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Research Article

Positive Periodic Solutions in Shifts $\delta_\pm$ for a Class of Higher-Dimensional Functional Dynamic Equations with Impulses on Time Scales

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Let $\mathbb{T} \subset \mathbb{R}$ be a periodic time scale in shifts $\delta_\pm$ with period $P \in (t_0, \infty)_{\mathbb{T}}$ and $t_0 \in \mathbb{T}$ is nonnegative and fixed. By using a multiple fixed point theorem in cones, some criteria are established for the existence and multiplicity of positive solutions in shifts $\delta_\pm$ for a class of higher-dimensional functional dynamic equations with impulses on time scales of the following form: $x^\Delta(t) = A(t)x(t) + b(t)f(t, x(g(t)))$, $t \neq t_j$, $t \in \mathbb{T}$, $x(t_j^+) = x(t_j^-) + I_j(x(t_j))$, where $A(t) = (a_{ij}(t))_{n\times n}$ is a nonsingular matrix with continuous real-valued functions as its elements. Finally, numerical examples are presented to illustrate the feasibility and effectiveness of the results.

1. Introduction

As is known to all, both continuous and discrete systems are very important in implementation and application. The study of dynamic equations on time scales, which unifies differential, difference, h-difference, and q-differences equations and more, has received much attention; see, for example, [1–16] and the references therein. The theory of dynamic equations on time scales was introduced by Hilger in his PhD thesis in 1988 [5]. The existence problem of periodic solutions is an important topic in qualitative analysis of functional dynamic equations. Up to now, there are only a few results concerning periodic solutions of dynamic equations on time scales; see, for example, [6–9]. In these papers, authors considered the existence of periodic solutions for dynamic equations on time scales satisfying the condition "there exists a $\omega > 0$ such that $t \pm \omega \in \mathbb{T}$, $\forall t \in \mathbb{T}$." Under this condition, all periodic time scales are unbounded above and below. However, there are many time scales such as $\overline{q^{\mathbb{Z}}} = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$ and $\sqrt{\mathbb{N}} = \{\sqrt{n} : n \in \mathbb{N}\}$ which do not satisfy this condition. Adivar and Raffoul introduced a new periodicity concept on time scales which does not oblige the time scale to be closed under the operation $t \pm \omega$ for a fixed $\omega > 0$. They defined a new periodicity concept with the aid of shift operators $\delta_\pm$ which are first defined in [10] and then generalized in [11].

Recently, based on a fixed-point theorem in cones, Çetin and Serap Topal studied the existence of positive periodic solutions in shifts $\delta_\pm$ for some nonlinear first-order functional dynamic equation on time scales; see [12, 13]. However, to the best of our knowledge, there are few papers published on the existence of positive periodic solutions in shifts $\delta_\pm$ for higher-dimensional functional dynamic equations with impulses, especially systems with the coefficient matrix being an arbitrary nonsingular $n \times n$ matrix.

Motivated by the above, in the present paper, we consider the following system:

$$x^\Delta(t) = A(t)x(t) + b(t)f(t, x(g(t))), \quad t \neq t_j,\ t \in \mathbb{T},$$
$$x(t_j^+) = x(t_j^-) + I_j(x(t_j)), \tag{1}$$

where $\mathbb{T} \subset \mathbb{R}$ is a periodic time scale in shifts $\delta_\pm$ with period $P \in [t_0, \infty)_{\mathbb{T}}$ and $t_0 \in \mathbb{T}$ is nonnegative and fixed; $A = (a_{ij})_{n\times n}$ is a nonsingular matrix with continuous real-valued functions as its elements, $A \in \mathscr{R}$, and $a_{ij} \in C(\mathbb{T}, \mathbb{R})$ is Δ-periodic in shifts $\delta_\pm$ with period ω; $b = \operatorname{diag}(b_1, b_2, \ldots, b_n)$ and $b_i \in C(\mathbb{T}, \mathbb{R})$ is Δ-periodic in shifts $\delta_\pm$ with period ω; $f = (f_1, f_2, \ldots, f_n)^T$ and $f_i \in C(\mathbb{T} \times \mathbb{R}^n, \mathbb{R})$ is periodic in shifts $\delta_\pm$ with period ω with respect to the first variable;

$g \in C(\mathbb{T}, \mathbb{T})$ is periodic in shifts $\delta_{\pm}$ with period ω; $x(t_j^+)$ and $x(t_j^-)$ represent the right and the left limit of $x(t_j)$ in the sense of time scales; in addition, if t_j is right-scattered, then $x(t_j^+) = x(t_j)$, whereas if t_j is left-scattered, then $x(t_j^-) = x(t_j)$; $I_j = (I_j^1, I_j^2, \ldots, I_j^n)^T$ and $I_j^i \in C(\mathbb{R}^n, \mathbb{R})$. Assume that there exists a positive constant q such that $t_{j+q} = \delta_+^{\omega}(t_j)$, $I_{j+q} = I_j$, $j \in \mathbb{Z}$. For each interval $\mathbb{I}$ of $\mathbb{R}$, we denote $\mathbb{I}_{\mathbb{T}} = \mathbb{I} \cap \mathbb{T}$; without loss of generality, set $[t_0, \delta_+^{\omega}(t_0))_{\mathbb{T}} \cap \{t_j, j \in \mathbb{Z}\} = \{t_1, t_2, \ldots, t_q\}$.

In [14], Li and Hu studied the existence of positive periodic solutions of system (1) on a periodic time scale $\mathbb{T}$ with $b(t) = 1$. The time scale $\mathbb{T}$ considered in [14] is unbounded above and below. Moreover, the condition (P_4) in [14] is too strict so that it cannot be satisfied even if the coefficient matrix A is a diagonal matrix. Therefore, the results in [14] are less applicable.

The main purpose of this paper is to study the existence and multiplicity of positive periodic solutions in shifts $\delta_{\pm}$ of system (1) under more general assumptions. By using Leggett-Williams fixed point theorem, sufficient conditions for the existence of at least three positive periodic solutions in shifts $\delta_{\pm}$ of system (1) will be established. The results presented in this paper improve and generalize the results in [14].

