

Then it follows from [15, 21] that subsystems (7) and (11) have no periodic solutions in the boundary planes. Furthermore, it is not difficult to see that system (1) satisfies conditions (2) of Theorem 3 and the other hypotheses in Theorem 9. Thus, all species in system (1) can coexist as time goes away. Figure 4

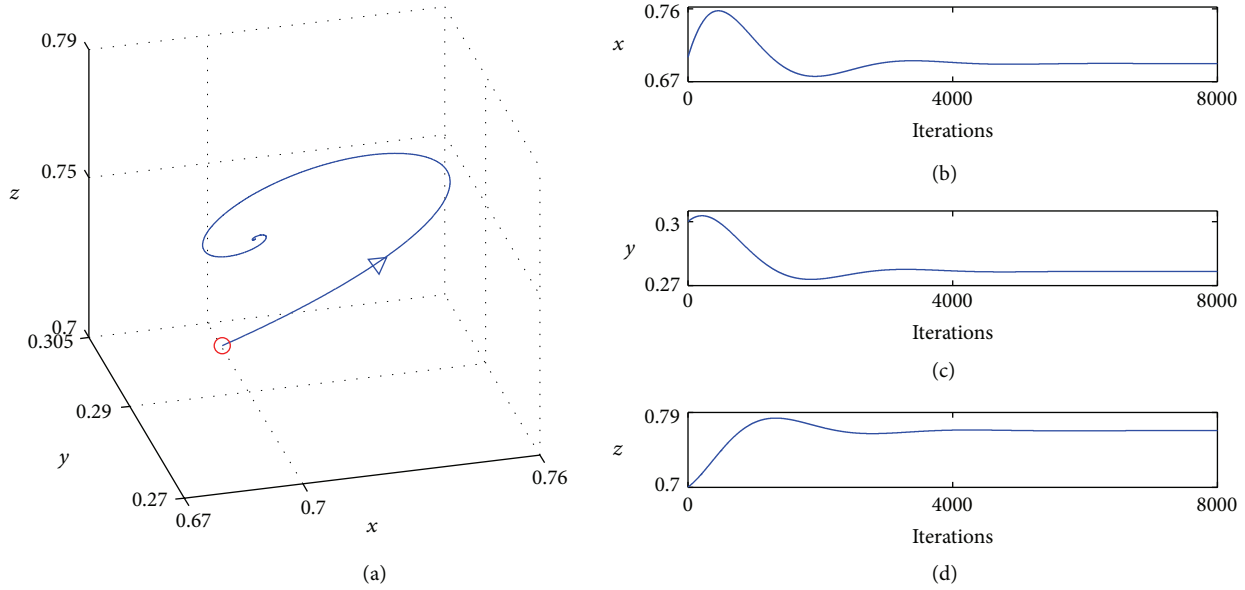


FIGURE 4: (a) A phase portrait of system (1) with initial condition (0.7, 0.3, 0.7) and (b)–(d) time series for $x_1(t)$, $x_2(t)$, and $y(t)$, respectively, when $a_1 = 0.8$, $b_1 = 0.5$, $a_2 = 1.0$, $b_2 = 1.0$, $c = 0.8$, $d = 0.199$, $e_1 = 0.3$, $e_2 = 0.5$, $\alpha = 1.0$, and $\beta = 0.7$.

shows a phase portrait and time series of all species of system (1) with initial condition (0.7, 0.3, 0.7).

Theorem 11. Suppose that conditions (31) and (32) are satisfied and system (1) has a finite number of limit cycles in the x_1y plane or in the x_2y plane. Then system (1) persists if

$$\int_0^T \left(a_1 - \frac{v_2(t)}{c} \right) dt > 0, \tag{34}$$

$$\int_0^T \left(a_2 - \frac{v_1(t)}{\alpha + \beta v_1(t)} \right) dt > 0 \tag{35}$$

hold for each of the limit cycles $(0, u_2(t), v_2(t))$ and $(u_1(t), 0, v_1(t))$ in the x_2y plane and in the x_1y plane, respectively, if it exists. Here, T means the period of the limit cycle.

