Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 516452, 7 pages
http://dx.doi.org/10.1155/2014/516452

Research Article

Some Existence Results of Positive Solution to Second-Order Boundary Value Problems

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Received 4 February 2014; Accepted 16 March 2014; Published 20 May 2014

Academic Editor: Xinan Hao

We study the existence of positive and monotone solution to the boundary value problem $u''(t) + f(t, u(t)) = 0$, $0 \leqslant t \leqslant 1$, $u(0) = \xi u(1)$, $u'(1) = \eta u'(0)$, where $\xi, \eta \in (0, 1) \cup (1, \infty)$. The main tool is the fixed point theorem of cone expansion and compression of functional type by Avery, Henderson, and O'Regan. Finally, four examples are provided to demonstrate the availability of our main results.

1. Introduction

Boundary value problems for ordinary differential equations play a very important role in both theory and applications. They are used to describe a large number of physical, biological, and chemical phenomena. In recent years many papers have been devoted to second-order two-point boundary value problem. For a small sample of such work, we refer the reader to the monographs of Agarwal [1], Agarwal et al. [2], and Guo and Lakshmikantham [3], the papers of Avery et al. [4] and Henderson and Thompson [5], and references therein along this line. In the literature, many attempts have been made by researchers to develop criteria which guarantee the existence and uniqueness of positive solutions to ordinary differential equations; this subject has attracted a lot of interests; see, for example, Cid et al. [6], Ehme [7], Ehme and Lanz [8], Ibrahim and Momani [9], Kong [10], Ma and An [11], Zhang and Liu [12], Zhang et al. [13], and Zhong and Zhang [14].

In this paper, we study the existence of positive and monotone solution for the second-order two-point boundary value problem

$$\begin{gathered} u''(t) + f(t, u(t)) = 0, \quad 0 \leqslant t \leqslant 1, \\ u(0) = \xi u(1), \qquad u'(1) = \eta u'(0), \end{gathered} \tag{1}$$

where $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous function and $\xi, \eta \in (0, 1) \cup (1, \infty)$ are two constants. The boundary conditions in problem (1) are closely related to some other boundary conditions. If $\xi = \eta = 1$, the boundary conditions in problem (1) reduce to periodic boundary conditions. If $\xi = \eta = -1$, the boundary conditions in problem (1) reduce to antiperiodic boundary conditions. If $\xi = \eta = 0$, problem (1) reduces to second-order right focal boundary value problem. In a recent paper [15], by applying a fixed point theorem by Avery et al. [16], Sun studied the existence of monotone positive solutions to problem (1). In this paper we will prove some new existence results for problem (1) by using the new fixed point theorem of cone expansion and compression of functional type by Avery et al. [17].

This paper is organized as follows. In Section 2 we present some notations, definitions, and lemmas. In Section 3 we establish some sufficient conditions which guarantee the existence of positive solutions to problem (1). In Section 4 we give four examples to illustrate the effectiveness and applications of the results presented in Section 3.

2. Preliminary Results

For the convenience of the reader, we present here the necessary definitions and background results. We also state the fixed point theorem of cone expansion and compression of functional type by Avery, Henderson, and O'Regan.

Definition 1. Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a *cone* of E if it satisfies the following two conditions:

(1) $u \in P$, $\lambda \geqslant 0$, implies $\lambda u \in P$;

(2) $u \in P$, $-u \in P$, implies $u = 0$.

Every cone $P \subset E$ induces an ordering in E given by $u \leqslant v$ if and only if $v - u \in P$.

Definition 2. Let E be a real Banach space. An operator $T : E \to E$ is said to be completely continuous if it is continuous and maps bounded sets into precompact sets.

Definition 3. A map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if $\alpha : P \to [0, +\infty)$ is continuous and

$$\alpha(\lambda u + (1-\lambda)v) \geqslant \lambda\alpha(u) + (1-\lambda)\alpha(v), \quad u, v \in P,\ 0 \leqslant \lambda \leqslant 1. \tag{2}$$

Similarly we said the map β is a nonnegative continuous convex functional on a cone P of a real Banach space E if $\beta : P \to [0, +\infty)$ is continuous and

$$\beta(\lambda u + (1-\lambda)v) \leqslant \lambda\beta(u) + (1-\lambda)\beta(v), \quad u, v \in P,\ 0 \leqslant \lambda \leqslant 1. \tag{3}$$

We say that the map γ is sublinear functional if

$$\gamma(\lambda u) \leqslant \lambda\gamma(u), \quad u \in P,\ 0 \leqslant \lambda \leqslant 1. \tag{4}$$

All the concepts discussed above can be found in [3].