In this paper, for each $x = (x_1, x_2, \ldots, x_n)^T \in C(\mathbb{T}, \mathbb{R}^n)$, the norm of x is defined as $\|x\| = \sup_{t \in [t_0, \delta_+^{\omega}(t_0)]_{\mathbb{T}}} |x(t)|_0$, where $|x(t)|_0 = \sum_{i=1}^{n} |x_i(t)|$ and when it comes to the fact that x is continuous, delta derivative, delta integrable, and so forth; we mean that each element x_i is continuous, delta derivative, delta integrable, and so forth.

The organization of this paper is as follows. In Section 2, we introduce some notations and definitions and state some preliminary results needed in later sections. Besides, in Section 2, we give some lemmas about the exponential function with shift operators, and Green's function of system (1). In Section 3, we establish our main results for positive periodic solutions in shifts $\delta_{\pm}$ by applying Leggett-Williams fixed point theorem. In Section 4, numerical examples are presented to illustrate that our results are feasible and more general.

2. Preliminaries

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$
$$\mu(t) = \sigma(t) - t. \tag{2}$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$; otherwise, $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$; otherwise, $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}^n$ is right-dense continuous provided that it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be a continuous function on $\mathbb{T}$. The set of continuous functions $f : \mathbb{T} \to \mathbb{R}^n$ will be denoted by $C(\mathbb{T}) = C(\mathbb{T}, \mathbb{R}^n)$.

For the basic theories of calculus on time scales, see [15].

Definition 1 (see [15]). An $n\times n$-matrix-valued function A on a time scale $\mathbb{T}$ is called regressive (with respect to $\mathbb{T}$) provided that

$$I + \mu(t) A(t) \tag{3}$$

is invertible for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$.

Definition 2 (see [15]). Let $t_0 \in \mathbb{T}$ and assume that A is a regressive $n \times n$-matrix-valued function. The unique matrix-valued solution of the IVP is

$$Y^{\Delta} = A(t) Y, \qquad Y(t_0) = I, \tag{4}$$

where I denotes as usual the $n\times n$-identity matrix, is called the matrix exponential function (at t_0), and is denoted by $e_A(\cdot, t_0)$.

Lemma 3 (see [15]). *If A is a regressive $n \times n$-matrix-valued function on $\mathbb{T}$, then*

(i) $e_0(t, s) \equiv I$ *and* $e_A(t, t) \equiv I$;

(ii) $e_A(\sigma(t), s) = (I + \mu(t)A(t))e_A(t, s)$;

(iii) $e_A(t, s) = e_A^{-1}(s, t)$;

(iv) $e_A(t, s)e_A(s, r) = e_A(t, r)$.

Lemma 4 (see [15])). *Let A be a regressive $n\times n$-matrix-valued function on $\mathbb{T}$ and suppose that $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and*

$$y^{\Delta} = A(t) y + f(t), \qquad y(t_0) = y_0, \tag{5}$$

has a unique solution $y : \mathbb{T} \to \mathbb{R}^n$. Moreover, the solution is given by

$$y(t) = e_A(t, t_0) y_0 + \int_{t_0}^{t} e_A(t, \sigma(\tau)) f(\tau) \Delta\tau. \tag{6}$$

The following definitions and lemmas about the shift operators and the new periodicity concept for time scales can be found in [16].

Let $\mathbb{T}^*$ be a nonempty subset of the time scale $\mathbb{T}$ and let $t_0 \in \mathbb{T}^*$ be a fixed number; define operators $\delta_{\pm} : [t_0, \infty) \times \mathbb{T}^* \to \mathbb{T}^*$. The operators δ_+ and δ_- associated with $t_0 \in \mathbb{T}^*$(called the initial point) are said to be forward and backward shift operators on the set $\mathbb{T}^*$, respectively. The variable $s \in [t_0, \infty)_{\mathbb{T}}$ in $\delta_{\pm}(s, t)$ is called the shift size. The values $\delta_+(s, t)$ and $\delta_-(s, t)$ in $\mathbb{T}^*$ indicate s units translation of the term $t \in \mathbb{T}^*$ to the right and left, respectively. The sets

$$\mathbb{D}_{\pm} := \{(s, t) \in [t_0, \infty)_{\mathbb{T}} \times \mathbb{T}^* : \delta_{\mp}(s, t) \in \mathbb{T}^*\} \tag{7}$$

are the domains of the shift operator $\delta_{\pm}$, respectively. Hereafter, $\mathbb{T}^*$ is the largest subset of the time scale $\mathbb{T}$ such that the shift operators $\delta_{\pm} : [t_0, \infty) \times \mathbb{T}^* \to \mathbb{T}^*$ exist.

Definition 5 (see [16], periodicity in shifts $\delta_\pm$). Let $\mathbb{T}$ be a time scale with the shift operators $\delta_\pm$ associated with the initial point $t_0 \in \mathbb{T}^*$. The time scale $\mathbb{T}$ is said to be periodic in shifts $\delta_\pm$ if there exists $p \in (t_0, \infty)_{\mathbb{T}^*}$ such that $(p,t) \in \mathbb{D}_\pm$ for all $t \in \mathbb{T}^*$. Furthermore, if

$$P := \inf\{p \in (t_0,\infty)_{\mathbb{T}^*} : (p,t) \in \delta_\pm, \forall t \in \mathbb{T}^*\} \neq t_0, \quad (8)$$

then P is called the period of the time scale $\mathbb{T}$.

Definition 6 (see [16], periodic function in shifts $\delta_\pm$). Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. We say that a real-valued function f defined on $\mathbb{T}^*$ is periodic in shifts $\delta_\pm$ if there exists $\omega \in [P,\infty)_{\mathbb{T}^*}$ such that $(\omega,t) \in \mathbb{D}_\pm$ and $f(\delta_\pm^\omega(t)) = f(t)$ for all $t \in \mathbb{T}^*$, where $\delta_\pm^\omega := \delta_\pm(\omega,t)$. The smallest number $\omega \in [P,\infty)_{\mathbb{T}^*}$ is called the period of f.

