

Research Article

Crank-Nicolson Fully Discrete H^1 -Galerkin Mixed Finite Element Approximation of One Nonlinear Integrodifferential Model

Fengxin Chen

School of Science, Shandong Jiaotong University, Jinan 250357, China

Correspondence should be addressed to Fengxin Chen; cfx_1981@163.com

Received 9 March 2014; Accepted 30 April 2014; Published 25 May 2014

Academic Editor: Xinan Hao

Copyright © 2014 Fengxin Chen. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We consider a fully discrete H^1 -Galerkin mixed finite element approximation of one nonlinear integrodifferential model which often arises in mathematical modeling of the process of a magnetic field penetrating into a substance. We adopt the Crank-Nicolson discretization for time derivative. Optimal order a priori error estimates for the unknown function in L^2 and H^1 norm and its gradient function in L^2 norm are presented. A numerical example is given to verify the theoretical results.

1. Introduction

The objective of this paper is to discuss a Crank-Nicolson fully discrete H^1 -mixed finite element scheme for the following nonlinear integrodifferential model:

$$\begin{aligned}u_t - (1 + \lambda(t))u_{xx} &= f(x, t), \quad (x, t) \in I \times (0, T], \\u(0, t) &= 0, \quad u(1, t) = 0, \quad 0 \leq t \leq T, \\u(x, 0) &= u_0(x), \quad x \in I,\end{aligned}\tag{1}$$

where $\lambda(t) = \int_0^t \int_0^1 (\partial u / \partial x)^2 dx ds$ and $I = [0, 1]$. $u_0(x)$ and $f(x, t)$ are given functions.

The above equations have been widely used to describe the process of a magnetic field penetrating into a substance, which is a generalization of the model proposed in [1–4]. The existence and uniqueness of a weak solution to the above boundary value problems were proved in [5].

During the last decades, many numerical methods were developed to discretize this kind of problems. For the finite difference approximation of the above model one can refer to [6–11]. For Galerkin finite element approximation of model (1) we can refer to [11], where the authors developed error estimates for semidiscretization in the energy norm. Note that the coefficient in (1) depends on the derivative of u . When

finite difference method and Galerkin method were used to solve this model, one needed to differentiate the numerical solution to determine the coefficient. This would generate an inaccurate coefficient, which then reduces the accuracy of the numerical approximation for u . In order to overcome this question an H^1 -Galerkin mixed finite element discrete scheme was proposed in [12]. Optimal order error estimates in L^2 norm and H^1 norm were presented. For more references with respect to H^1 -Galerkin mixed finite element method one can refer to [13–17].

In [12] the backward Euler method was used to discretize the time derivative. Note that problem (1) is nonlocal due to the integration term in the coefficient. To improve the convergence order for time discretization and save the storage we construct a Crank-Nicolson H^1 -mixed finite element scheme for problem (1). By using elliptic projection and the boundness of the numerical solutions we prove optimal a priori error estimates for the scalar unknown function and its flux. Finally we carry out a numerical example to verify our theoretical results.

The rest of this paper is organized as follows. In Section 2 a Crank-Nicolson H^1 -mixed finite element scheme is constructed. Optimal a priori error estimates are deduced in Section 3. In Section 4 a numerical example is carried out to verify our theoretical results.

Throughout the paper, we use the standard notation $W^{m,q}(\Omega)$ for Sobolev space on Ω with a norm $\|\cdot\|_{m,q}$ and a seminorm $|\cdot|_{m,q}$. For $q = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $\|\cdot\|_m = \|\cdot\|_{m,2}$, and for $m = 0$, we denote $\|\cdot\| = \|\cdot\|_0$. Moreover, the inner products in $L^2(\Omega)$ are indicated by (\cdot, \cdot) . Let X be a Banach space and $\varphi(t) : [0, T] \mapsto X$; we set

$$\|\varphi\|_{L^2(X)}^2 = \int_0^T \|\varphi(s)\|_X^2 ds, \quad \|\varphi\|_{L^\infty(X)} = \operatorname{ess\,sup}_{0 \leq t \leq T} \|\varphi\|_X. \quad (2)$$

In addition, C denotes a generic constant independent of the spatial mesh parameter h and time discretization parameter τ , and ε denotes an arbitrarily small positive constant.

2. Crank-Nicolson Discrete Scheme

In this section we first briefly describe the weak formulation for problem (1) and then construct a Crank-Nicolson discrete scheme for it.

2.1. Weak Formulation. In order to define a fully discrete H^1 -Galerkin mixed finite element procedure for problem (1), we firstly split (1) into a first order system. Let $\sigma = u_x$; then (1) reduces to

$$\begin{aligned} \sigma &= u_x, \\ u_t &= (1 + \lambda(t))\sigma_x + f(x, t), \end{aligned} \quad (3)$$

where $\lambda(t) = \int_0^t \int_0^1 \sigma^2 dx ds$.