Property A1. Let P be a cone in a real Banach space E and Ω a bounded open subset of E with $0 \in \Omega$. Then a continuous functional $\beta : P \to [0, \infty)$ is said to satisfy Property A1 if one of the following conditions holds:

(a) β is convex, $\beta(0) = 0$, and $\beta(u) \neq 0$ if $u \neq 0$ and $\inf_{u \in P \cap \partial\Omega} \beta(u) > 0$,

(b) β is sublinear, $\beta(0) = 0$, and $\beta(u) \neq 0$ if $u \neq 0$ and $\inf_{u \in P \cap \partial\Omega} \beta(u) > 0$,

(c) β is concave and unbounded.

Property A2. Let P be a cone in a real Banach space E and Ω a bounded open subset of E with $0 \in \Omega$. Then a continuous functional $\beta : P \to [0, \infty)$ is said to satisfy Property A2 if one of the following conditions hold:

(a) β is convex, $\beta(0) = 0$, and $\beta(u) \neq 0$ if $u \neq 0$,

(b) β is sublinear, $\beta(0) = 0$, and $\beta(u) \neq 0$ if $u \neq 0$,

(c) $\beta(u + v) \geqslant \beta(u) + \beta(v)$ for all $u, v \in P$, $\beta(0) = 0$, and $\beta(u) \neq 0$ if $u \neq 0$.

To prove our results, we will need the following fixed point theorem, which is presented by Avery et al. [17].

Theorem 4. *Let Ω_1 and Ω_2 be two bounded open sets in a Banach space E such that $0 \in \Omega_1$ and $\Omega_1 \subseteq \Omega_2$ and P is a cone in E. Suppose that $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator, α and γ are nonnegative continuous functional on P, and one of the two conditions*

(K1) α satisfies Property A1 with $\alpha(Tu) \geqslant \alpha(u)$, for all $u \in P \cap \partial\Omega_1$, and γ satisfies Property A2 with $\gamma(Tu) \leqslant \gamma(u)$, for all $u \in P \cap \partial\Omega_2$; or

(K2) γ satisfies Property A2 with $\gamma(Tu) \leqslant \gamma(u)$, for all $u \in P \cap \partial\Omega_1$, and α satisfies Property A1 with $\alpha(Tu) \geqslant \alpha(u)$, for all $u \in P \cap \partial\Omega_2$

is satisfied. Then T has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

To study problem (1), we need the following lemmas (see [15]).

Lemma 5. *Green's function $G : [0,1] \times [0,1] \to [0, \infty)$ for the BVP (1.1) is given by*

$$G(t,s) = \frac{1}{(1-\xi)(1-\eta)} \times \begin{cases} s + \eta(t-s) + \xi\eta(1-t), & 0 \leqslant s \leqslant t \leqslant 1, \\ t + \xi(s-t) + \xi\eta(1-s), & 0 \leqslant t \leqslant s \leqslant 1. \end{cases} \tag{5}$$

Lemma 6. *Suppose that $\xi, \eta, \delta \in (0,1)$. Then Green's function $G(t,s)$ defined by (5) has the following properties:*

(a) *$G(t,s) \geqslant 0$, $\partial G(t,s)/\partial t \geqslant 0$, $\forall t, s \in [0,1]$;*

(b) *$tG(1,s) \leqslant G(t,s) \leqslant G(1,s)$, $\forall t, s \in [0,1]$;*

(c) *$\max_{0 \leqslant t \leqslant 1} \int_0^1 G(t,s)ds = \int_0^1 G(1,s)ds = (1+\eta)/2(1-\xi)(1-\eta)$;*

(d) *$\min_{\delta \leqslant t \leqslant 1} \int_\delta^1 G(t,s)ds = \int_\delta^1 G(\delta,s)ds = (1-\delta)\delta/(1-\eta) + (1-\delta)(1+\delta+\eta-\eta\delta)\xi/2(1-\xi)(1-\eta) \geqslant (1-\delta)\delta/(1-\eta)$.*