Definition 7 (see [16], Δ-periodic function in shifts $\delta_\pm$). Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. We say that a real-valued function f defined on $\mathbb{T}^*$ is Δ-periodic in shifts $\delta_\pm$ if there exists $\omega \in [P,\infty)_{\mathbb{T}^*}$ such that $(\omega,t) \in \mathbb{D}_\pm$ for all $t \in \mathbb{T}^*$ and the shifts $\delta_\pm^\omega$ are Δ-differentiable with rd-continuous derivatives and $f(\delta_\pm^\omega(t))\delta_\pm^{\Delta\omega}(t) = f(t)$ for all $t \in \mathbb{T}^*$, where $\delta_\pm^\omega := \delta_\pm(\omega,t)$. The smallest number $\omega \in [P,\infty)_{\mathbb{T}^*}$ is called the period of f.

Lemma 8 (see [16]). *Consider $\delta_+^\omega(\sigma(t)) = \sigma(\delta_+^\omega(t))$ and $\delta_-^\omega(\sigma(t)) = \sigma(\delta_-^\omega(t))$ for all $t \in \mathbb{T}^*$.*

Lemma 9 (see [16]). *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P, and let f be a Δ-periodic function in shifts $\delta_\pm$ with the period $\omega \in [P,\infty)_{\mathbb{T}^*}$. Suppose that $f \in C_{rd}(\mathbb{T})$, then*

$$\int_{t_0}^{t} f(s)\,\Delta s = \int_{\delta_\pm^\omega(t_0)}^{\delta_\pm^\omega(t)} f(s)\,\Delta s. \quad (9)$$

Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$. If one takes $\nu(t) = \delta_\pm^\omega(t)$, then one has $\nu(\mathbb{T}) = \mathbb{T}$ and $[f(\nu(t))]^\Delta = (f^\Delta \circ \nu)(t)\nu^\Delta(t)$.

Now, we prove two properties of the exponential functions $e_A(t,t_0)$ and shift operators on time scales.

Lemma 10. *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. Suppose that the shifts $\delta_\pm^\omega$ are Δ-differentiable on $t \in \mathbb{T}^*$, where $\omega \in [P,\infty)_{\mathbb{T}^*}$ and $A \in \mathcal{R}$ is Δ-periodic in shifts $\delta_\pm$ with the period ω. Then*

$$e_A(\delta_\pm^\omega(t), \delta_\pm^\omega(t_0)) = e_A(t,t_0) \quad \text{for } t, t_0 \in \mathbb{T}^*. \quad (10)$$

Proof. Let $Y(t) = F(\delta_\pm^\omega(t))$, where $F(t) = e_A(t, \delta_\pm^\omega(t_0))$, then

$$\begin{aligned} Y^\Delta(t) &= [F(\delta_\pm^\omega(t))]^\Delta = (F^\Delta \circ \delta_\pm^\omega)(t)\,\delta_\pm^{\Delta\omega}(t) \\ &= A(\delta_\pm^\omega(t))\,\delta_\pm^{\Delta\omega}(t)\,e_A(\delta_\pm^\omega(t), \delta_\pm^\omega(t_0)) \\ &= A(t)\,e_A(\delta_\pm^\omega(t), \delta_\pm^\omega(t_0)) \\ &= A(t)\,Y(t), \end{aligned} \quad (11)$$

and $Y(t_0) = e_A(\delta_\pm^\omega(t_0), \delta_\pm^\omega(t_0)) = I$. Hence, Y solves the IVP,

$$Y^\Delta(t) = A(t)\,Y(t), \qquad Y(t_0) = I, \quad (12)$$

which has exactly one solution according to Lemma 4, and therefore we have

$$e_A(\delta_\pm^\omega(t), \delta_\pm^\omega(t_0)) = e_A(t,t_0) \quad \text{for } t, t_0 \in \mathbb{T}^*. \quad (13)$$

This completes the proof. □

Lemma 11. *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. Suppose that the shifts $\delta_\pm^\omega$ are Δ-differentiable on $t \in \mathbb{T}^*$, where $\omega \in [P,\infty)_{\mathbb{T}^*}$ and $A \in \mathcal{R}$ is Δ-periodic in shifts $\delta_\pm$ with the period ω. Then*

$$e_A(\delta_\pm^\omega(t), \sigma(\delta_\pm^\omega(s))) = e_A(t, \sigma(s)) \quad \text{for } t, s \in \mathbb{T}^*. \quad (14)$$

Proof. From Lemma 8, we know $\delta_\pm^\omega(\sigma(t)) = \sigma(\delta_\pm^\omega(t))$. By Lemmas 10 and 3, we can obtain

$$e_A(\delta_\pm^\omega(t), \sigma(\delta_\pm^\omega(s))) = e_A(t, \sigma(s)) \quad \text{for } t, s \in \mathbb{T}^*. \quad (15)$$

This completes the proof. □

Define

$$\begin{aligned} PC(\mathbb{T}) &= \Big\{x = (x_1, x_2, \ldots, x_n)^T : \mathbb{T} \longrightarrow \mathbb{R}^n \,\big|\, x_i|_{(t_j,t_{j+1})} \in C(t_j, t_{j+1}), \\ &\quad \exists x(t_j^-) = x(t_j), x(t_j^+),\ j \in \mathbb{Z},\ i = 1,2,\ldots,n\Big\}. \end{aligned} \quad (16)$$

Set

$$X = \{x(t) : x(t) \in PC(\mathbb{T}), x(\delta_+^\omega(t)) = x(t)\} \quad (17)$$

with the norm defined by $\|x\| = \sup_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} |x(t)|_0$, where $|x(t)|_0 = \sum_{i=1}^n |x_i(t)|$; then X is a Banach space.

