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Research Article

Symmetry Reductions of (2 + 1)-Dimensional CDGKS Equation and Its Reduced Lax Pairs

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With the aid of symbolic computation, we obtain the symmetry transformations of the (2 + 1)-dimensional Caudrey-Dodd-Gibbon-Kotera-Sawada (CDGKS) equation by Lou's direct method which is based on Lax pairs. Moreover, we use the classical Lie group method to seek the symmetry groups of both the CDGKS equation and its Lax pair and then reduce them by the obtained symmetries. In particular, we consider the reductions of the Lax pair completely. As a result, three reduced (1 + 1)-dimensional equations with their new Lax pairs are presented and some group-invariant solutions of the equation are given.

1. Introduction

In modern mathematics with ramifications of several fields of mathematics, physics, and other sciences, it is getting more and more popular to study the symmetry analysis of differential equations, especially high-dimensional ones, such as finding symmetries, symmetry groups of transformation, symmetry reductions, and construction group invariant solutions.

Nowadays, there are three basic methods for finding symmetry reductions of the given nonlinear systems [1], namely, the classical Lie group method [2, 3], the nonclassical Lie group method [4], and the Clarkson and Kruskal's direct method [5]. Then Lou improved the direct method [6], which was based on Lax pairs. With the classical Lie group method, Zhi [7, 8] studied symmetry reductions of the Lax pair for the (2+1)-dimensional Konopelchenko-Dubrovsky equation and found that the reduced Lax pairs do not always lead to the reduced KD equations.

In [9], the first two ZS-AKNS members, the coupled nonlinear Schrödinger, and the mKdV equations yield special solutions to the KP equation. This means the assembling of (1 + 1)-dimensions into (2 + 1)-dimensions. The technique is applied to the KdV hierarchy. The assembling of the first two KdV equations leads to the (2 + 1)-dimensional CDGKS equation:

$$\begin{aligned} w_t = & -\frac{1}{9}\left(w_{xxxx} + 5u_{xx} + 5ww_{xx} + \frac{5}{3}w^3\right)_x \\ & -\frac{5}{9}\left(wu_x + w_x\partial_x^{-1}w_y - \partial_x^{-1}w_{yy}\right), \end{aligned} \tag{1}$$

which is a higher-order generalization of the celebrated Korteweg-de Vries (KdV) equation. Equation (1) was first introduced in [10], and its (1+1)-dimensional version was studied by Sawada and Kotera [11] and Caudrey et al. [12]. The (1+1)-dimensional CDGKS equation is not a member of the Lax hierarchy of the Korteweg-de Vries equation and has some distinct properties, as reported in [13]. In [14], the algebraic-geometric solutions to (1) were obtained. The Lax pair of linear equations of the (2 + 1)-dimensional CDGKS equation (1) is as follows:

$$\begin{aligned} \Phi_y &= -\Phi_{xxx} - w\Phi_x, \\ \Phi_t &= \Phi_{xxxxx} + \frac{5}{3}\left(w\Phi_{xxx} + 15w_x\Phi_{xx}\right) \\ &\quad + \frac{5}{9}\left(2w_{xx} + w^2 - \partial_x^{-1}w_y\right)\Phi_x. \end{aligned} \tag{2}$$

The plan of the present paper is as follows. Section 2 presents the symmetry transformations of the CDGKS equation by means of its Lax pair with Lou's direct method. Section 3 gives the symmetry reductions of the CDGKS equation and its Lax pair, based on the symmetries obtained by the classical Lie group method. A short summary is in Section 4.

2. Symmetry Transformations by the Direct Method

In this section, we will seek the symmetry transformations of the CDGKS equation and will determine the Lie group of transformations of (1) with the direct method based on the Lax pair due to Lou.

By a transformation, (1) is equivalent to the following system:

$$\begin{aligned} w_t &= -\frac{1}{9}\left(w_{xxxx} + 5u_{xx} + 5ww_{xx} + \frac{5}{3}w^3\right)_x \\ &\quad - \frac{5}{9}\left(wu_x + w_x u - u_y\right), \\ w_y &= u_x, \end{aligned} \tag{3}$$

which possesses the Lax pair

$$\begin{aligned} \Phi_y &= -\Phi_{xxx} - w\Phi_x, \\ \Phi_t &= \Phi_{xxxxx} + \frac{5}{3}\left(w\Phi_{xxx} + 15w_x\Phi_{xx}\right) \\ &\quad + \frac{5}{9}\left(2w_{xx} + w^2 - u\right)\Phi_x. \end{aligned} \tag{4}$$