Lemma 7. *Suppose that $\xi, \eta \in (1, \infty)$, $\delta \in (0,1)$. Then Green's function $G(t,s)$ defined by (5) has the following properties:*

(a) *$G(t,s) \geqslant 0$, $\partial G(t,s)/\partial t \leqslant 0$, $\forall t, s \in [0,1]$;*

(b) *$(1-t)G(0,s) \leqslant G(t,s) \leqslant G(0,s)$, $\forall t, s \in [0,1]$;*

(c) *$\max_{0 \leqslant t \leqslant 1} \int_0^1 G(t,s)ds = \int_0^1 G(0,s)ds = (1+\eta)\xi/2(\xi-1)(\eta-1)$;*

(d) *$\min_{0 \leqslant t \leqslant 1-\delta} \int_0^{1-\delta} G(t,s)ds = \int_0^{1-\delta} G(1-\delta,s)ds = \eta(1-\delta)\delta/(\eta-1) + (1-\delta)(1+\eta-\delta+\eta\delta)/2(\xi-1)(\eta-1) \geqslant \eta(1-\delta)\delta/(\eta-1)$.*

3. Main Results

In this section, we will apply Theorem 4 to study the existence of positive and monotonic solution to problem (1).

3.1. Case I: $\xi, \eta \in (0,1)$ $u(0)=u'(1)=u''(0)=u'''(1)=0$. In this case we define the cone P by

$$P = \{u \in C[0,1] : u(t) \geqslant 0, \\ u(t) \text{ is increasing on } [0,1] \text{ and } u(t) \geqslant t\|u\|, \\ t \in [0,1]\}. \quad (6)$$

Then P is a normal cone of E. Define the operator $T : P \to E$ by

$$(Tu)(t) = \int_0^1 G(t,s) f(s,u(s))\, ds, \quad t \in [0,1]. \quad (7)$$

Then by Lemma 6 and Ascoli-Arzela Theorem we know that $T(P) \subseteq P$ and T is a completely continuous operator. Let us define two continuous functionals α and γ on the cone P by

$$\alpha(u) := \min_{t\in[\delta,1]} u(t) = u(\delta), \\ \gamma(u) := \max_{t\in[0,1]} u(t) = u(1) = \|u\|. \quad (8)$$

It is clear that $\alpha(u) \leqslant \gamma(u)$ for all $u \in P$.

Theorem 8. *Suppose that $f \in C([0,1]\times[0,\infty),[0,\infty))$ and there exist $r, R \in (0,\infty)$, $\delta \in (0,1)$ with $r < \delta R$ such that the following conditions are satisfied:*

(A1) $f(t,x) \geqslant (1-\eta)r/(1-\delta)\delta$, for all $(t,x) \in [\delta,1]\times[r,R]$;

(A2) $f(t,x) \leqslant 2(1-\xi)(1-\eta)R/(1+\eta)$, for all $(t,x) \in [0,1]\times[0,R]$.

Then problem (1) *admits a positive and increasing solution u^* such that*

$$r \leqslant \min_{t\in[\delta,1]} u^*(t), \qquad \max_{t\in[0,1]} u^*(t) \leqslant R. \quad (9)$$

Proof. Let

$$\Omega_1 = \{u : \alpha(u) < r\}, \qquad \Omega_2 = \{u : \gamma(u) < R\}, \quad (10)$$

it is easy to see that $0 \in \Omega_1$, and Ω_1 and Ω_2 are bounded open subsets of E. Let $u \in \overline{\Omega}_1$; then we have

$$r \geqslant \alpha(u) = \min_{t\in[\delta,1]} u(t) \geqslant \delta\|u\| = \delta\gamma(u). \quad (11)$$

Thus $R > r/\delta \geqslant \gamma(u)$; that is, $u \in \Omega_2$, so $\overline{\Omega}_1 \subseteq \Omega_2$.