Lemma 12. *The function $x \in X$ is an ω-periodic solution in shifts $\delta_\pm$ of system* (1) *if and only if x is an ω-periodic solution in shifts $\delta_\pm$ of*

$$\begin{aligned} x(t) &= \int_t^{\delta_+^\omega(t)} G(t,s)\,b(s)\,f(s, x(g(s)))\,\Delta s \\ &\quad + \sum_{j:t_j\in[t,\delta_+^\omega(t))_{\mathbb{T}}} G(t,t_j)\,e_A(\sigma(t_j), t_j)\,I_j(x(t_j)), \end{aligned} \quad (18)$$

where

$$G(t,s) = [e_A(t_0, \delta_+^\omega(t_0)) - I]^{-1} e_A(t, \sigma(s)) := (G_{ik})_{n\times n}. \quad (19)$$

Proof. If $x(t)$ is an ω-periodic solution in shifts $\delta_\pm$ of system (1), for any $t \in \mathbb{T}$, there exists $j \in \mathbb{Z}$ such that t_j is the first impulsive point after t. By using Lemma 4, for $s \in [t, t_j]_{\mathbb{T}}$, we have

$$x(s) = e_A(s,t)\,x(t) + \int_t^s e_A(s,\sigma(\theta))\,b(\theta)\,f(\theta, x(g(\theta)))\,\Delta\theta. \tag{20}$$

Then

$$x(t_j) = e_A(t_j,t)\,x(t) + \int_t^{t_j} e_A(t_j,\sigma(\theta))\,b(\theta)\,f(\theta, x(g(\theta)))\,\Delta\theta. \tag{21}$$

Again, using Lemma 4 and (21), for $s \in (t_j, t_{j+1}]_{\mathbb{T}}$, then

$$\begin{aligned} x(s) &= e_A(s,t_j)\,x(t_j^+) + \int_{t_j}^s e_A(s,\sigma(\theta))\,b(\theta)\,f(\theta, x(g(\theta)))\,\Delta\theta \\ &= e_A(s,t_j)\,x(t_j) + \int_{t_j}^s e_A(s,\sigma(\theta))\,b(\theta)\,f(\theta, x(g(\theta)))\,\Delta\theta + e_A(s,t_j)\,I_j(x(t_j)) \\ &= e_A(s,t)\,x(t) + \int_t^s e_A(s,\sigma(\theta))\,b(\theta)\,f(\theta, x(g(\theta)))\,\Delta\theta + e_A(s,t_j)\,I_j(x(t_j)). \end{aligned} \tag{22}$$

Repeating the above process for $s \in [t, \delta_+^\omega(t)]_{\mathbb{T}}$, we have

$$x(s) = e_A(s,t)\,x(t) + \int_t^s e_A(s,\sigma(\theta))\,b(\theta)\,f(\theta, x(g(\theta)))\,\Delta\theta + \sum_{j:t_j\in[t,s)_{\mathbb{T}}} e_A(s,t_j)\,I_j(x(t_j)). \tag{23}$$

Let $s = \delta_+^\omega(t)$ in the above equality; we have

$$x(\delta_+^\omega(t)) = e_A(\delta_+^\omega(t),t)\,x(t) + \int_t^{\delta_+^\omega(t)} e_A(\delta_+^\omega(t),\sigma(\theta))\,b(\theta)\,f(\theta, x(g(\theta)))\,\Delta\theta + \sum_{j:t_j\in[t,\delta_+^\omega(t))_{\mathbb{T}}} e_A(\delta_+^\omega(t),t_j)\,I_j(x(t_j)). \tag{24}$$

Noticing that $x(\delta_+^\omega(t)) = x(t)$ and $e_A(t,\delta_+^\omega(t)) = e_A(t_0,\delta_+^\omega(t_0))$, by Lemma 3, then x satisfies (18).

Let x be an ω-periodic solution in shifts $\delta_\pm$ of (18). If $t \neq t_i$, $i \in \mathbb{Z}$, then, by (18) and Lemma 8, we have

$$\begin{aligned} x^\Delta(t) &= A(t)\,x(t) + G(\sigma(t),\delta_+^\omega(t))\,b(\delta_+^\omega(t))\,\delta_+^{\Delta\omega}(t) \times f(\delta_+^\omega(t), x(g(\delta_+^\omega(t)))) - G(\sigma(t),t)\,b(t)\,f(t,x(g(t))) \\ &= A(t)\,x(t) + b(t)\,f(t,x(g(t))). \end{aligned} \tag{25}$$

If $t = t_i$, $i \in \mathbb{Z}$, then, by (18), we have

$$\begin{aligned} x(t_i^+) - x(t_i^-) &= \sum_{j:t_j\in[t_i^+,\delta_+^\omega(t_i^+))_{\mathbb{T}}} G(t_i,t_j)\,e_A(\sigma(t_j),t_j)\,I_j(x(t_j)) - \sum_{j:t_j\in[t_i^-,\delta_+^\omega(t_i^-))_{\mathbb{T}}} G(t_i,t_j)\,e_A(\sigma(t_j),t_j)\,I_j(x(t_j)) \\ &= G(t_i,\delta_+^\omega(t_i))\,e_A(\sigma(\delta_+^\omega(t_i)),\delta_+^\omega(t_i))\,I_i(x(\delta_+^\omega(t_i))) - G(t_i,t_i)\,e_A(\sigma(t_i),t_i)\,I_i(x(t_i)) \\ &= I_i(x(t_i)). \end{aligned} \tag{26}$$

So, x is an ω-periodic solution in shifts $\delta_\pm$ of system (1). This completes the proof. □

By using Lemmas 10 and 11, it is easy to verify that Green's function $G(t,s)$ satisfies

$$G(\delta_+^\omega(t),\delta_+^\omega(s)) = G(t,s), \quad \forall t \in \mathbb{T}^*,\ s \in [t,\delta_+^\omega(t)]_{\mathbb{T}}. \tag{27}$$

For convenience, we introduce the following notations:

$$\begin{aligned} &G(t,s)\,e_A(\sigma(s),s) := E(t,s) = (E_{ik})_{n\times n}, \quad \forall t,s\in\mathbb{T},\ i,k = 1,2,\dots,n; \\ &A_1 := \min_{1\le k\le n}\ \inf_{s,t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} \left|\sum_{i=1}^n G_{ik}(t,s)\right|, \\ &B_1 := \max_{1\le k\le n}\ \sup_{s,t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} \left|\sum_{i=1}^n G_{ik}(t,s)\right|, \\ &A_2 := \min_{1\le k\le n}\ \inf_{s,t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} \left|\sum_{i=1}^n E_{ik}(t,s)\right|, \\ &B_2 := \max_{1\le k\le n}\ \sup_{s,t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} \left|\sum_{i=1}^n E_{ik}(t,s)\right|, \\ &A_3 := \min\{A_1,A_2\}, \qquad B_3 := \max\{B_1,B_2\}. \end{aligned} \tag{28}$$

Hereafter, we assume that

(P_1) $A_3 > 0$, $B_3 > 0$;

(P_2) $G_{ik}b_k f_k \ge 0$, $E_{ik}I_j^k \ge 0$, $\forall i,k = 1,2,\dots,n$, $j \in \mathbb{Z}$.