Let $H_0^1(I) = \{v \in H^1(I); v(0) = v(1) = 0\}$. It is natural to state the weak formulation for problem (1) in the following form:

$$\begin{aligned} (u_x, v_x) &= (\sigma, v_x), \quad v \in H_0^1(I), \\ (\sigma_t, w) + ((1 + \lambda(t))\sigma_x, w_x) + (f, w_x) &= 0, \quad w \in H^1(I). \end{aligned} \quad (4)$$

2.2. The Crank-Nicolson Discrete Scheme. First we introduce two finite element spaces. Let V_h and W_h denote the finite dimensional subspaces of $H_0^1(I)$ and $H^1(I)$, respectively, with the following approximation properties:

$$\begin{aligned} \inf_{\psi_h \in V_h} \{ \|\psi - \psi_h\|_{0,p} + h\|\psi - \psi_h\|_{1,p} \} &\leq Ch^{k+1} \|\psi\|_{k+1,p}, \\ \psi &\in H_0^1(I) \cap W^{k+1,p}(I), \\ \inf_{w_h \in W_h} \{ \|w - w_h\|_{0,p} + h\|w - w_h\|_{1,p} \} &\leq Ch^{r+1} \|w\|_{r+1,p}, \\ w &\in W^{r+1,p}(I), \end{aligned} \quad (5)$$

where $1 \leq p \leq \infty$. k, r are positive integers.

To define the fully discrete scheme we also need a time mesh grid. Let $0 = t^0 < t^1 < \dots < t^N = T$ be a given partition of the time interval $[0, T]$ with step length $\tau = T/N$, for some

positive integers N . Define $t^n = n\tau$ and $t^{n-(1/2)} = (n - (1/2))\tau$. For convenience we set $\phi^n = \phi(t^n)$ and $\phi^{n-(1/2)} = (\phi^n + \phi^{n-1})/2$ for a smooth function ϕ .

Let U^n and Q^n denote the discrete counterpart of u and σ at $t = t^n$ which satisfy the following Crank-Nicolson discrete scheme:

$$\begin{aligned} (U_x^n, v_{hx}) &= (Q^n, v_{hx}), \quad v_h \in V_h, \\ \left(\frac{Q^n - Q^{n-1}}{\tau}, w_h \right) &+ \left(\left(1 + \sum_{i=1}^n \delta_{n,i} \frac{\|Q^i\|^2 + \|Q^{i-1}\|^2}{2} \right) \frac{Q_x^n + Q_x^{n-1}}{2}, w_{hx} \right) \\ &+ (f^{n-(1/2)}, w_{hx}) = 0, \quad w_h \in W_h, \end{aligned} \quad (6)$$

where

$$\delta_{n,i} = \begin{cases} \tau, & i = 1, 2, \dots, n-1, \\ \frac{\tau}{2}, & i = n, \end{cases} \quad (8)$$

and U^0, Q^0 are to be defined later.

The existence and uniqueness of the discrete solution for the above problems can be guaranteed by the theory presented in [18, page 237-239].

To discretize the time integration we used the following integroformula:

$$\int_0^{t^{n-(1/2)}} g(\tau) ds \approx \sum_{i=1}^n \delta_{n,i} g^{i-(1/2)}. \quad (9)$$

Its truncation error can be estimated as follows:

$$\begin{aligned} \left| \int_0^{t^{n-(1/2)}} g(\tau) ds - \sum_{i=1}^n \delta_{n,i} g^{n-(1/2)} \right| \\ \leq C \left(\tau \int_{t^{n-1}}^{t^n} |g_t| + \tau^2 \int_0^{t^n} |g_{tt}| ds \right). \end{aligned} \quad (10)$$

3. Error Analysis

3.1. Preliminaries. We begin by recalling some preliminary knowledge that will be used in the following convergence analysis.

We define the following elliptic projections: $\tilde{u}_h(t) \in V_h$, $\tilde{\sigma}_h(t) \in W_h$, which satisfy

$$\begin{aligned} (u_x - \tilde{u}_{hx}, v_{hx}) &= 0, \quad \forall v_h \in V_h, \\ (\sigma_x - \tilde{\sigma}_{hx}, w_{hx}) + \alpha(\sigma - \tilde{\sigma}_h, w_h) &= 0, \quad \forall w_h \in W_h. \end{aligned} \quad (11)$$

Here α is chosen to guarantee the H^1 -coercivity of the bilinear form in the second equations. Moreover, it is easy to check that the bilinear form is bounded.

Let $\eta = u - \tilde{u}_h, \rho = \sigma - \tilde{\sigma}_h$; then η and ρ satisfy the following estimates from [19]:

$$\begin{aligned} \|\eta(t)\|_j + \|\eta_t(t)\|_j &\leq Ch^{k+1-j} (\|u\|_{k+1} + \|u_t\|_{k+1}), \quad j = 0, 1, \\ \|\rho(t)\|_j + \|\rho_t(t)\|_j &\leq Ch^{r+1-j} (\|q\|_{r+1} + \|q_t\|_{r+1}), \quad j = 0, 1. \end{aligned} \quad (12)$$

To derive the error estimates we also need the following discrete Gronwall inequality.