Proof. First, consider a limit cycle $(0, u_2(t), v_2(t))$ in x_2y plane. Then the variational matrix about the limit cycle $(0, u_2(t), v_2(t))$ can be written as

$$V_1 = \begin{pmatrix} a_1 - \frac{v_2(t)}{c} & 0 & 0 \\ 0 & a_2 - 2b_2u(t) - \frac{v_2(t)(\alpha + \beta v_2(t))}{(\alpha + u_2(t) + \beta v_2(t))^2} & -\frac{u_2(t)(\alpha + u_2(t))}{(\alpha + u_2(t) + \beta v_2(t))^2} \\ \frac{e_1 v_2(t)}{c} & -\frac{e_2 v_2(t)(\alpha + \beta v_2(t))}{(\alpha + u_2(t) + \beta v_2(t))^2} & -d + \frac{u_2(t)(\alpha + u_2(t))}{(\alpha + u_2(t) + \beta v_2(t))^2} \end{pmatrix}. \tag{36}$$

Now, let (x_1, x_2, y) be a solution of system (1) with positive initial condition (z_1, z_2, z_3) sufficiently close to the limit cycle. It is easily obtained from the variational matrix V_1 that $\partial x_1 / \partial z_1$ is a solution of system $dx_1 / dt = (a_1 - (v_2(t)/c))x_1$ with $x_1(0) = 1$. Thus, we obtain

$$\frac{\partial x_1}{\partial z_1}(t, z_1, z_2, z_3) = \exp\left(\int_0^t \left(a_1 - \frac{v_2(s)}{c} \right) ds\right). \tag{37}$$

From Taylor expansion, we get

$$x_1(t, z_1, z_2, z_3) - x_1(t, 0, z_2, z_3) \approx \exp\left(\int_0^t \left(a_1 - \frac{v_2(s)}{c} \right) ds\right) z_1. \tag{38}$$

Thus x_1 increases or decreases as the value of $\int_0^t (a_1 - (v_2(s)/c)) ds$ is positive or negative, respectively. Since E_5 and the limit cycle $(0, u_2(t), v_2(t))$ are the only possible limit in the x_2y plane of trajectories with positive initial condition,

the trajectories go away from the x_2y plane if conditions (31) and (34) are satisfied.

Similar argument can be applied to each limit cycle $(u_1(t), 0, v_1(t))$ to obtain the fact that the trajectories go away

from the x_1y plane if conditions (32) and (35) hold by considering the variational matrix V_2 about the limit cycle $(u_1(t), 0, v_1(t))$ as follows:

$$V_2 = \begin{pmatrix} a_1 - 2b_1u_1 - \frac{cv_1(t)}{(c+u_1)^2} & 0 & -\frac{u_1}{c+u_1} \\ 0 & a_2 - \frac{v_1(t)}{\alpha + \beta v_1(t)} & 0 \\ \frac{ce_1v_1(t)}{(c+u_1)^2} & \frac{e_2v_1(t)}{\alpha + \beta v_1(t)} & -d + \frac{e_1u_1(t)}{c+u_1(t)} \end{pmatrix}. \quad (39)$$

Therefore the proof is complete. \square

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5. Conclusions and Remarks

In this paper, we have considered a predator-prey system with two preys and one predator with two different types of functional responses, Holling type and Beddington-DeAngelis type. Until now, many researches for two-prey and one-predator systems have dealt with the same functional responses to describe the relationship between prey and predator even if the two preys are different from each other. Thus in this research we adopted two different types of functional responses to model the relationship between two different preys and predator. We investigated stabilities of system about equilibrium points by virtue of stabilities of two subsystems without one of the preys of system. Also, we found out conditions that guarantee that the system is persistent. In addition, some numerical examples are illustrated to substantiate our theoretical results.

Generally speaking, if food is abundant the predators do not interfere with each other to get it; otherwise there is intense competition between predators to get food. In order to describe such kind of phenomenon, we use the mixed functional responses. Thus, due to the mixed functional responses, one can see from Theorem 5 that the value β of the impact of the predator interference to catch the prey x_2 has an effect on the extinction of another prey x_1 . Thus even though ecological systems have two different functional responses, they have a variety of dynamical behaviors.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