Let

$$K = \{x \in X : |x(t)|_0 \geq \xi \|x\|, t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}\}, \tag{29}$$

where $\xi = A_3/B_3 \in (0,1)$. Obviously, K is a cone in X.

Define an operator H by

$$(Hx)(t) = \int_t^{\delta_+^\omega(t)} G(t,s)\, b(s)\, f(s, x(g(s)))\,\Delta s + \sum_{j: t_j \in [t, \delta_+^\omega(t))_{\mathbb{T}}} G(t, t_j)\, e_A(\sigma(t_j), t_j)\, I_j(x(t_j)); \tag{30}$$

that is,

$$(Hx)(t) = \int_t^{\delta_+^\omega(t)} G(t,s)\, b(s)\, f(s, x(g(s)))\,\Delta s + \sum_{j: t_j \in [t, \delta_+^\omega(t))_{\mathbb{T}}} E(t, t_j)\, I_j(x(t_j)), \tag{31}$$

for all $x \in K$, $t \in \mathbb{T}$, where $G(t,s)$ is defined by (19), and

$$(Hx)(t) = ((H_1x)(t), (H_2x)(t), \ldots, (H_nx)(t))^T, \tag{32}$$

where

$$(H_ix)(t) = \int_t^{\delta_+^\omega(t)} \sum_{k=1}^n G_{ik} b_k(s)\, f_k(s, x(g(s)))\,\Delta s + \sum_{j: t_j \in [t, \delta_+^\omega(t))_{\mathbb{T}}} \sum_{k=1}^n E_{ik} I_j^k(x(t_j)), \quad i = 1, 2, \ldots, n. \tag{33}$$

In the following, we will give some lemmas concerning K and H defined by (29) and (31), respectively.

Lemma 13. *Assume that (P_1)-(P_2) hold; then $H : K \to K$ is well defined.*

Proof. For any $x \in K$, it is clear that $Hx \in PC(\mathbb{T})$. In view of (31), by Lemma 9 and (27), for $t \in \mathbb{T}$, we obtain

$$\begin{aligned}
&(Hx)(\delta_+^\omega(t)) \\
&= \int_{\delta_+^\omega(t)}^{\delta_+^\omega(\delta_+^\omega(t))} G(\delta_+^\omega(t), s)\, b(s)\, f(s, x(g(s)))\,\Delta s \\
&\quad + \sum_{j: t_j \in [\delta_+^\omega(t), \delta_+^\omega(\delta_+^\omega(t)))_{\mathbb{T}}} G(\delta_+^\omega(t), t_j)\, e_{-a} \times (\sigma(t_j), t_j)\, I_j(x(t_j)) \\
&= \int_t^{\delta_+^\omega(t)} G(\delta_+^\omega(t), \delta_+^\omega(s))\, b(\delta_+^\omega(s))\, \delta_+^{\Delta\omega}(s) \times f(\delta_+^\omega(s), x(g(\delta_+^\omega(s))))\,\Delta s \\
&\quad + \sum_{k: t_k \in [t, \delta_+^\omega(t))_{\mathbb{T}}} G(\delta_+^\omega(t), \delta_+^\omega(t_k))\, e_{-a} \times (\sigma(\delta_+^\omega(t_k)), \delta_+^\omega(t_k))\, I_k(x(\delta_+^\omega(t_k))) \\
&= \int_t^{\delta_+^\omega(t)} G(t,s)\, b(s)\, f(s, x(g(s)))\,\Delta s \\
&\quad + \sum_{k: t_k \in [t, \delta_+^\omega(t))_{\mathbb{T}}} G(t, t_k)\, e_{-a}(\sigma(t_k), t_k)\, I_k(x(t_k)) \\
&= (Hx)(t);
\end{aligned} \tag{34}$$

that is, $Hx \in X$.

Furthermore, for any $x \in K, \forall t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}$, by (P_2), we have

$$\begin{aligned}
&|(Hx)(t)|_0 \\
&= \left| \int_t^{\delta_+^\omega(t)} G(t,s)\, b(s)\, f(s, x(g(s)))\,\Delta s + \sum_{j: t_j \in [t, \delta_+^\omega(t))_{\mathbb{T}}} E(t, t_j)\, I_j(x(t_j)) \right|_0 \\
&= \sum_{i=1}^n \left| \int_t^{\delta_+^\omega(t)} \sum_{k=1}^n G_{ik} b_k(s)\, f_k(s, x(g(s)))\,\Delta s + \sum_{j: t_j \in [t, \delta_+^\omega(t))_{\mathbb{T}}} \sum_{k=1}^n E_{ik} I_j^k(x(t_j)) \right| \\
&\geq A_3 \int_t^{\delta_+^\omega(t)} \sum_{k=1}^n |b_k(s)\, f_k(s, x(g(s)))|\,\Delta s + A_3 \sum_{k=1}^n \sum_{j=1}^q |I_j(x(t_j))| \\
&= A_3 \int_{t_0}^{\delta_+^\omega(t_0)} |b(s)\, f(s, x(g(s)))|_0 \Delta s + A_3 \sum_{j=1}^q |I_j(x(t_j))|_0 \\
&= \frac{A_3}{B_3} \left[B_3 \int_{t_0}^{\delta_+^\omega(t_0)} |b(s)\, f(s, x(g(s)))|_0 \Delta s + B_3 \sum_{j=1}^q |I_j(x(t_j))|_0 \right] \\
&\geq \xi \|Hx\|;
\end{aligned} \tag{35}$$

that is, $Hx \in K$. This completes the proof. □

Define

$$B^m := \min_{1\le i\le n} \int_{t_0}^{\delta_+^\omega(t_0)} |b_i(s)|\,\Delta s,$$
$$B^M := \max_{1\le i\le n} \int_{t_0}^{\delta_+^\omega(t_0)} |b_i(s)|\,\Delta s. \tag{36}$$