Let

$$\Phi = G\phi\left(\xi,\eta,\tau\right), \tag{5}$$

where G, ξ, η, and τ are functions of (x,y,t) and $\phi(\xi,\eta,\tau)$ has the same equations as (4). Consider

$$\begin{aligned} \phi_\eta &= -\phi_{\xi\xi\xi} - \widetilde{w}\left(\xi,\eta,\tau\right)\phi_\xi, \\ \phi_\tau &= \phi_{\xi\xi\xi\xi\xi} + \frac{5}{3}\left(\widetilde{w}\left(\xi,\eta,\tau\right)\phi_{\xi\xi\xi} + \widetilde{w}_\xi\left(\xi,\eta,\tau\right)\phi_{\xi\xi}\right) \\ &\quad + \frac{5}{9}\left(2\widetilde{w}_{\xi\xi}\left(\xi,\eta,\tau\right) + \widetilde{w}^2\left(\xi,\eta,\tau\right) - \widetilde{u}\left(\xi,\eta,\tau\right)\right)\phi_\xi. \end{aligned} \tag{6}$$

Substitution of (5) and (6) into (4) leads to a system of differential equations. Comparing the different derivatives of ϕ, we get the restricted equations of G, ξ, η, and τ. Solving these equations, we obtain

$$\begin{aligned} \xi &= \tau_t^{1/5}x + \frac{9}{50}\tau_{tt}\tau_t^{-4/5}y^2 + \frac{3}{5}\dot{\eta}_1\left(t\right)\tau_t^{-2/5}y + \xi_1\left(t\right), \\ \eta &= \tau_t^{3/5}y + \eta_1\left(t\right), \quad \tau = \tau\left(t\right), \quad G = \text{constant}, \end{aligned} \tag{7}$$

and the relations of w, u and $\widetilde{w}(\xi,\eta,\tau)$, $\widetilde{u}(\xi,\eta,\tau)$ are as follows:

$$\begin{aligned} w &= \tau_t^{2/5}\widetilde{w}\left(\xi,\eta,\tau\right) - \frac{9}{25}\tau_{tt}\tau_t^{-1}y - \frac{3}{5}\dot{\eta}_1\left(t\right)\tau_t^{-3/5}, \\ u &= \tau_t^{4/5}\widetilde{u}\left(\xi,\eta,\tau\right) + \left(\frac{9}{25}\tau_{tt}\tau_t^{-1}y + \frac{3}{5}\dot{\eta}_1\left(t\right)\tau_t^{-3/5}y\right)w \\ &\quad - \frac{9}{25}\tau_{tt}\tau_t^{-1}x - \left(\frac{81}{250}\tau_{ttt}\tau_t^{-1} - \frac{324}{625}\tau_{tt}^2\tau_t^{-2}\right)y^2 \\ &\quad - \left(\frac{27}{25}\ddot{\eta}_1\left(t\right)\tau_t^{-3/5} - \frac{162}{125}\tau_{tt}\dot{\eta}_1\tau_t^{-8/5}\right)y \\ &\quad - \frac{9}{5}\dot{\xi}_1\left(t\right)\tau_t^{-1/5} + \frac{18}{25}\dot{\eta}_1^2\left(t\right)\tau_t^{-6/5}, \end{aligned} \tag{8}$$

where ξ_1, η_1, and τ are arbitrary functions of t. In this paper, the dots denote differentiation with respect to t.

From the above results one can get the following symmetry group theorem for the CDGKS equation.

Theorem 1. *If $w = w(x,y,t)$, $u = u(x,y,t)$ are a solution of the CDGKS equation, then so is $(\widetilde{w},\widetilde{u})$, given by* (8) *and* (7).

From Theorem 1, let $\xi_1 = \epsilon f(t)$, $\eta_1 = \epsilon g(t)$, and $\tau = t + \epsilon h(t)$ in (8), with infinitesimal parameter ϵ; we can obtain the Lie point symmetry structure again, $\widetilde{w} = w + \epsilon\sigma(w)$, $\widetilde{u} = u + \epsilon\sigma(u)$. Furthermore, we have

$$\begin{aligned} \sigma\left(w\right) &= -\frac{2}{5}\dot{h}w + \frac{9}{25}\ddot{h}y + \frac{3}{5}\dot{g}, \\ \sigma\left(u\right) &= \frac{9}{5}\dot{f} + \frac{9}{25}\ddot{h}x + \frac{81}{250}\dddot{h}y^2 \\ &\quad + \frac{27}{25}\ddot{g}y - \left(\frac{9}{25}\ddot{h}y + \frac{3}{5}\dot{g}\right)w - \frac{4}{5}\dot{h}u. \end{aligned} \tag{9}$$