- [1] M. Baurmann, T. Gross, and U. Feudel, "Instabilities in spatially extended predator-prey systems: spatio-temporal patterns in the neighborhood of Turing-Hopf bifurcations," *Journal of Theoretical Biology*, vol. 245, no. 2, pp. 220–229, 2007.
- [2] P. A. Braza, "The bifurcation structure of the Holling-Tanner model for predator-prey interactions using two-timing," *SIAM Journal on Applied Mathematics*, vol. 63, no. 3, pp. 889–904, 2003.
- [3] C. Consner, D. L. DeAngelis, J. S. Ault, and D. B. Olson, "Effects of spatial grouping on the functional response of predators," *Theoretical Population Biology*, vol. 56, no. 1, pp. 65–75, 1999.
- [4] S. Ruan and D. Xiao, "Global analysis in a predator-prey system with nonmonotonic functional response," *SIAM Journal on Applied Mathematics*, vol. 61, no. 4, pp. 1445–1472, 2000/01.
- [5] H. Baek, "Species extinction and permanence of an impulsively controlled two-prey one-predator system with seasonal effects," *BioSystems*, vol. 98, no. 1, pp. 7–18, 2009.
- [6] M. Fan and Y. Kuang, "Dynamics of a nonautonomous predator-prey system with the Beddington-DeAngelis functional response," *Journal of Mathematical Analysis and Applications*, vol. 295, no. 1, pp. 15–39, 2004.
- [7] S. Gakkhar and R. K. Naji, "Seasonally perturbed prey-predator system with predator-dependent functional response," *Chaos, Solitons and Fractals*, vol. 18, no. 5, pp. 1075–1083, 2003.
- [8] R. Arditi and L. R. Ginzburg, "Coupling in predator-prey dynamics: ratio-dependence," *Journal of Theoretical Biology*, vol. 139, no. 3, pp. 311–326, 1989.
- [9] H. I. Freedman, *Deterministic Mathematical Models in Population Ecology*, Marcel Dekker, New York, NY, USA, 1980.
- [10] S. Gakkhar and R. K. Naji, "Order and chaos in predator to prey ratio-dependent food chain," *Chaos, Solitons and Fractals*, vol. 18, no. 2, pp. 229–239, 2003.
- [11] A. Hastings and T. Powell, "Chaos in a three-species food chain," *Ecology*, vol. 72, no. 3, pp. 896–903, 1991.
- [12] S. Hsu, T. Hwang, and Y. Kuang, "A ratio-dependent food chain model and its applications to biological control," *Mathematical Biosciences*, vol. 181, no. 1, pp. 55–83, 2003.

- [13] A. Klebanoff and A. Hastings, "Chaos in three-species food chains," *Journal of Mathematical Biology*, vol. 32, no. 5, pp. 427–451, 1994.
- [14] S. Lv and M. Zhao, "The dynamic complexity of a three species food chain model," *Chaos, Solitons and Fractals*, vol. 37, no. 5, pp. 1469–1480, 2008.
- [15] R. K. Naji and A. T. Balasim, "Dynamical behavior of a three species food chain model with Beddington-DeAngelis functional response," *Chaos, Solitons and Fractals*, vol. 32, no. 5, pp. 1853–1866, 2007.
- [16] C. Shen, "Permanence and global attractivity of the food-chain system with Holling IV type functional response," *Applied Mathematics and Computation*, vol. 194, no. 1, pp. 179–185, 2007.
- [17] J. A. Vano, J. C. Wildenberg, M. B. Anderson, J. K. Noel, and J. C. Sprott, "Chaos in low-dimensional Lotka-Volterra models of competition," *Nonlinearity*, vol. 19, no. 10, pp. 2391–2404, 2006.
- [18] M. Zhao and S. Lv, "Chaos in a three-species food chain model with a Beddington-DeAngelis functional response," *Chaos, Solitons and Fractals*, vol. 40, no. 5, pp. 2305–2316, 2009.
- [19] H. I. Freedman and P. Waltman, "Persistence in models of three interacting predator-prey populations," *Mathematical Biosciences*, vol. 68, no. 2, pp. 213–231, 1984.
- [20] F. Brauer and C. Castillo-Chávez, *Mathematical Models in Population Biology and Epidemiology*, vol. 40 of *Texts in applied mathematics*, Springer, New York, NY, USA, 2001.
- [21] J. Sugie, R. Kohno, and R. Miyazaki, "On a predator-prey system of Holling type," *Proceedings of the American Mathematical Society*, vol. 125, no. 7, pp. 2041–2050, 1997.