Lemma 14. *Assume that* (P_1)*-*(P_2) *hold; then* $H : K \to K$ *is completely continuous.*

Proof. We first show that H is continuous. Because of the continuity of f and I_j, $j \in \mathbb{Z}$, for any $\nu > 0$ and $\varepsilon > 0$, there exists a $\eta > 0$ such that

$$\{\phi, \psi \in C(\mathbb{T}, \mathbb{R}^n), \|\phi\| \le \nu, \|\psi\| \le \nu, \|\phi - \psi\| < \eta\} \tag{37}$$

imply that

$$|f(s, \phi(g(s))) - f(s, \psi(g(s)))|_0 < \frac{\varepsilon}{2B_3B^M},$$
$$|I_j(\phi) - I_j(\psi)|_0 < \frac{\varepsilon}{2B_3q}, \quad j \in \mathbb{Z}. \tag{38}$$

Therefore, if $x, y \in K$ with $\|x\| \le \nu$, $\|y\| \le \nu$, $\|x - y\| < \eta$, then

$$\begin{aligned}
&|(Hx)(t) - (Hy)(t)|_0 \\
&\le \sum_{i=1}^n \Bigg| \int_t^{\delta_+^\omega(t)} \sum_{k=1}^n G_{ik} b_k(s) f_k(s, x(g(s)))\,\Delta s \\
&\qquad + \sum_{j: t_j \in [t, \delta_+^\omega(t))_{\mathbb{T}}} \sum_{k=1}^n E_{ik} I_j^k(x(t_j)) \\
&\qquad - \int_t^{\delta_+^\omega(t)} \sum_{k=1}^n G_{ik} b_k(s) f_k(s, y(g(s)))\,\Delta s \\
&\qquad + \sum_{j: t_j \in [t, \delta_+^\omega(t))_{\mathbb{T}}} \sum_{k=1}^n E_{ik} I_j^k(y(t_j)) \Bigg| \\
&\le \int_t^{\delta_+^\omega(t)} \sum_{k=1}^n \left| \sum_{i=1}^n G_{ik} \right| |b_k(s) f_k(s, x(g(s))) \\
&\qquad - b_k(s) f_k(s, y(g(s)))|\,\Delta s \\
&\quad + \sum_{j: t_j \in [t, \delta_+^\omega(t))_{\mathbb{T}}} \sum_{k=1}^n \left| \sum_{i=1}^n E_{ik} \right| |I_j^k(x(t_j)) - I_j^k(y(t_j))| \\
&< B_3 \Bigg(\int_t^{\delta_+^\omega(t)} |b(s) f(s, x(g(s))) - b(s) f(s, y(g(s)))|_0 \Delta s \\
&\qquad + \sum_{j=1}^p |I_j(x) - I_j(y)|_0 \Bigg) \\
&< B_3 \left(B^M \frac{\varepsilon}{2B_3B^M} + q \frac{\varepsilon}{2B_3q} \right) \\
&= \varepsilon,
\end{aligned} \tag{39}$$

for all $t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}$, which yields

$$\|Hx - Hy\| = \sup_{t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}} |(Hx)(t) - (Hy)(t)|_0 \le \varepsilon; \tag{40}$$

that is, H is continuous.

Next, we show that H maps any bounded sets in K into relatively compact sets. We first prove that f maps bounded sets into bounded sets. Indeed, let $\varepsilon = 1$; for any $\nu > 0$, there exists $\eta > 0$ such that $\{x, y \in K, \|x\| \le \nu, \|y\| \le \nu, \|x - y\| < \eta, s \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}\}$ imply that

$$|f(s, x(g(s))) - f(s, y(g(s)))|_0 < 1,$$
$$|I_j(x(t_j)) - I_j(y(t_j))|_0 < 1, \quad j \in \mathbb{Z}. \tag{41}$$

Choose a positive integer N such that $(\nu/N) < \eta$. Let $x \in K$ and define $x^k(\cdot) = x(\cdot)k/N$, $k = 0, 1, 2, \ldots, N$. If $\|x\| < \nu$, then

$$\begin{aligned}
&\|x^k - x^{k-1}\| \\
&= \sup_{t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}} \left| \frac{x(\cdot)k}{N} - \frac{x(\cdot)(k-1)}{N} \right|_0 \le \|x\| \frac{1}{N} \le \frac{\nu}{N} < \eta.
\end{aligned} \tag{42}$$

So

$$|f(s, x^k(g(s))) - f(s, x^{k-1}(g(s)))|_0 < 1, \tag{43}$$

for all $s \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}$, and

$$|I_j(x^k(t_j)) - I_j(x^k(t_j))|_0 < 1, \quad j \in \mathbb{Z}, \tag{44}$$

and these yield

$$\begin{aligned}
&\left|f(s,x(g(s)))\right|_0 \\
&= \left|f\left(s,x^N(g(s))\right)\right|_0 \\
&\le \sum_{k=1}^{N}\left|f\left(s,x^k(g(s))\right)-f\left(s,x^{k-1}(g(s))\right)\right|_0+\left|f(s,0)\right|_0 \\
&< N+\sup_{s\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}}\left|f(s,0)\right|_0 =: W, \\
&\left|I_j\left(x\left(t_j\right)\right)\right|_0 \\
&\quad= \left|I_j\left(x^N\left(t_j\right)\right)\right|_0 \\
&\quad\le \sum_{k=1}^{N}\left|I_j(x^N(t_j))-I_j(x^{N-1}(t_j))\right|_0+\left|I_j(0)\right|_0 \\
&\quad< N+\max_{1\le j\le q}\left|I_j(0)\right|_0 =: U, \quad j\in\mathbb{Z}.
\end{aligned} \tag{45}$$

It follows from (32) and (45) that, for $t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}$,

$$\|Hx\| = \sup_{t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}}\sum_{i=1}^{n}\left|(H_ix)(t)\right| \le B_3\left(WB^M+pU\right) := D. \tag{46}$$

Finally, for $t\in\mathbb{T}$, we have

$$(Hx)^{\Delta}(t) = A(t)(Hx)(t)+b(t)f(t,x(g(t))). \tag{47}$$

So

$$\begin{aligned}
\left|(Hx)^{\Delta}(t)\right|_0 &= \left|A(t)(Hx)(t)+b(t)f(t,x(g(t)))\right|_0 \\
&\le A^uD+B^uW,
\end{aligned} \tag{48}$$

where $A^u := \max_{1\le j\le n}\sup_{t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}}\sum_{i=1}^{n}|a_{ij}(t)|$, $B^u := \max_{1\le j\le n}\sup_{t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}}|b_i(t)|$.