Claim 1 (if $u \in P \cap \partial\Omega_1$, then $\alpha(Tu) \geqslant \alpha(u)$). To see this let $u \in P\cap\partial\Omega_1$; then $R = \gamma(u) \geqslant u(s) \geqslant \alpha(u) = r$, $s \in [\delta,1]$. It follows from (A1) and Lemmas 6(d) and (7) that

$$\begin{aligned}\alpha(Tu) = (Tu)(\delta) &= \int_0^1 G(\delta,s) f(s,u(s))\, ds \\ &\geqslant \int_\delta^1 G(\delta,s) f(s,u(s))\, ds \\ &\geqslant \frac{(1-\eta)r}{(1-\delta)\delta}\int_\delta^1 G(\delta,s)\, ds \\ &\geqslant \frac{(1-\eta)r}{(1-\delta)\delta}\cdot\frac{(1-\delta)\delta}{1-\eta} = r = \alpha(u).\end{aligned} \quad (12)$$

Claim 2 (if $u \in P\cap\partial\Omega_2$, then $\gamma(Tu) \leqslant \gamma(u)$). To see this let $u \in P\cap\partial\Omega_2$; then $u(s) \leqslant \gamma(u) = R$, $s \in [0,1]$. Thus condition (A2) and Lemma 6(c) yield that

$$\begin{aligned}\gamma(Tu) = (Tu)(1) &= \int_0^1 G(1,s) f(s,u(s))\, ds \\ &\leqslant \frac{2(1-\xi)(1-\eta)R}{1+\eta}\int_0^1 G(1,s)\, ds \\ &= \frac{2(1-\xi)(1-\eta)R}{1+\eta}\cdot\frac{1+\eta}{2(1-\xi)(1-\eta)} = R = \gamma(u).\end{aligned} \quad (13)$$

Clearly α satisfies Property A1(c) and γ satisfies Property A2(a). Therefore hypothesis (K1) of Theorem 4 is satisfied and hence T has at least one fixed point $u^* \in P\cap(\overline{\Omega}_2\setminus\Omega_1)$; that is, problem (1) has at least one positive solution $u^*(t) \in P$ such that

$$r \leqslant \min_{t\in[\delta,1]} u^*(t), \qquad \max_{t\in[0,1]} u^*(t) \leqslant R. \quad (14)$$

This completes the proof. □

Theorem 9. *Suppose that $f \in C([0,1]\times[0,\infty),[0,\infty))$ and there exist $r, R \in (0,\infty)$ with $r < R$ such that the following conditions are satisfied:*

(A3) $f(t,x) \leqslant 2(1-\xi)(1-\eta)r/(1+\eta)$, for $(t,x) \in [0,1]\times[0,r]$;

(A4) $f(t,x) \geqslant (1-\eta)R/(1-\delta)\delta$, for $(t,x) \in [\delta,1]\times[R,R/\delta]$.

Then problem (1) *admits a positive and increasing solution $u^* \in P$ such that*

$$r \leqslant \max_{t\in[0,1]} u^*(t), \qquad \min_{t\in[\delta,1]} u^*(t) \leqslant R. \quad (15)$$

Proof. For all $u \in P$ we have $\alpha(u) \leqslant \gamma(u)$. Thus if we let

$$\Omega_3 = \{u : \gamma(u) < r\}, \qquad \Omega_4 = \{u : \alpha(u) < R\}, \quad (16)$$

we have that $0 \in \Omega_3$ and $\overline{\Omega}_3 \subseteq \Omega_4$, with Ω_3 and Ω_4 being bounded open subsets of E.

Claim 1 (if $u \in P\cap\partial\Omega_3$, then $\gamma(Tu) \leqslant \gamma(u)$). To see this let $u \in P\cap\partial\Omega_3$; then $u(s) \leqslant \gamma(u) = r$, $s \in [0,1]$. Thus condition (A3) and Lemma 6(c) yield that

$$\begin{aligned}\gamma(Tu) &= (Tu)(1) \\ &= \int_0^1 G(1,s) f(s,u(s))\, ds \\ &\leqslant \frac{2(1-\xi)(1-\eta)r}{1+\eta}\int_0^1 G(1,s)\, ds \\ &= \frac{2(1-\xi)(1-\eta)r}{1+\eta}\cdot\frac{1+\eta}{2(1-\xi)(1-\eta)} = r = \gamma(u).\end{aligned} \quad (17)$$