Lemma 1 (discrete Gronwall inequality; see [20]). *Let $\tau, B_1, C_1 > 0$ and let a_n, b_n, c_n and d_n be sequences of nonnegative numbers satisfying*

$$\forall n \geq 0, \quad a_n + \tau \sum_{i=0}^n b_i \leq B_1 + C_1 \tau \sum_{i=0}^n a_i + \tau \sum_{i=0}^n c_i. \quad (13)$$

Then, if $C_1 \tau < 1$,

$$\forall n \geq 0, \quad a_n + \tau \sum_{i=0}^n b_i \leq e^{C_1(n+1)\tau} \left(B_1 + \tau \sum_{i=0}^n c_i \right). \quad (14)$$

3.2. Error Analysis. To estimate the errors, we firstly decompose the errors into

$$\begin{aligned} u(t^n) - U^n &= u(t^n) - \tilde{u}_h(t^n) + \tilde{u}_h(t^n) - U^n = \eta^n + \zeta^n, \\ \sigma(t^n) - Q^n &= \sigma(t^n) - \tilde{\sigma}_h(t^n) + \tilde{\sigma}_h(t^n) - Q^n = \rho^n + \xi^n. \end{aligned} \quad (15)$$

Note that the estimates of η^n and ρ^n can be found out easily from (12) at $t = t^n$. Therefore it remains to estimate ζ^n and ξ^n .

Setting $t = t^{n-(1/2)}$ in (4) and combining (6) and (7) with auxiliary projections, we deduce the following error equations with respect to ζ^n and ξ^n :

$$\begin{aligned} (\zeta_x^n, v_{hx}) &= (\rho^n, v_{hx}) + (\xi^n, v_{hx}), \quad (16) \\ \left(\frac{\zeta^n - \zeta^{n-1}}{\tau}, w_h \right) &+ \left(\frac{\xi^n + \xi^{n-1}}{2}, w_{hx} \right) \\ &= \left(\frac{\sigma^n - \sigma^{n-1}}{\tau} - \sigma_t^{n-(1/2)}, w_h \right) - \left(\frac{\rho^n - \rho^{n-1}}{\tau}, w_h \right) \\ &+ \alpha \left(\frac{\rho^n + \rho^{n-1}}{2}, w_h \right) \\ &- \left(\lambda^{n-(1/2)} \sigma_x^{n-(1/2)} - \left(\sum_{i=1}^n \delta_{n,i} \frac{\|Q^i\|^2 + \|Q^{i-1}\|^2}{2} \right) \right. \\ &\quad \left. \times \frac{Q_x^n + Q_x^{n-1}}{2}, w_{hx} \right) \\ &+ \left(\frac{\sigma_x^n + \sigma_x^{n-1}}{2} - \sigma_x^{n-(1/2)}, w_{hx} \right). \end{aligned} \quad (17)$$

Theorem 2. *Suppose that $U^0 = \tilde{u}_h(0), Q^0 = \tilde{\sigma}_h(0)$, and $1 \leq J \leq N$. Then there exists a positive constant C independent of h and τ such that for sufficiently small τ*

$$\begin{aligned} \|u^J - U^J\| + h \|u^J - U^J\|_1 + \|\sigma^J - Q^J\| \\ \leq C (h^{\min\{k+1, r+1\}} + \tau^2). \end{aligned} \quad (18)$$

Here $k, r \geq 1$ are positive integers.

Proof. Choosing $v_h = \zeta^n$ in (16) yields

$$(\zeta_x^n, \zeta_x^n) = (\rho^n, \zeta_x^n) + (\xi^n, \zeta_x^n), \quad (19)$$

which implies

$$\|\zeta_x^n\| \leq \|\rho^n\| + \|\xi^n\|. \quad (20)$$

Setting $w_h = (\zeta^n + \zeta^{n-1})/2$ in (17) gives

$$\begin{aligned} \left(\frac{\zeta^n - \zeta^{n-1}}{\tau}, \frac{\zeta^n + \zeta^{n-1}}{2} \right) &+ \left(\frac{\xi^n + \xi^{n-1}}{2}, \frac{\xi^n + \xi^{n-1}}{2} \right) \\ &+ \left(\lambda^{n-(1/2)} \sigma_x^{n-(1/2)} - \left(\sum_{i=1}^n \delta_{n,i} \frac{\|Q^i\|^2 + \|Q^{i-1}\|^2}{2} \right) \right. \\ &\quad \left. \times \frac{Q_x^n + Q_x^{n-1}}{2}, \frac{\xi^n + \xi^{n-1}}{2} \right) \\ &= \left(\frac{\sigma^n - \sigma^{n-1}}{\tau} - \sigma_t^{n-(1/2)}, \frac{\xi^n + \xi^{n-1}}{2} \right) \\ &- \left(\frac{\rho^n - \rho^{n-1}}{\tau}, \frac{\xi^n + \xi^{n-1}}{2} \right) \\ &+ \alpha \left(\frac{\rho^n + \rho^{n-1}}{2}, \frac{\xi^n + \xi^{n-1}}{2} \right) \\ &+ \left(\frac{\sigma_x^n + \sigma_x^{n-1}}{2} - \sigma_x^{n-(1/2)}, \frac{\xi^n + \xi^{n-1}}{2} \right). \end{aligned} \quad (21)$$