The equivalent vector expression of the symmetries can be expressed as

$$X = X_1\left(f\right) + X_2\left(g\right) + X_3\left(h\right), \tag{10}$$

where

$$\begin{aligned} X_1\left(f\right) &= f\frac{\partial}{\partial x} + \frac{9}{5}\dot{f}\frac{\partial}{\partial u}, \\ X_2\left(g\right) &= \frac{3}{5}\dot{g}y\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + \frac{3}{5}\dot{g}\frac{\partial}{\partial w} \\ &\quad + \left(\frac{27}{25}\ddot{g}y - \frac{3}{5}\dot{g}w\right)\frac{\partial}{\partial u}, \\ X_3\left(h\right) &= \left(\frac{1}{5}\dot{h}x + \frac{9}{50}\ddot{h}y^2\right)\frac{\partial}{\partial x} + \frac{3}{5}\dot{h}y\frac{\partial}{\partial y} \\ &\quad + h\frac{\partial}{\partial t} + \left(\frac{9}{25}\ddot{h}y - \frac{2}{5}\dot{h}w\right)\frac{\partial}{\partial w} \\ &\quad + \left(\frac{9}{25}\ddot{h}x + \frac{81}{250}\dddot{h}y^2 - \frac{9}{25}\ddot{h}yw - \frac{4}{5}\dot{h}u\right)\frac{\partial}{\partial u}. \end{aligned} \tag{11}$$

The associated Lie algebras between any two vector fields become

$$\begin{aligned} \left[X_1\left(f_1\right), X_3\left(f_2\right)\right] &= X_1\left(\frac{1}{5}f_1\dot{f}_2 - \dot{f}_1 f_2\right), \\ \left[X_2\left(f_1\right), X_2\left(f_2\right)\right] &= \frac{3}{5}X_1\left(f_1\dot{f}_2 - \dot{f}_1 f_2\right), \end{aligned}$$

$$[X_2(f_1), X_3(f_2)] = X_2\left(\frac{3}{5}f_1\dot{f}_2 - \dot{f}_1 f_2\right),$$
$$[X_3(f_1), X_3(f_2)] = X_3\left(f_1\dot{f}_2 - \dot{f}_1 f_2\right),$$
$$[X_1(f_1), X_2(f_2)] = [X_1(f_1), X_1(f_2)] = 0. \quad (12)$$

It is easy to show that $\{X_1, X_2, X_3\}$ constructs a Kac-Moody algebra.

3. Symmetry Reductions of the CDGKS Equation and Its Reduced Lax Pairs

In this section, we will use the classical Lie group method to seek some symmetries of the CDGKS equation and its Lax pair. The Lie point symmetry algebra admitted by its corresponding Lax pair (4) is

$$X = \xi_1\frac{\partial}{\partial x} + \xi_2\frac{\partial}{\partial y} + \xi_3\frac{\partial}{\partial t} + \phi_1\frac{\partial}{\partial q} + \phi_2\frac{\partial}{\partial u} + \phi_3\frac{\partial}{\partial \Phi}, \quad (13)$$

where ξ_1, ξ_2, ξ_3, ϕ_1, ϕ_2, and ϕ_3 are functions of x, y, t, w, u, and Φ.

With the aid of Maple, we obtain the following infinitesimals:

$$\xi_1 = 9\ddot{f}y^2 + 15\dot{g}y + 10\dot{f}x + 5h, \qquad \xi_2 = 30\dot{f}y + 25g,$$
$$\xi_3 = 50f, \qquad \phi_1 = 18\ddot{f}y + 15\dot{g} - 20\dot{f}w,$$
$$\phi_2 = 18\ddot{f}x - 3\left(6\ddot{f}y + 5\dot{g}\right)w + \frac{81}{5}\dddot{f}y^2$$
$$+ 27\ddot{g}y - 40\dot{f}u + 9\dot{h},$$
$$\phi_3 = C_1\Phi + C_2, \quad (14)$$

where C_1, C_2 are arbitrary constants and f, g, and h are arbitrary functions of t. Also, we can obtain the Lie point symmetry algebra admitted by (3), and we find that the CDGKS equation and its Lax pair admit the same symmetry transformations of the independent variables except the eigenfunction ϕ_3.