Claim 2 (if $u \in P \cap \partial\Omega_4$, then $\alpha(Tu) \geqslant \alpha(u)$). To see this let $u \in P \cap \partial\Omega_4$; then $R/\delta = \alpha(u)/\delta \geqslant \gamma(u) \geqslant u(s) \geqslant \alpha(u) = R$, $s \in [\delta, 1]$. Thus it follows from (A4) and Lemma 6(d) that

$$\begin{aligned} \alpha(Tu) = (Tu)(\delta) &= \int_0^1 G(\delta,s) f(s,u(s))\, ds \\ &\geqslant \int_\delta^1 G(\delta,s) f(s,u(s))\, ds \\ &\geqslant \frac{(1-\eta)R}{(1-\delta)\delta} \int_\delta^1 G(\delta,s)\, ds \\ &\geqslant \frac{(1-\eta)R}{(1-\delta)\delta} \cdot \frac{(1-\delta)\delta}{1-\eta} = R = \alpha(u). \end{aligned} \tag{18}$$

Clearly α satisfies Property A1(c) and γ satisfies Property A2(a). Therefore hypothesis (K2) of Theorem 4 is satisfied and hence T has at least one fixed point $u^* \in P \cap (\overline{\Omega}_4 \setminus \Omega_3)$; that is, problem (1) has at least one positive solution $u^* \in P$ such that

$$r \leqslant \max_{t\in[0,1]} u^*(t), \qquad \min_{t\in[\delta,1]} u^*(t) \leqslant R. \tag{19}$$

This completes the proof. □

3.2. Case II: $\xi,\eta \in (1,\infty)$. In this case we define the cone P by

$$\begin{aligned} P = \{u \in C[0,1] : u(t) \geqslant 0,& \\ &u(t) \text{ is decreasing on } [0,1], \\ &\text{and } u(t) \geqslant (1-t)\|u\|, \\ &t \in [0,1]\}. \end{aligned} \tag{20}$$

Then P is a normal cone of E. Define the operator $T : P \to E$ by (7). Then by Lemma 7 and Ascoli-Arzela Theorem we know that $T(P) \subseteq P$ and T is a completely continuous operator. Let us define two continuous functionals α and γ on the cone P by

$$\begin{aligned} \alpha(u) &:= \min_{t\in[0,1-\delta]} u(t) = u(1-\delta), \\ \gamma(u) &:= \max_{t\in[0,1]} u(t) = u(0) = \|u\|. \end{aligned} \tag{21}$$

It is clear that $\alpha(u) \leqslant \gamma(u)$ for all $u \in P$.

Theorem 10. *Suppose that $f \in C([0,1]\times[0,\infty),[0,\infty))$ and there exist $r, R \in (0,\infty)$, $\delta \in (0,1)$ with $r < \delta R$ such that the following conditions are satisfied:*

(B1) $f(t,x) \geqslant (\eta-1)r/\eta(1-\delta)\delta$, for all $(t,x) \in [0,1-\delta]\times[r,R]$;

(B2) $f(t,x) \leqslant 2(\xi-1)(\eta-1)R/(1+\eta)\xi$, for all $(t,x) \in [0,1]\times[0,R]$.

Then problem (1) *admits a positive and decreasing solution u^* such that*

$$r \leqslant \min_{t\in[0,1-\delta]} u^*(t), \qquad \max_{t\in[0,1]} u^*(t) \leqslant R. \tag{22}$$

Proof. Letting

$$\Omega_1 = \{u : \alpha(u) < r\}, \qquad \Omega_2 = \{u : \gamma(u) < R\}, \tag{23}$$

then $0 \in \Omega_1$, Ω_1, and Ω_2 are bounded open subsets of E. Let $u \in \overline{\Omega}_1$; then we have

$$r \geqslant \alpha(u) = \min_{t\in[0,1-\delta]} u(t) \geqslant \delta\|u\| = \delta\gamma(u). \tag{24}$$

Thus $R > r/\delta \geqslant \gamma(u)$; that is, $u \in \Omega_2$, so $\overline{\Omega}_1 \subseteq \Omega_2$.