For the third term on the left side we have

$$\begin{aligned} \left(\lambda^{n-(1/2)} \sigma_x^{n-(1/2)} - \left(\sum_{i=1}^n \delta_{n,i} \frac{\|Q^i\|^2 + \|Q^{i-1}\|^2}{2} \right) \right. \\ \left. \times \frac{Q_x^n + Q_x^{n-1}}{2}, \frac{\xi^n + \xi^{n-1}}{2} \right) \\ = \left(\left(\sum_{i=1}^n \delta_{n,i} \frac{\|Q^i\|^2 + \|Q^{i-1}\|^2}{2} \right) \frac{\xi^n + \xi^{n-1}}{2}, \frac{\xi^n + \xi^{n-1}}{2} \right) \end{aligned}$$

$$\begin{aligned}
& + \left(\left(\lambda^{n-(1/2)} - \left(\sum_{i=1}^n \delta_{n,i} \frac{\|Q^i\|^2 + \|Q^{i-1}\|^2}{2} \right) \right) \sigma_x^{n-(1/2)}, \right. \\
& \quad \left. \frac{\xi_x^n + \xi_x^{n-1}}{2} \right) \\
& + \left(\left(\sum_{i=1}^n \delta_{n,i} \frac{\|Q^i\|^2 + \|Q^{i-1}\|^2}{2} \right) \left(\sigma_x^{n-(1/2)} - \frac{\sigma_x^n + \sigma_x^{n-1}}{2} \right), \right. \\
& \quad \left. \frac{\xi_x^n + \xi_x^{n-1}}{2} \right) \\
& - \alpha \left(\left(\sum_{i=1}^n \delta_{n,i} \frac{\|Q^i\|^2 + \|Q^{i-1}\|^2}{2} \right) \frac{\rho^n + \rho^{n-1}}{2}, \frac{\xi^n + \xi^{n-1}}{2} \right). \tag{22}
\end{aligned}$$

Then we obtain

$$\begin{aligned}
& \frac{1}{2\tau} \left(\|\xi^n\|^2 - \|\xi^{n-1}\|^2 \right) + \left(1 + \sum_{i=1}^n \delta_{n,i} \frac{\|Q^i\|^2 + \|Q^{i-1}\|^2}{2} \right) \\
& \quad \times \left\| \frac{\xi_x^n + \xi_x^{n-1}}{2} \right\|^2 \\
& = - \left(\left(\lambda^{n-(1/2)} - \left(\sum_{i=1}^n \delta_{n,i} \frac{\|Q^i\|^2 + \|Q^{i-1}\|^2}{2} \right) \right) \sigma_x^{n-(1/2)}, \right. \\
& \quad \left. \frac{\xi_x^n + \xi_x^{n-1}}{2} \right) \\
& + \left(\left(\sum_{i=1}^n \delta_{n,i} \frac{\|Q^i\|^2 + \|Q^{i-1}\|^2}{2} \right) \left(\frac{\sigma_x^n + \sigma_x^{n-1}}{2} - \sigma_x^{n-(1/2)} \right), \right. \\
& \quad \left. \frac{\xi_x^n + \xi_x^{n-1}}{2} \right) \\
& + \left(\frac{\sigma^n - \sigma^{n-1}}{\tau} - \sigma_t^{n-(1/2)}, \frac{\xi^n + \xi^{n-1}}{2} \right) \\
& - \left(\frac{\rho^n - \rho^{n-1}}{\tau}, \frac{\xi^n + \xi^{n-1}}{2} \right)
\end{aligned}$$

$$\begin{aligned}
& + \alpha \left(\left(1 + \sum_{i=1}^n \delta_{n,i} \frac{\|Q^i\|^2 + \|Q^{i-1}\|^2}{2} \right) \frac{\rho^n + \rho^{n-1}}{2}, \right. \\
& \quad \left. \frac{\xi^n + \xi^{n-1}}{2} \right) \\
& + \left(\frac{\sigma_x^n + \sigma_x^{n-1}}{2} - \sigma_{hx}^{n-(1/2)}, \frac{\xi_x^n + \xi_x^{n-1}}{2} \right). \tag{23}
\end{aligned}$$

Now it remains to estimate the terms on the right side of (23). It is easy to prove that

$$\begin{aligned}
& \left\| \frac{\rho^n - \rho^{n-1}}{\tau} \right\|^2 \leq C \frac{1}{\tau} \int_{t^{n-1}}^{t^n} \|\rho_t\|^2 ds, \\
& \left\| \frac{\sigma^n - \sigma^{n-1}}{\tau} - \sigma_t^{n-(1/2)} \right\|^2 \leq C\tau^3 \int_{t^{n-1}}^{t^n} \|\sigma_{ttt}\|^2 ds, \tag{24} \\
& \left\| \frac{\sigma_x^n + \sigma_x^{n-1}}{2} - \sigma_x^{n-(1/2)} \right\|^2 \leq C\tau^3 \int_{t^{n-1}}^{t^n} \|\sigma_{xttt}\|^2 ds.
\end{aligned}$$

Here the Taylor formula with integral remainder was used.

Note that

$$\begin{aligned}
& \left| \lambda^{n-(1/2)} - \left(\sum_{i=1}^n \delta_{n,i} \frac{\|Q^i\|^2 + \|Q^{i-1}\|^2}{2} \right) \right| \\
& = \left| \int_0^1 \left(\int_0^{t^{n-(1/2)}} \sigma^2 ds - \sum_{i=1}^n \delta_{n,i} \frac{(Q^i)^2 + (Q^{i-1})^2}{2} \right) dx \right| \\
& \leq \left| \int_0^1 \int_0^{t^{n-(1/2)}} (\sigma^2 - \tilde{\sigma}_h^2) ds dx \right| \\
& + \left| \int_0^1 \left(\int_0^{t^{n-(1/2)}} \tilde{\sigma}_h^2 ds - \sum_{i=1}^n \delta_{n,i} \frac{(\tilde{\sigma}_h^i)^2 + (\tilde{\sigma}_h^{i-1})^2}{2} \right) dx \right| \\
& + \left| \int_0^1 \sum_{i=1}^n \delta_{n,i} \left(\frac{(\tilde{\sigma}_h^i)^2 + (\tilde{\sigma}_h^{i-1})^2}{2} - \frac{(Q^i)^2 + (Q^{i-1})^2}{2} \right) dx \right| \\
& := I_1 + I_2 + I_3. \tag{25}
\end{aligned}$$