To sum up, $\{Hx : x\in K, \|x\|\le\nu\}$ is a family of uniformly bounded and equicontinuous functionals on $[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}$. By a theorem of Arzela-Ascoli, we know that the functional H is completely continuous. This completes the proof. □

3. Main Results

In this section, we will state and prove our main results about the existence of at least three positive periodic solutions of system (1) via Leggett-Williams fixed point theorem.

Let X be a Banach space with cone K. A map α is said to be a nonnegative continuous concave functional on K if $\alpha : K\to[0,+\infty)$ is continuous and

$$\begin{aligned}
\alpha(\lambda x+(1-\lambda)y) &\ge \lambda\alpha(x)+(1-\lambda)\alpha(y), \\
&\forall x,y\in K, \quad 0<\lambda<1.
\end{aligned} \tag{49}$$

Let a,b be two numbers such that $0<a<b$ and let α be a nonnegative continuous concave functional on K. We define the following convex sets:

$$\begin{aligned}
K_a &= \{x\in K : \|x\|<a\}, \\
K(\alpha,a,b) &= \{x\in K : a\le\alpha(x), \|x\|\le b\}.
\end{aligned} \tag{50}$$

Lemma 15 (see [17] Leggett-Williams fixed point theorem). *Let $H : \overline{K}_c\to\overline{K}_c$ be completely continuous and let α be a nonnegative continuous concave functional on K such that $\alpha(x)\le\|x\|$ for all $x\in\overline{K}_c$. Suppose that there exist $0<d<a<b\le c$ such that*

(1) *$\{x\in K(\alpha,a,b) : \alpha(x)>a\}\ne\emptyset$ and $\alpha(Hx)>a$ for $x\in K(\alpha,a,b)$;*

(2) *$\|Hx\|<d$ for all $\|x\|\le d$;*

(3) *$\alpha(Hx)>a$ for all $x\in K(\alpha,a,c)$ with $\|H(x)\|>b$.*

Then H has at least three fixed points $x_1,x_2,x_3\in\overline{K}_c$ satisfying $\|x_1\|<d$, $a<\alpha(x_2)$, $\|x_3\|>d$ and $\alpha(x_3)<a$.

For convenience, we introduce the following notations:

$$\begin{aligned}
f^{\vartheta} &:= \limsup_{\|u\|\to\vartheta}\sup_{t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}}\frac{\left|f(t,u)\right|_0}{\|u\|}, \\
I^{\vartheta} &:= \limsup_{\|u\|\to\vartheta}\sum_{j=1}^{q}\frac{\left|I_j(u)\right|_0}{\|u\|}, \\
f_b &:= \min_{\xi b\le|u|_0\le b}\inf_{t\in[t_0,\delta_+^{\omega}(t_0)]_{\mathbb{T}}}\left|f(t,u)\right|_0, \\
I_b &:= \min_{\xi b\le|u|_0\le b}\sum_{j=1}^{q}\left|I_j(u)\right|_0.
\end{aligned} \tag{51}$$

Theorem 16. *Assume that (P_1)-(P_2) hold, and there exists a number $b>0$ such that the following conditions*

(i) $f^0+I^0<1/B_3$, $f^{\infty}+I^{\infty}<1/B_3$,

(ii) $B^mf_b+I_b>\xi b/A_3$ *for* $\xi b\le|u|_0\le b$, $t\in\mathbb{T}$,

hold. Then system (1) *has at least three positive ω-periodic solutions in shifts $\delta_{\pm}$.*

Proof. By the condition $f^{\infty}+I^{\infty}<1/B_3$ of (i), one can find that, for

$$0<\varepsilon<\frac{(1/B_3)-(f^{\infty}+I^{\infty})}{2}, \tag{52}$$

there exists a $c_0>b$ such that

$$\left|f(s,u)\right|_0 \le \frac{f^{\infty}+\varepsilon}{B^M}\|u\|, \qquad \sum_{j=1}^{q}\left|I_j(u)\right|_0 \le (I^{\infty}+\varepsilon)\|u\|, \tag{53}$$

where $\|u\|>c_0$.

Let $c_1 = c_0/\xi$; if $x \in K$, $\|x\| > c_1$, then $\|x\| > c_0$, and we have

$$\begin{aligned} &|(Hx)(t)|_0 \\ &= \left| \int_t^{\delta_+^\omega(t)} G(t,s)\, b(s)\, f(s, x(g(s)))\, \Delta s \right. \\ &\quad \left. + \sum_{j:t_j \in [t,\delta_+^\omega(t))_{\mathbb{T}}} E(t,t_j) I_j(x(t_j)) \right|_0 \\ &\le B_3 \sum_{k=1}^{n} \int_t^{\delta_+^\omega(t)} \left| b_k(s)\, f_k(s, x(g(s))) \right| \Delta s \\ &\quad + B_3 \sum_{k=1}^{n} \sum_{j=1}^{q} \left| I_j(x(t_j)) \right| \\ &\le B_3 B^M \left| f(s, x(g(s))) \right|_0 \\ &\quad + B_3 \sum_{j=1}^{q} \left| I_j(x(t_j)) \right|_0 \\ &\le B_3 \left[f^\infty + I^\infty + 2\varepsilon \right] \|x\| \\ &< \|x\| . \end{aligned} \tag{54}$$

Take $k_{c_1} = \{x \mid x \in K, \|x\| \le c_1\}$; then the set k_{c_1} is a bounded set. According to the fact that H is completely continuous, then H maps bounded sets into bounded sets and there exists a number c_2 such that

$$\|Hx\| \le c_2, \quad \forall x \in k_{c_1}. \tag{55}$$

If $c_2 \le c_1$, we deduce that $H : k_{c_1} \to k_{c_1}$ is completely continuous. If $c_2 < c_1$, then, from (54), we know that for any $x \in k_{c_2} \setminus k_{c_1}$ and $\|Hx\| < \|x\| < c_2$ hold. Thus we have $H : k_{c_2} \to k_{c_2}$ is completely continuous. Now, take $c = \max\{c_1, c_2\}$; then $c > b$, so $H : k_c \to k_c$ is completely continuous.