After determining the infinitesimals (14), the symmetry variables are found by solving the corresponding characteristic equations:

$$\frac{dx}{\xi_1} = \frac{dy}{\xi_2} = \frac{dt}{\xi_3} = \frac{dw}{\phi_1} = \frac{du}{\phi_2} = \frac{d\Phi}{\phi_3}. \quad (15)$$

While solving the above characteristic equation one has to distinguish between the cases in which some of the functions f, g, h and the constants C_1, C_2 are identical to zero and cases where they are not. This leads to different relations between the similarity variables $(\tilde{x}, \tilde{y}, P, Q, \Psi)$ and the original variables (x, y, t, w, u, Φ). As a result we obtain the following cases.

Case 1 ($f(t) \neq 0$). Integrating (15), we get the following similarity variables:

$$\tilde{x} = f^{-1/5}x - \frac{9}{50}\dot{f}f^{-6/5}y^2 - \frac{3}{10}gf^{-6/5}y - \frac{1}{10}\int hf^{-6/5}dt + \frac{3}{20}\int g^2 f^{-11/5}dt,$$
$$\tilde{y} = f^{-3/5}y - \frac{1}{2}\int gf^{-8/5}dt,$$
$$w = f^{-2/5}P + \frac{9}{25}\dot{f}f^{-1}y + \frac{3}{10}gf^{-1},$$
$$u = f^{-4/5}Q - \frac{54}{125}\dot{f}f^{-2}gy - \frac{162}{625}\dot{f}^2 f^{-2}y^2 - \frac{9}{25}\dot{f}f^{-7/5}yP - \frac{9}{50}f^{-2}g^2 - \frac{3}{10}f^{-7/5}gP + \frac{9}{25}\dot{f}f^{-1}x + \frac{81}{250}\ddot{f}f^{-1}y^2 + \frac{27}{50}f^{-1}\dot{g}y + \frac{9}{50}f^{-1}h, \quad (16)$$

where P and Q are symmetry reduction fields with respect to the group invariants $\tilde{x}, \tilde{y}$ and f, g, and h are arbitrary functions of t. The reduced equation (3) turns out to be

$$\frac{1}{9}\left(P_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + 5Q_{\tilde{x}\tilde{x}} + 5PP_{\tilde{x}\tilde{x}} + \frac{5}{3}P^3\right)_{\tilde{x}} + \frac{5}{9}\left(PQ_{\tilde{x}} + P_{\tilde{x}}Q - Q_{\tilde{y}}\right) = 0,$$
$$P_{\tilde{y}} = Q_{\tilde{x}}. \quad (17)$$

In the following reductions, we find that C_1, C_2 play the role of spectral parameter in the reduced Lax pair. In this case, the following three cases should be considered; namely, (i) $C_1 \neq 0$, (ii) $C_1 = 0$, $C_2 \neq 0$, and (iii) $C_1 = C_2 = 0$.

(i) One has $C_1 \neq 0$.

From (15), we can obtain the eigenfunction

$$\Phi = -\frac{C_2}{C_1} + e^{\int (C_1/50f(t))dt}\Psi(\tilde{x}, \tilde{y}). \quad (18)$$

Substituting (16) and (18) into (4), we obtain the first type of the reduced Lax pair

$$\Psi_{\tilde{y}} + \Psi_{\tilde{x}\tilde{x}\tilde{x}} + P\Psi_{\tilde{x}} = 0,$$
$$450\Psi_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + 750P\Psi_{\tilde{x}\tilde{x}\tilde{x}} + 750P_{\tilde{x}}\Psi_{\tilde{x}\tilde{x}} + 500P_{\tilde{x}\tilde{x}}\Psi_{\tilde{x}} + 250P^2\Psi_{\tilde{x}} - 250Q\Psi_{\tilde{x}} - 9C_1\Psi = 0. \quad (19)$$