By Hölder inequality and the bound of the projection $\tilde{\sigma}_h$ we have

$$\begin{aligned}
I_1 & = \left| \int_0^1 \int_0^{t^{n-(1/2)}} (\sigma^2 - \tilde{\sigma}_h^2) ds dx \right| \\
& \leq \|\sigma - \tilde{\sigma}_h\|_{L^2(0,t^n;L^2(I))} \left(\|\sigma\|_{L^2(0,t^n;L^2(I))} + \|\tilde{\sigma}_h\|_{L^2(0,t^n;L^2(I))} \right) \\
& \leq C\|\rho\|_{L^2(0,t^n;L^2(I))}. \tag{26}
\end{aligned}$$

Using the error formula (10) and the bound of the projection $\tilde{\sigma}_h$ we derive

$$\begin{aligned}
 I_2 &= \left| \int_0^1 \left(\int_0^{t^{n-(1/2)}} \tilde{\sigma}_h^2 ds - \sum_{i=1}^n \delta_{n,i} (\tilde{\sigma}_h^{i-(1/2)})^2 \right) dx \right| \\
 &\leq C \left| \left(\int_0^1 \tau \int_{t^{n-1}}^{t^n} \frac{d}{dt} (\tilde{\sigma}_h^2) ds + \tau^2 \int_0^{t^n} \frac{d^2}{dt^2} (\tilde{\sigma}_h^2) ds \right) dx \right| \\
 &= C \left| \int_0^1 \left(\tau \int_{t^{n-1}}^{t^n} 2\tilde{\sigma}_h \tilde{\sigma}_{ht} ds + \tau^2 \int_0^{t^n} (2\tilde{\sigma}_{ht}^2 + 2\tilde{\sigma}_{ht} \tilde{\sigma}_{htt}) ds \right) dx \right| \\
 &\leq C\tau \int_{t^{n-1}}^{t^n} \|\tilde{\sigma}_h\| \|\tilde{\sigma}_{ht}\| ds \\
 &\quad + C\tau^2 \left(\|\tilde{\sigma}_{ht}\|_{L^2(0,t^n;L^2(I))} \right. \\
 &\quad \left. + \|\tilde{\sigma}_h\|_{L^2(0,t^n;L^2(I))} \|\tilde{\sigma}_{htt}\|_{L^2(0,t^n;L^2(I))} \right) \\
 &\leq C \left(\tau^{3/2} \left(\int_{t^{n-1}}^{t^n} \|\tilde{\sigma}_h\|^2 \|\tilde{\sigma}_{ht}\|^2 ds \right)^{1/2} + \tau^2 \right). \tag{27}
 \end{aligned}$$

For I_3 we have

$$\begin{aligned}
 I_3 &= \frac{1}{2} \left| \int_0^1 \sum_{i=1}^n \delta_{n,i} \left((\tilde{\sigma}_h^i)^2 - (Q^i)^2 + (\tilde{\sigma}_h^{i-1})^2 - (Q^{i-1})^2 \right) dx \right| \\
 &= \frac{1}{2} \left| \int_0^1 \sum_{i=1}^n \delta_{n,i} \left((\tilde{\sigma}_h^i - Q^i) (\tilde{\sigma}_h^i + Q^i) \right. \right. \\
 &\quad \left. \left. + (\tilde{\sigma}_h^{i-1} - Q^{i-1}) (\tilde{\sigma}_h^{i-1} + Q^{i-1}) \right) dx \right| \\
 &\leq \frac{1}{2} \sum_{i=1}^n \delta_{n,i} \left(\|\xi^i\| (\|\tilde{\sigma}_h^i\| + \|Q^i\|) + \|\xi^{i-1}\| (\|\tilde{\sigma}_h^{i-1}\| + \|Q^{i-1}\|) \right). \tag{28}
 \end{aligned}$$

To bound I_3 we need to derive the boundness of Q^i . From (7) we can deduce

$$\begin{aligned}
 &\left(\left(1 + \sum_{i=1}^n \delta_{n,i} \frac{\|Q^i\|^2 + \|Q^{i-1}\|^2}{2} \right) \frac{Q_x^n + Q_x^{n-1}}{2}, \frac{Q_x^n + Q_x^{n-1}}{2} \right) \\
 &+ \left(\frac{Q^n - Q^{n-1}}{\tau}, \frac{Q^n + Q^{n-1}}{2} \right) + \left(f^{n-(1/2)}, \frac{Q_x^n + Q_x^{n-1}}{2} \right) = 0. \tag{29}
 \end{aligned}$$

By ε inequality we have

$$\begin{aligned}
 &\frac{1}{2\tau} (\|Q^n\|^2 - \|Q^{n-1}\|^2) + \left(\frac{1}{2} + \sum_{i=1}^n \delta_{n,i} \frac{\|Q^i\|^2 + \|Q^{i-1}\|^2}{2} \right) \\
 &\times \left\| \frac{Q_x^n + Q_x^{n-1}}{2} \right\|^2 \leq \|f^{n-(1/2)}\|^2. \tag{30}
 \end{aligned}$$

Then, multiplying by 2τ and summing from 1 to l we conclude

$$\|Q^l\|^2 \leq \tau \sum_{i=1}^l \|f^{n-(1/2)}\|^2 + \|Q^0\|^2, \tag{31}$$

which implies Q^l is bounded.