Claim 1 (if $u \in P \cap \partial\Omega_1$, then $\alpha(Tu) \geqslant \alpha(u)$). To see this let $u \in P \cap \partial\Omega_1$; then $R = \gamma(u) \geqslant u(s) \geqslant \alpha(u) = r$, $s \in [0,1-\delta]$. Thus it follows from (A1) and Lemmas 7(d) and (7) that

$$\begin{aligned} \alpha(Tu) &= (Tu)(1-\delta) \\ &= \int_0^1 G(1-\delta,s) f(s,u(s))\, ds \\ &\geqslant \int_0^{1-\delta} G(1-\delta,s) f(s,u(s))\, ds \\ &\geqslant \frac{(\eta-1)r}{\eta(1-\delta)\delta} \int_0^{1-\delta} G(1-\delta,s)\, ds \\ &\geqslant \frac{(\eta-1)r}{\eta(1-\delta)\delta} \cdot \frac{\eta(1-\delta)\delta}{\eta-1} = r = \alpha(u). \end{aligned} \tag{25}$$

Claim 2 (if $u \in P \cap \partial\Omega_2$, then $\gamma(Tu) \leqslant \gamma(u)$). To see this let $u \in P \cap \partial\Omega_2$; then $u(s) \leqslant \gamma(u) = R$, $s \in [0,1]$. Thus condition (B2) and Lemma 7(c) yield

$$\begin{aligned} \gamma(Tu) = (Tu)(0) &= \int_0^1 G(0,s) f(s,u(s))\, ds \\ &\leqslant \frac{2(\xi-1)(\eta-1)R}{(1+\eta)\xi} \int_0^1 G(0,s)\, ds \\ &= \frac{2(\xi-1)(\eta-1)R}{(1+\eta)\xi} \cdot \frac{(1+\eta)\xi}{2(\xi-1)(\eta-1)} \\ &= R = \gamma(u). \end{aligned} \tag{26}$$

Clearly α satisfies Property A1(c) and γ satisfies Property A2(a). Therefore hypothesis (K1) of Theorem 4 is satisfied and hence T has at least one fixed point $u^* \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$; that is, problem (1) has at least one positive solution $u^*(t) \in P$ such that

$$r \leqslant \min_{t\in[0,1-\delta]} u^*(t), \qquad \max_{t\in[0,1]} u^*(t) \leqslant R. \tag{27}$$

This completes the proof. □

Theorem 11. *Suppose that $f \in C([0,1]\times[0,\infty),[0,\infty))$, and there exist $r, R \in (0,\infty)$ with $r < R$ such that the following conditions are satisfied:*

(B3) $f(t,x) \leqslant 2(\xi-1)(\eta-1)r/(1+\eta)\xi$, for $(t,x) \in [0,1]\times[0,r]$;

(B4) $f(t,x) \geqslant (\eta-1)R/\eta(1-\delta)\delta$, for $(t,x) \in [0,1-\delta] \times [R, R/\delta]$.

Then problem (1) *admits a positive and decreasing solution $u^* \in P$ such that*

$$r \leqslant \max_{t\in[0,1]} u^*(t), \qquad \min_{t\in[\delta,1]} u^*(t) \leqslant R. \tag{28}$$

Proof. For all $u \in P$ we have $\alpha(u) \leqslant \gamma(u)$. Thus if we let

$$\Omega_3 = \{u : \gamma(u) < r\}, \qquad \Omega_4 = \{u : \alpha(u) < R\}, \tag{29}$$

we have that $0 \in \Omega_3$ and $\overline{\Omega}_3 \subseteq \Omega_4$, with Ω_3 and Ω_4 being bounded open subsets of E.

Claim 1 (if $u \in P \cap \partial\Omega_3$, then $\gamma(Tu) \leqslant \gamma(u)$). To see this let $u \in P \cap \partial\Omega_3$; then $u(s) \leqslant \gamma(u) = r$, $s \in [0,1]$. Thus condition (B3) and Lemma 7(c) yield

$$\begin{aligned}\gamma(Tu) &= (Tu)(0)\\ &= \int_0^1 G(0,s) f(s,u(s))\,ds \leqslant \frac{2(\xi-1)(\eta-1)r}{(1+\eta)\xi}\\ &\quad \times \int_0^1 G(0,s)\,ds\\ &= \frac{2(\xi-1)(\eta-1)r}{(1+\eta)\xi} \cdot \frac{(1+\eta)\xi}{2(\xi-1)(\eta-1)} = r = \gamma(x).\end{aligned} \tag{30}$$