By direct computation, from (19) we can obtain

$$\frac{5}{3}\left(Q_{\tilde{x}} - P_{\tilde{y}}\right)\Psi_{\tilde{x}\tilde{x}\tilde{x}} + \frac{5}{3}\left(Q_{\tilde{x}\tilde{x}} - P_{\tilde{y}\tilde{x}}\right)\Psi_{\tilde{x}\tilde{x}} + \frac{10}{9}\left(Q_{\tilde{x}\tilde{x}\tilde{x}} - P_{\tilde{y}\tilde{x}\tilde{x}}\right)\Psi_{\tilde{x}} + \frac{10}{9}\left(Q_{\tilde{x}} - P_{\tilde{y}}\right)P\Psi_{\tilde{x}}$$

$$-\left[\frac{1}{9}\left(P_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{x}}+5Q_{\tilde{x}\tilde{x}}+5PP_{\tilde{x}\tilde{x}}+\frac{5}{3}P^3\right)_{\tilde{x}}\right.$$
$$\left.+\frac{5}{9}\left(PQ_{\tilde{x}}+P_{\tilde{x}}Q-Q_{\tilde{y}}\right)\right]\Psi_{\tilde{x}}=0. \tag{20}$$

It is easy to check that the reduced (1 + 1)-dimensional equation (17) is the compatibility condition of the reduced Lax pair (19).

(ii) One has $C_1 = 0$, $C_2 \neq 0$.

The eigenfunction is

$$\Phi=\int\frac{C_2}{50f(t)}dt+\Psi(\tilde{x},\tilde{y}). \tag{21}$$

We obtain the second type of the reduced Lax pair

$$\Psi_{\tilde{y}}+\Psi_{\tilde{x}\tilde{x}\tilde{x}}+P\Psi_{\tilde{x}}=0,$$
$$450\Psi_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{x}}+750P\Psi_{\tilde{x}\tilde{x}\tilde{x}}+750P_{\tilde{x}}\Psi_{\tilde{x}\tilde{x}}+500P_{\tilde{x}\tilde{x}}\Psi_{\tilde{x}} \tag{22}$$
$$+250P^2\Psi_{\tilde{x}}-250Q\Psi_{\tilde{x}}-9C_2=0.$$

Similarly, the reduced equation (17) is the compatibility condition of the reduced Lax pair (22).

(iii) One has $C_1 = C_2 = 0$.

The eigenfunction is

$$\Phi=\Psi(\tilde{x},\tilde{y}). \tag{23}$$

We obtain the third type of the reduced Lax pair

$$\Psi_{\tilde{y}}+\Psi_{\tilde{x}\tilde{x}\tilde{x}}+P\Psi_{\tilde{x}}=0,$$
$$9\Psi_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{x}}+15P\Psi_{\tilde{x}\tilde{x}\tilde{x}}+15P_{\tilde{x}}\Psi_{\tilde{x}\tilde{x}}+10P_{\tilde{x}\tilde{x}}\Psi_{\tilde{x}} \tag{24}$$
$$+5P^2\Psi_{\tilde{x}}-5Q\Psi_{\tilde{x}}=0.$$

Equation (17) is the compatibility condition of the Lax pair (24).

The reduction equation (17) is much simpler than the original equation (3). It is easy to obtain the solutions of (17). Consider

$$P(\tilde{x},\tilde{y})=C_4-6C_2^2\tanh^2(C_1+C_2\tilde{x}+C_3\tilde{y}),$$
$$Q(\tilde{x},\tilde{y})=-6C_2C_3\tanh^2(C_1+C_2\tilde{x}+C_3\tilde{y})$$
$$-C_2^{-1}C_3C_4-C_4^2+8C_2^2C_4 \tag{25}$$
$$-\frac{76}{5}C_2^4-8C_2C_3+C_2^{-2}C_3^2,$$

where C_1, C_2, C_3, and C_4 are arbitrary constants.

According to (16), we can get the group-invariant solutions of (3). Consider

$$q=f^{-2/5}\left(C_4-6C_2^2\tanh^2(Z)\right)+\frac{9}{25}\dot{f}f^{-1}y+\frac{3}{10}gf^{-1},$$
$$u=f^{-4/5}\left(C_4-6C_2^2\tanh^2(Z)\right)-\frac{54}{125}\dot{f}f^{-2}gy$$
$$-\frac{162}{625}\dot{f}^2f^{-2}y^2-\frac{9}{25}\dot{f}f^{-7/5}yP-\frac{9}{50}f^{-2}g^2$$
$$-\frac{3}{10}f^{-7/5}g\Big(-6C_2C_3\tanh^2(Z)-C_2^{-1}C_3C_4-C_4^2$$
$$+8C_2^2C_4-\frac{76}{5}C_2^4-8C_2C_3+C_2^{-2}C_3^2\Big)$$
$$+\frac{9}{25}\dot{f}f^{-1}x+\frac{81}{250}\ddot{f}f^{-1}y^2+\frac{27}{50}f^{-1}\dot{g}y+\frac{9}{50}f^{-1}h, \tag{26}$$

with

$$Z=C_1+C_2\Big(f^{-1/5}x-\frac{9}{50}\dot{f}f^{-6/5}y^2-\frac{3}{10}gf^{-6/5}y$$
$$-\frac{1}{10}\int hf^{-6/5}dt+\frac{3}{20}\int g^2f^{-11/5}dt\Big) \tag{27}$$
$$+C_3\left(f^{-3/5}y-\frac{1}{2}\int gf^{-8/5}dt\right),$$

and f, g, and h are arbitrary functions of t.