Combining (31) with the above estimate of I_3 and using the boundness of $\tilde{\sigma}_h$, we can get

$$I_3 \leq C\tau \sum_{i=1}^n \|\xi^i\|, \tag{32}$$

where $\xi^0 = 0$ was used.

Collecting the above estimates for $I_1 \sim I_3$ and using ε inequality, we obtain

$$\begin{aligned}
 &\left| \left(\left(\lambda^{n-(1/2)} - \left(\sum_{i=1}^n \delta_{n,i} \frac{\|Q^i\|^2 + \|Q^{i-1}\|^2}{2} \right) \right) \sigma_x^{n-(1/2)}, \right. \right. \\
 &\quad \left. \left. \frac{\xi_x^n + \xi_x^{n-1}}{2} \right) \right| \\
 &\leq C \left(\|\rho\|_{L^2(0,t^n;L^2(I))} + \tau^{3/2} \left(\int_{t^{n-1}}^{t^n} \|\tilde{\sigma}_h\|^2 \|\tilde{\sigma}_{ht}\|^2 ds \right)^{1/2} + \tau^2 \right) \\
 &\quad \times \left\| \frac{\xi_x^n + \xi_x^{n-1}}{2} \right\| + \tau \sum_{i=1}^n \|\xi^i\| \left\| \frac{\xi_x^n + \xi_x^{n-1}}{2} \right\| \\
 &\leq C \left(\|\rho\|_{L^2(0,t^n;L^2(I))}^2 + \tau^3 \int_{t^{n-1}}^{t^n} \|\tilde{\sigma}_h\|^2 \|\tilde{\sigma}_{ht}\|^2 ds + \tau^4 \right) \\
 &\quad + \tau \sum_{i=1}^n \|\xi^i\|^2 + \varepsilon \left\| \frac{\xi_x^n + \xi_x^{n-1}}{2} \right\|^2 \\
 &\leq C \left(\|\rho\|_{L^2(0,t^n;L^2(I))}^2 + \tau^3 \int_{t^{n-1}}^{t^n} \|\tilde{\sigma}_h\|^2 \|\tilde{\sigma}_{ht}\|^2 ds \right. \\
 &\quad \left. + \tau^4 + \tau \sum_{i=1}^n \|\xi^i\|^2 \right) \\
 &\quad + \varepsilon \left\| \frac{\xi_x^n + \xi_x^{n-1}}{2} \right\|^2, \\
 &\left| \left(\left(\sum_{i=1}^n \delta_{n,i} \frac{\|Q^i\|^2 + \|Q^{i-1}\|^2}{2} \right) \left(\frac{\sigma_x^n + \sigma_x^{n-1}}{2} - \sigma_x^{n-(1/2)} \right), \right. \right. \\
 &\quad \left. \left. \frac{\xi_x^n + \xi_x^{n-1}}{2} \right) \right| \\
 &\leq C \left\| \frac{\sigma_x^n + \sigma_x^{n-1}}{2} - \sigma_x^{n-(1/2)} \right\| \left\| \frac{\xi_x^n + \xi_x^{n-1}}{2} \right\|
 \end{aligned}$$

TABLE 1: The errors of $\|u^J - U^J\|$ at different time.

$h = \tau$	Time							
	$t = 0.2$		$t = 0.4$		$t = 0.6$		$t = 0.8$	
	Error	Order	Error	Order	Error	Order	Error	Order
1/20	$1.0174e - 004$	\	$1.1819e - 004$	\	$1.2690e - 004$	\	$1.2445e - 004$	\
1/40	$2.5472e - 005$	1.9979	$2.9553e - 005$	1.9997	$3.1723e - 005$	2.0001	$3.1112e - 005$	2.0000
1/80	$6.3711e - 006$	1.9993	$7.3900e - 006$	1.9997	$7.9329e - 006$	1.9996	$7.7813e - 006$	1.9994

TABLE 2: The errors of $\|u_x^J - U_x^J\|$ at different time.

$h = \tau$	Time							
	$t = 0.2$		$t = 0.4$		$t = 0.6$		$t = 0.8$	
	Error	Order	Error	Order	Error	Order	Error	Order
1/20	0.0020	\	0.0024	\	0.0025	\	0.0025	\
1/40	0.0010	1.0000	0.0012	1.0000	0.0013	0.9434	0.0012	1.0589
1/80	$5.0969e - 04$	0.9723	$5.9120e - 04$	1.0213	$6.3463e - 04$	1.0345	$6.2251e - 04$	0.9469

TABLE 3: The errors of $\|\sigma^J - Q^J\|$ at different time.