Claim 2 (if $u \in P \cap \partial\Omega_4$, then $\alpha(Tu) \geqslant \alpha(u)$). To see this let $u \in P \cap \partial\Omega_4$; then $R/\delta = \alpha(u)/\delta \geqslant \gamma(u) \geqslant u(s) \geqslant \alpha(u) = R$, $s \in [0, 1-\delta]$. Thus it follows from (B4) and Lemma 7(d) that

$$\begin{aligned}\alpha(Tu) &= (Tu)(1-\delta) = \int_0^1 G(1-\delta,s) f(s,u(s))\,ds\\ &\geqslant \int_0^{1-\delta} G(1-\delta,s) f(s,u(s))\,ds \geqslant \frac{(\eta-1)R}{\eta(1-\delta)\delta}\\ &\quad \times \int_0^{1-\delta} G(1-\delta,s)\,ds\\ &\geqslant \frac{(\eta-1)R}{\eta(1-\delta)\delta} \cdot \frac{\eta(1-\delta)\delta}{\eta-1} = R = \alpha(u).\end{aligned} \tag{31}$$

Clearly α satisfies Property A1(c) and γ satisfies Property A2(a). Therefore hypothesis (K2) of Theorem 4 is satisfied and hence T has at least one fixed point $u^* \in P \cap (\overline{\Omega}_4 \setminus \Omega_3)$; that is, problem (1) has at least one positive solution $u^* \in P$ such that

$$r \leqslant \max_{t\in[0,1]} u^*(t), \qquad \min_{t\in[0,1-\delta]} u^*(t) \leqslant R. \tag{32}$$

This completes the proof. □

4. Examples

At the end of the paper, we present some examples to illustrate the usefulness of our main results.

Example 1. Consider the second-order boundary value problem

$$\begin{gathered}u''(t) + \frac{1+t}{144}u^2(t) + \frac{1}{24}u(t) + \frac{1}{8}t^2 + \frac{1}{4}t + \frac{1}{4} = 0,\\ 0 \leqslant t \leqslant 1,\\ u(0) = \frac{1}{2}u(1), \qquad u'(1) = \frac{1}{2}u'(0).\end{gathered} \tag{33}$$

In this case, $\xi = \eta = 1/2$. Let $\delta = 1/2$, $r = 1/6$, $R = 3$; observe that $r < \delta R$. Setting

$$f(t,x) = \frac{1+t}{144}x^2 + \frac{1}{24}x + \frac{1}{8}t^2 + \frac{1}{4}t + \frac{1}{4}, \tag{34}$$

then $f \in C([0,1] \times [0,\infty), [0,\infty))$, and for $(t,x) \in [0,1] \times [0,R]$, we have

$$f(t,x) \leqslant f(1,R) = \frac{7}{8} < 1 = \frac{2(1-\xi)(1-\eta)R}{1+\eta}. \tag{35}$$

For $(t,x) \in [\delta,1] \times [r,R]$, we have

$$f(t,x) \geqslant f(\delta,r) = \frac{1429}{3456} > \frac{1}{3} = \frac{(1-\eta)r}{(1-\delta)\delta}. \tag{36}$$

Clearly, all the assumptions of Theorem 8 hold and consequently problem (33) has at least one positive and increasing solution $u^*(t)$ such that

$$\frac{1}{6} \leqslant \min_{t\in[1/2,1]} u^*(t), \qquad \max_{t\in[0,1]} u^*(t) \leqslant 3. \tag{37}$$

Example 2. Consider the second-order boundary value problem

$$\begin{gathered}u''(t) + \frac{2+2t}{9}u^3(t) + \frac{1}{9}u(t) + \frac{1}{9}t = 0, \quad 0 \leqslant t \leqslant 1,\\ u(0) = \frac{1}{4}u(1), \qquad u'(1) = \frac{1}{3}u'(0).\end{gathered} \tag{38}$$