Case 2 ($f(t) = 0$, $g(t) \neq 0$). In this case, integrating (15) with $f = 0$ leads to the following similarity variables:

$$\tilde{x}=5gx-\frac{3}{2}\dot{g}y^2-hy,\qquad \tilde{y}=t,$$
$$w=\frac{3\dot{g}}{5g}y+P, \tag{28}$$
$$u=\frac{27\ddot{g}g-9\dot{g}^2}{50g^2}y^2+\frac{9\dot{h}-15\dot{g}P}{25g}y+Q,$$

where P and Q are the similarity reduction fields with respect to $\tilde{x}$ and $\tilde{y}$. Substituting (28) into (3) yields the second type of similarity reductions:

$$9gP_{\tilde{y}}+3125g^6P_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{x}}+\left(625g^4F-125g^3h\right)P_{\tilde{x}\tilde{x}\tilde{x}}$$
$$+625g^4P_{\tilde{x}}P_{\tilde{x}\tilde{x}}+6\dot{g}P-\frac{9}{5}\dot{h}+\frac{3\dot{g}h}{5g}$$
$$+\left(9\tilde{x}\dot{g}+25g^2P^2-5ghP+25g^2Q-h^2\right)P_{\tilde{x}}=0, \tag{29}$$
$$-3\dot{g}+5ghP_{\tilde{x}}+25g^2Q_{\tilde{x}}=0,$$

with $g = g(\tilde{y})$, $h = h(\tilde{y})$.

(i) One has $C_1 \neq 0$.

We can obtain the eigenfunction

$$\Phi = -\frac{C_2}{C_1} + e^{C_1 y/25g(t)}\Psi(\tilde{x}, \tilde{y}). \quad (30)$$

Substituting (28) and (30) into (4), in this case we obtain the first type of the reduced Lax pair:

$$\begin{aligned} &3125g^4\Psi_{\tilde{x}\tilde{x}\tilde{x}} + 125g^2P\Psi_{\tilde{x}} - 25gh\Psi_{\tilde{x}} + C_1\Psi = 0, \\ &28125g^6\Psi_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + 1875g^4P\Psi_{\tilde{x}\tilde{x}\tilde{x}} \\ &\quad + 1875g^4P_{\tilde{x}}\Psi_{\tilde{x}\tilde{x}} + 1250g^4P_{\tilde{x}\tilde{x}}\Psi_{\tilde{x}} \\ &\quad + 25g^2P^2\Psi_{\tilde{x}} - 25g^2Q\Psi_{\tilde{x}} \\ &\quad - 9\dot{g}\tilde{x}\Psi_{\tilde{x}} - 9g\Psi_{\tilde{y}} = 0. \end{aligned} \quad (31)$$

From (31), we obtain

$$\begin{aligned} &- 3C_1\left(-3\dot{g} + 5ghP_{\tilde{x}} + 25g^2Q_{\tilde{x}}\right)\Psi \\ &+ 9375g^4\left(5ghP_{\tilde{x}\tilde{x}} + 25g^2Q_{\tilde{x}\tilde{x}}\right)\Psi_{\tilde{x}\tilde{x}} \\ &+ \Bigg[50gh\left(-3\dot{g} + 5ghP_{\tilde{x}} + 25g^2Q_{\tilde{x}}\right) \\ &\quad - 250g^2P\left(-3\dot{g} + 5ghP_{\tilde{x}} + 25g^2Q_{\tilde{x}}\right) \\ &\quad + 3125g^4\left(5ghP_{\tilde{x}\tilde{x}\tilde{x}} + 25g^2Q_{\tilde{x}\tilde{x}\tilde{x}}\right) \\ &\quad - \frac{25}{2}g\Bigg(9gP_{\tilde{y}} + 3125g^6P_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{x}} \\ &\qquad + \left(625g^4F - 125g^3h\right)P_{\tilde{x}\tilde{x}\tilde{x}} \\ &\qquad + 625g^4P_{\tilde{x}}P_{\tilde{x}\tilde{x}} + 6\dot{g}P + \frac{9}{5}\dot{h} - \frac{3\dot{g}h}{5g} \\ &\qquad + \left(9\tilde{x}\dot{g} + 25g^2P^2 \right. \\ &\qquad\qquad \left. - 5ghP + 25g^2Q - h^2\right)P_{\tilde{x}}\Bigg)\Bigg]\Psi_{\tilde{x}} = 0. \end{aligned} \quad (32)$$