$h = \tau$	Time							
	$t = 0.2$		$t = 0.4$		$t = 0.6$		$t = 0.8$	
	Error	Order	Error	Order	Error	Order	Error	Order
1/20	$1.9532e - 004$	\	$3.5339e - 004$	\	$5.4745e - 004$	\	$7.6700e - 004$	\
1/40	$4.7504e - 005$	2.0397	$8.6496e - 005$	2.0306	$1.3441e - 004$	2.0261	$1.8884e - 004$	2.0221
1/80	$1.1699e - 005$	2.0217	$2.1381e - 005$	2.0163	$3.3284e - 005$	2.0137	$4.6818e - 005$	2.0120

$$\begin{aligned} &\leq C \left\| \frac{\sigma_x^n + \sigma_x^{n-1}}{2} - \sigma_x^{n-(1/2)} \right\|^2 + \varepsilon \left\| \frac{\xi_x^n + \xi_x^{n-1}}{2} \right\|^2 \\ &\leq C\tau^3 \int_{t^{n-1}}^{t^n} \|\sigma_{xttt}\|^2 ds + \varepsilon \left\| \frac{\xi_x^n + \xi_x^{n-1}}{2} \right\|^2. \end{aligned}$$

(33)

Inserting (24) and (33) into (23) leads to

$$\begin{aligned} &\frac{1}{2\tau} (\|\xi^n\|^2 - \|\xi^{n-1}\|^2) \\ &\leq C \left(\|\rho\|_{L^2(0,t^n;L^2(I))}^2 + \|\rho^n\|^2 + \|\rho^{n-1}\|^2 \right. \\ &\quad \left. + \tau^3 \int_{t^{n-1}}^{t^n} \|\tilde{\sigma}_h\|^2 ds + \tau^4 + \tau \sum_{i=1}^n \|\xi^i\|^2 \right) \\ &\quad + C \frac{1}{\tau} \int_{t^{n-1}}^{t^n} \|\rho_t\|^2 ds + C\tau^3 \int_{t^{n-1}}^{t^n} \|\sigma_{xttt}\|^2 ds \\ &\quad + C\tau^3 \int_{t^{n-1}}^{t^n} \|\sigma_{ttt}\|^2 ds \\ &\quad + C \left\| \frac{\xi^n + \xi^{n-1}}{2} \right\|^2. \end{aligned}$$

(34)

Multiplying by 2τ and summing from 1 to J leads to

$$\begin{aligned} \|\xi^J\|^2 &\leq C\|\rho\|_{L^\infty(0,t^J;L^2(I))}^2 + C\tau^4 \int_0^{t^J} \|\tilde{\sigma}_h\|^2 ds + C\tau^4 \\ &\quad + C \int_0^{t^J} \|\rho_t\|^2 ds + C\tau^4 \int_0^{t^J} \|\sigma_{xttt}\|^2 ds \\ &\quad + C\tau^4 \int_0^{t^J} \|\sigma_{ttt}\|^2 ds \\ &\quad + C\tau^2 \sum_{n=1}^J \sum_{i=1}^n \|\xi^i\|^2 + C\tau \sum_{n=1}^J \left\| \frac{\xi^n + \xi^{n-1}}{2} \right\|^2 \\ &\leq C\|\rho\|_{L^\infty(0,t^J;L^2(I))}^2 + C \int_0^{t^J} \|\rho_t\|^2 ds + C\tau \sum_{n=1}^J \|\xi^n\|^2 \quad (35) \\ &\quad + C\tau^4 \left(\int_0^{t^J} \|\tilde{\sigma}_h\|^2 ds + \int_0^{t^J} \|\sigma_{xttt}\|^2 ds \right. \\ &\quad \left. + \int_0^{t^J} \|\sigma_{ttt}\|^2 ds + 1 \right) \\ &\leq C\|\rho\|_{L^\infty(0,t^J;L^2(I))}^2 + C \int_0^{t^J} \|\rho_t\|^2 ds + C\tau \sum_{n=1}^J \|\xi^n\|^2 \\ &\quad + C\tau^4. \end{aligned}$$

Taking τ_1 , let $0 < \tau \leq \tau_1$, such that $1 - C\tau > 0$; then by discrete Gronwall's lemma we obtain that

$$\|\xi^J\|^2 \leq C\|\rho\|_{L^\infty(0,t^J;L^2(I))}^2 + C \int_0^{t^J} \|\rho_t\|^2 ds + C\tau^4. \quad (36)$$

Substituting (36) into (20) yields

$$\|\zeta_x^J\|^2 \leq C\|\rho\|_{L^\infty(0,t^J;L^2(I))}^2 + C \int_0^{t^J} \|\rho_t\|^2 ds + C\tau^4 \quad (37)$$

Combining (36), (37), and the estimates of η^n , ρ^n and using the triangle inequality, we can complete the proof. \square

4. Numerical Example

In this section a numerical example is carried out to verify the theorems presented in this paper.

Example 1. Let us consider the initial and boundary problem (1) with the initial value $u(x, 0) = x(1 - x) \sin(x)$ and the right hand term

$$\begin{aligned} f = & x(1 - x) \cos(x + t) \\ & - \left(1 + \frac{11}{60}t - \frac{1}{8} \cos(t) \sin(t) \right. \\ & \left. - \frac{1}{8} \cos(t + 1) \sin(t + 1) + \frac{1}{8} \cos(1) \sin(1) \right) \\ & \times (-2 \sin(x + t) + 2(1 - x) \cos(x + t) \\ & - 2x \cos(x + t) - x(1 - x) \sin(x + t)). \end{aligned} \quad (38)$$

This example is taken from [11].