In this case, $\xi = 1/4$ and $\eta = 1/3$. Let $\delta = 1/2$, $R = 3$, $r = 1$, and

$$f(t,x) = \frac{2+2t}{9}x^3 + \frac{1}{9}x + \frac{1}{9}t. \tag{39}$$

Then $f \in C([0,1] \times [0,\infty), [0,\infty))$, and for $(t,x) \in [0,1] \times [0,r]$,

$$f(t,x) \leqslant f(1,r) = \frac{2}{3} < \frac{3}{4} = \frac{2(1-\xi)(1-\eta)r}{1+\eta}. \tag{40}$$

For $(t,x) \in [\delta,1] \times [R, R/\delta]$,

$$f(t,x) \geqslant f(\delta,R) = 9\frac{7}{18} > 8 = \frac{(1-\eta)R}{(1-\delta)\delta}. \tag{41}$$

Hence, by Theorem 9, problem (38) has at least one positive and increasing solution $u^*(t)$ such that

$$1 \leqslant \max_{t\in[0,1]} u^*(t), \qquad \min_{t\in[1/2,1]} u^*(t) \leqslant 3. \tag{42}$$

Example 3. Consider the second-order boundary value problem

$$u''(t) + \frac{1+t}{144}u^2(t) + \frac{1}{24}u(t) + \frac{1}{8}t^2 + \frac{1}{2} = 0, \quad 0 \leqslant t \leqslant 1,$$
$$u(0) = 2u(1), \qquad u'(1) = 2u'(0). \tag{43}$$

In this case, $\xi = \eta = 2$. Let $\delta = 1/2$, $r = 1/6$, $R = 3$; it is evident that $r < \delta R$. Set

$$f(t,x) = \frac{1+t}{144}x^2 + \frac{1}{24}x + \frac{1}{8}t^2 + \frac{1}{2}. \tag{44}$$

Then $f \in C([0,1]\times[0,\infty),[0,\infty))$, and for $(t,x) \in [0,1]\times[0,R]$, we have

$$f(t,x) \leqslant f(1,R) = \frac{7}{8} < 1 = \frac{2(\xi-1)(\eta-1)R}{(1+\eta)\xi}. \tag{45}$$

For $(t,x) \in [0,1-\delta]\times[r,R]$, we have

$$f(t,x) \geqslant f(0,r) > \frac{1}{2} > \frac{1}{3} = \frac{(\eta-1)r}{\eta(1-\delta)\delta}. \tag{46}$$

So all conditions of Theorem 10 are satisfied and consequently problem (43) has at least one positive and decreasing solution $u^*(t)$ such that

$$\frac{1}{6} \leqslant \min_{t\in[1/2,1]} u^*(t), \qquad \max_{t\in[0,1]} u^*(t) \leqslant 3. \tag{47}$$

Example 4. Consider the second-order boundary value problem

$$u''(t) + \frac{1+t}{9}u^4(t) + \frac{t}{9}u(t) + \frac{1}{3} = 0, \quad 0 \leqslant t \leqslant 1,$$
$$u(0) = 4u(1), \qquad u'(1) = 3u'(0). \tag{48}$$

In this problem, $\xi = 4$, and $\eta = 3$. Let $\delta = 1/2$, $R = 3$, $r = 1$, and

$$f(t,x) = \frac{1+t}{9}x^4 + \frac{t}{9}x + \frac{1}{3}. \tag{49}$$

Then $f \in C([0,1]\times[0,\infty),[0,\infty))$, and for $(t,x) \in [0,1]\times[0,r]$,

$$f(t,x) \leqslant f(1,r) = \frac{2}{3} < \frac{3}{4} = \frac{2(\xi-1)(\eta-1)r}{(1+\eta)\xi}. \tag{50}$$

For $(t,x) \in [0,1-\delta]\times[R,R/\delta]$,

$$f(t,x) \geqslant f(0,R) = 9\frac{1}{3} > 8 = \frac{(\eta-1)R}{\eta(1-\delta)\delta}. \tag{51}$$

Hence, by Theorem 11, problem (48) has at least one positive and decreasing solution $u^*(t)$ such that

$$1 \leqslant \max_{t\in[0,1]} u^*(t), \qquad \min_{t\in[0,1/2]} u^*(t) \leqslant 3. \tag{52}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors thank the referee for her/his careful reading of the paper and useful suggestions. This work was supported by Zhejiang Provincial Natural Science Foundation of China (no. Y12A01012).

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