It is easy to prove that the reduced (1 + 1)-dimensional equation (29) is the compatibility condition of the reduced Lax pair (31).

(ii) One has $C_1 = 0$, $C_2 \neq 0$.

The eigenfunction is

$$\Phi = \frac{C_2 y}{25g(t)} + \Psi(\tilde{x}, \tilde{y}). \quad (33)$$

We obtain the second type of the reduced Lax pair

$$\begin{aligned} &3125g^4\Psi_{\tilde{x}\tilde{x}\tilde{x}} + 125g^2P\Psi_{\tilde{x}} - 25gh\Psi_{\tilde{x}} + C_2 = 0, \\ &28125g^6\Psi_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + 1875g^4P\Psi_{\tilde{x}\tilde{x}\tilde{x}} \\ &\quad + 1875g^4P_{\tilde{x}}\Psi_{\tilde{x}\tilde{x}} + 1250g^4P_{\tilde{x}\tilde{x}}\Psi_{\tilde{x}} \\ &\quad + 25g^2P^2\Psi_{\tilde{x}} - 25g^2Q\Psi_{\tilde{x}} - 9\dot{g}\tilde{x}\Psi_{\tilde{x}} - 9g\Psi_{\tilde{y}} = 0. \end{aligned} \quad (34)$$

Similarly, the reduced (1 + 1)-dimensional equation (29) is the compatibility condition of the reduced Lax pair (34).

(iii) One has $C_1 = C_2 = 0$.

The eigenfunction is

$$\Phi = \Psi(\tilde{x}, \tilde{y}). \quad (35)$$

We obtain the reduced Lax pair

$$\begin{aligned} &125g^4\Psi_{\tilde{x}\tilde{x}\tilde{x}} + 5g^2P\Psi_{\tilde{x}} - gh\Psi_{\tilde{x}} = 0, \\ &28125g^6\Psi_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + 1875g^4P\Psi_{\tilde{x}\tilde{x}\tilde{x}} \\ &\quad + 1875g^4P_{\tilde{x}}\Psi_{\tilde{x}\tilde{x}} + 1250g^4P_{\tilde{x}\tilde{x}}\Psi_{\tilde{x}} \\ &\quad + 25g^2P^2\Psi_{\tilde{x}} - 25g^2Q\Psi_{\tilde{x}} - 9\dot{g}\tilde{x}\Psi_{\tilde{x}} - 9g\Psi_{\tilde{y}} = 0. \end{aligned} \quad (36)$$

The compatibility condition of the reduced Lax pair is

$$\begin{aligned} &375g^3\left(5ghP_{\tilde{x}\tilde{x}} + 25g^2Q_{\tilde{x}\tilde{x}}\right)\Psi_{\tilde{x}\tilde{x}} \\ &\quad + \Bigg[2h\left(-3\dot{g} + 5ghP_{\tilde{x}} + 25g^2Q_{\tilde{x}}\right) \\ &\quad - 10gP\left(-3\dot{g} + 5ghP_{\tilde{x}} + 25g^2Q_{\tilde{x}}\right) \\ &\quad + 125g^3\left(5ghP_{\tilde{x}\tilde{x}\tilde{x}} + 25g^2Q_{\tilde{x}\tilde{x}\tilde{x}}\right) \\ &\quad - \frac{1}{2}\Bigg(9gP_{\tilde{y}} + 3125g^6P_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{x}} \\ &\qquad + \left(625g^4F - 125g^3h\right)P_{\tilde{x}\tilde{x}\tilde{x}} + 625g^4P_{\tilde{x}}P_{\tilde{x}\tilde{x}} \\ &\qquad + 6\dot{g}P + \frac{9}{5}\dot{h} - \frac{3\dot{g}h}{5g} \\ &\qquad + \left(9\tilde{x}\dot{g} + 25g^2P^2 - 5ghP + 25g^2Q - h^2\right)P_{\tilde{x}}\Bigg)\Bigg]\Psi_{\tilde{x}} \\ &= 0. \end{aligned} \quad (37)$$

The reduced (1 + 1)-dimensional equation (29) is just a subset of (37); then (36) is not the Lax pair of (29).