In this example we use piecewise linear finite element spaces to approximate the unknown functions u and σ , respectively. The Crank-Nicolson method is used to approximate the time derivative. Then the corresponding error estimates reduce to

$$\|u^J - U^J\| + h\|u^J - U^J\|_1 + \|\sigma^J - Q^J\| \leq C(h^2 + \tau^2). \quad (39)$$

In the numerical implementation we choose $h = \tau$. The errors and the corresponding rate of convergence for $u^J - U^J$ and $\sigma^J - Q^J$ are displayed in Tables 1, 2, and 3, respectively.

We can observe that the numerical results are in agreement with our theoretical results proposed in Section 3.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

References

[1] D. G. Gordeziani, T. A. Dzhangveladze, and T. K. Korshiya, "Existence and uniqueness of the solution of a class of nonlinear parabolic problems," *Differentsial'nye Uravneniya*, vol. 19, no. 7, pp. 1197–1207, 1983 (Russian).

[2] D. G. Gordeziani, T. A. Dzhangveladze, and T. K. Korshiya, "Existence and uniqueness of the solution of a class of nonlinear parabolic problems," *International Journal of Difference Equations*, vol. 19, no. 7, pp. 887–895, 1983.

[3] T. A. Dzhangveladze, "The first boundary value problem for a nonlinear equation of parabolic type," *Doklady Akademii Nauk SSSR*, vol. 269, no. 4, pp. 839–842, 1983 (Russian).

[4] T. A. Dzhangveladze, "The first boundary value problem for a nonlinear equation of parabolic type," *Soviet Physics—Doklady*, vol. 28, no. 4, pp. 323–324, 1983.

[5] T. Jangveladze, "On one class of nonlinear integro-differential parabolic equations," *Seminar of I. Vekua Institute of Applied Mathematics. Reports*, vol. 23, pp. 51–87, 1997.

[6] T. A. Dzhangveladze, "Convergence of a difference scheme for a system of nonlinear partial differential equations," *Soobshcheniya Akademii Nauk Gruzinskoi SSR*, vol. 126, no. 2, pp. 257–260, 1987 (Russian).

[7] T. Dzhangveladze, "The difference scheme for one system of nonlinear partial differential equations," *Seminar of I. Vekua Institute of Applied Mathematics. Reports*, vol. 2, pp. 40–43, 1986 (Russian).

[8] T. Jangveladze, Z. Kiguradze, and B. Neta, "Large time behavior of solutions and finite difference scheme to a nonlinear integro-differential equation," *Computers & Mathematics with Applications*, vol. 57, no. 5, pp. 799–811, 2009.

[9] T. Jangveladze, "Convergence of a difference scheme for a nonlinear integro-differential equation," *Proceedings of I. Vekua Institute of Applied Mathematics*, vol. 48, pp. 38–43, 1998.

[10] T. Jangveladze, Z. Kiguradze, and B. Neta, "Finite difference approximation of a nonlinear integro-differential system," *Applied Mathematics and Computation*, vol. 215, no. 2, pp. 615–628, 2009.

[11] T. Jangveladze, Z. Kiguradze, and B. Neta, "Galerkin finite element method for one nonlinear integro-differential model," *Applied Mathematics and Computation*, vol. 217, no. 16, pp. 6883–6892, 2011.

[12] Z. Zhou, F. Chen, and H. Chen, "Convergence analysis for H^1 -Galerkin mixed finite element approximation of one nonlinear integro-differential model," *Applied Mathematics and Computation*, vol. 220, pp. 783–791, 2013.

[13] A. K. Pani, "An H^1 -Galerkin mixed finite element method for parabolic partial differential equations," *SIAM Journal on Numerical Analysis*, vol. 22, no. 2, pp. 231–252, 2002.

[14] A. K. Pani and G. Fairweather, "An H^1 -Galerkin mixed finite element methods for parabolic partial integro-differential equations," *IMA Journal on Numerical Analysis*, vol. 25, no. 2, pp. 712–727, 1998.

[15] A. K. Pani, R. K. Sinha, and A. K. Ota, "An H^1 -Galerkin mixed method for second order hyperbolic equations," *International Journal of Numerical Analysis and Modeling*, vol. 1, no. 2, pp. 111–130, 2004.

[16] H.-Z. Chen and H. Wang, "An optimal-order error estimate on an H^1 -Galerkin mixed method for a nonlinear parabolic equation in porous medium flow," *Numerical Methods for Partial Differential Equations*, vol. 26, no. 1, pp. 188–205, 2010.

[17] L. Guo and H. Chen, " H^1 -Galerkin mixed finite element method for the regularized long wave equation," *Computing. Archives for Scientific Computing*, vol. 77, no. 2, pp. 205–221, 2006.

[18] V. Thomée, *Galerkin finite element methods for parabolic problems*, vol. 25, Springer, Berlin, Germany, 2nd edition, 2006.

- [19] M. F. Wheeler, "A priori L^2 error estimates for Galerkin approximations to parabolic partial differential equations," *SIAM Journal on Numerical Analysis*, vol. 10, pp. 723–759, 1973.
- [20] J. G. Heywood and R. Rannacher, "Finite-element approximation of the nonstationary navier-stokes problem—part IV. Error analysis for second-order time discretization," *SIAM Journal on Numerical Analysis*, vol. 27, no. 2, pp. 353–384, 1990.