Case 3 ($f(t) = g(t) = 0$, $h(t) \neq 0$). In this case, the characteristic equation becomes

$$\frac{dx}{5h} = \frac{du}{9\dot{h}}. \quad (38)$$

Then the group invariants are $\tilde{x} = y$, $\tilde{y} = t$ and the symmetry reduction fields are

$$w = P(\tilde{x}, \tilde{y}), \qquad u = \frac{9\dot{h}}{5h}x + Q(\tilde{x}, \tilde{y}), \tag{39}$$

where P and Q are symmetry reduction fields with respect to the group invariants $\tilde{x}$, $\tilde{y}$. Under the above similarity transformations, (3) is reduced to a system of PDE in two independent variables of $\tilde{x}$ and $\tilde{y}$. Consider

$$\begin{gathered} 9hP_{\tilde{y}} - 5hQ_{\tilde{x}} + 9\dot{h}P = 0, \\ 5hP_{\tilde{x}} - 9\dot{h} = 0. \end{gathered} \tag{40}$$

(i) One has $C_1 \neq 0$.

We can obtain the eigenfunction

$$\Phi = -\frac{C_2}{C_1} + e^{C_1 x/5h(t)}\Psi(\tilde{x}, \tilde{y}). \tag{41}$$

Substituting (39) and (41) into (4), in this case we obtain the first type of the reduced Lax pair:

$$\begin{gathered} 125h^3\Psi_{\tilde{x}} + 25C_1h^2P\Psi + C_1^3\Psi = 0, \\ 28125h^5\Psi_{\tilde{y}} + 3125C_1h^4P^2\Psi + 3125C_1h^4Q\Psi \\ - 375C_1^3h^2P\Psi - 9C_1^5\Psi = 0. \end{gathered} \tag{42}$$

The compatibility condition is

$$\begin{gathered} C_1h\left[3C_1^2\left(5hP_{\tilde{x}} - 9\dot{h}\right) + 50h^2P\left(5hP_{\tilde{x}} - 9\dot{h}\right) \right. \\ \left. + 25h^2\left(9hP_{\tilde{y}} - 5hQ_{\tilde{x}} + 9\dot{h}P\right)\right]\Psi = 0. \end{gathered} \tag{43}$$

It is easy to prove that the reduced $(1 + 1)$-dimensional equation (40) is the compatibility condition of the reduced Lax pair (42).

(ii) One has $C_1 = 0$, $C_2 \neq 0$.

The eigenfunction is

$$\Phi = \frac{C_2 x}{5h(t)} + \Psi(\tilde{x}, \tilde{y}). \tag{44}$$

We obtain the reduced Lax pair

$$\begin{gathered} 5h\Psi_{\tilde{x}} + C_2P = 0, \\ 9h\Psi_{\tilde{y}} - C_2P^2 + C_2Q = 0. \end{gathered} \tag{45}$$

The compatibility condition is

$$\frac{C_2}{h}\left[\left(9hP_{\tilde{y}} - 5hQ_{\tilde{x}} + 9\dot{h}P\right) + 2P\left(5hP_{\tilde{x}} - 9\dot{h}\right)\right] = 0. \tag{46}$$

We can see that the reduced $(1+1)$-dimensional equation (40) is just a subset of (46); then (45) is not the Lax pair of (40).

4. Summary

To understand the integrability aspects of the $(2 + 1)$-dimensional CDGKS equation, we carry out Lou's direct method and obtain the symmetry transformations of the equation. In fact, we can get infinitely many explicit solutions to (3) through the symmetry transformations. With the classical Lie group method, we obtain the Lie point symmetry groups of both the CDGKS equation and its Lax pair. By the obtained symmetries, we can reduce the dimensions and orders of the $(2 + 1)$-dimensional CDGKS equation and get three $(1 + 1)$-dimensional equations with their new Lax pairs. Since the reduced equations are much simpler than the original ones, it is easy to obtain some group-invariant solutions of the $(2+1)$-dimensional CDGKS equation. By the new Lax pairs, we can research the Darboux transformation and explicit solutions to the CDGKS equation as well. These topics will be considered in the future.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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