Denote the positive continuous concave functional $\alpha(x)$ as $\alpha(x) = \inf_{t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}} |x(t)|_0$. Firstly, let $a = \xi b$ and take $x \equiv (a+b)/2$, $x \in K(\alpha, a, b)$, $\alpha(x) > a$, then the set $\{x \in K(\alpha, a, b)\} \ne \emptyset$. By (ii), if $x \in K(\alpha, a, b)$, then $\alpha(x) \ge a$, and we have

$$\begin{aligned} &\alpha(Hx) \\ &= \inf_{t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}} |(Hx)(t)|_0 \\ &= \inf_{t \in [t_0, \delta_+^\omega(t_0)]_{\mathbb{T}}} \left| \int_t^{\delta_+^\omega(t)} G(t,s)\, b(s)\, f(s, x(g(s)))\, \Delta s \right. \\ &\quad \left. + \sum_{j:t_j \in [t,\delta_+^\omega(t))_{\mathbb{T}}} E(t,t_j) I_j(x(t_j)) \right|_0 \\ &\ge A_3 \sum_{k=1}^{n} \int_t^{\delta_+^\omega(t)} \left| b_k(s)\, f_k(s, x(g(s))) \right| \Delta s \\ &\quad + A_3 \sum_{k=1}^{n} \sum_{j=1}^{q} \left| I_j(x(t_j)) \right| \\ &\ge A_3 B^m \left| f(s, x(g(s))) \right|_0 \\ &\quad + A_3 \sum_{j=1}^{q} \left| I_j(x(t_j)) \right|_0 \\ &\ge A_3 (B^m f_b + I_b) \\ &> A_3 \frac{\xi b}{A_3} = a. \end{aligned} \tag{56}$$

Hence, condition (1) of Lemma 15 holds.

Secondly, by the condition $f^0 + I^0 < 1/B_3$ of (i), one can find that, for

$$0 < \varepsilon < \frac{(1/B_3) - (f^0 + I^0)}{2}, \tag{57}$$

there exists a d $(0 < d < a)$ such that

$$\begin{aligned} |f(s,u)|_0 &\le \frac{f^0 + \varepsilon}{B^M} \|u\| , \\ \sum_{j=1}^{q} |I_j(u)|_0 &\le (I^0 + \varepsilon) \|u\| , \end{aligned} \tag{58}$$

where $0 \le \|u\| \le d$. If $x \in K_d = \{x \mid \|x\| \le d\}$, we have

$$\begin{aligned} |(Hx)(t)|_0 &= \left| \int_t^{\delta_+^\omega(t)} G(t,s)\, b(s)\, f(s, x(g(s)))\, \Delta s \right. \\ &\quad \left. + \sum_{j:t_j \in [t,\delta_+^\omega(t))_{\mathbb{T}}} E(t,t_j)\, I_j(x(t_j)) \right|_0 \\ &\le B_3 \sum_{k=1}^{n} \int_t^{\delta_+^\omega(t)} \left| b_k(s)\, f_k(s, x(g(s))) \right| \Delta s \\ &\quad + B_3 \sum_{k=1}^{n} \sum_{j=1}^{q} \left| I_j(x(t_j)) \right| \\ &\le B_3 B^M \left| f(s, x(g(s))) \right|_0 \\ &\quad + B_3 \sum_{j=1}^{q} \left| I_j(x(t_j)) \right|_0 \\ &\le B_3 \left[f^0 + I^0 + 2\varepsilon \right] \|x\| \\ &< \|x\| \le d; \end{aligned} \tag{59}$$

that is, condition (2) of Lemma 15 holds.

Finally, if $x \in K(\alpha, a, c)$ with $\|Hx\| > b$, by the definition of the cone K, we have

$$\begin{aligned}\alpha(Hx) &= \inf_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} |(Hx)(t)|_0 \\ &\geq \inf_{t\in[t_0,\delta_+^\omega(t_0)]_{\mathbb{T}}} \xi\|Hx\| > \xi b = a,\end{aligned} \quad (60)$$

which implies that condition (3) of Lemma 15 holds.

To sum up, all conditions in Lemma 15 hold. By Lemma 15, the operator H has at least three fixed points in $\overline{K}_c$. Therefore, system (1) has at least three positive ω-periodic solutions in shifts $\delta_\pm$, and

$$\begin{aligned}&x_1 \in K_d, \qquad x_2 \in \{x \in K(\alpha, a, c), \alpha(x) > a\}, \\ &x_3 \in \overline{K}_c \setminus \alpha\left(K(\alpha, a, c) \cup \overline{K}_d\right).\end{aligned} \quad (61)$$

This completes the proof. □

Corollary 17. *Using the following*

(i^*) $f^0 = 0$, $I^0 = 0$, $f^\infty = 0$, $I^\infty = 0$,

instead of (i) *in Theorem 16, the conclusion of Theorem 16 remains true.*

4. Numerical Examples

Consider the following system with impulses on time scales:

$$\begin{aligned}&x^\Delta(t) = A(t)x(t) + b(s)f(t, x(g(t))), \quad t \neq t_j,\ t \in \mathbb{T}, \\ &x(t_j^+) = x(t_j^-) + I_j(x(t_j)).\end{aligned} \quad (62)$$

Example 1. Let

$$\begin{aligned}&A(t) = \begin{bmatrix} -1.5 & 1 \\ 1 & -1.5 \end{bmatrix}, \\ &b(t) = \operatorname{diag}(1 - 0.5\sin 4\pi t, 1 - 0.5\sin 4\pi t), \\ &f(t, x(g(t))) = \begin{bmatrix} |x(t)|_0 (0.05 - 0.03|\sin 2\pi t|) \\ (|x(t)|_0)^2 e^{-0.01|x(t)|_0} \end{bmatrix}, \\ &I_j^i(x(t_j)) = 0.01\left|\sin\left(\left|x(t_j)\right|_0\right)\right|, \\ &i = 1, 2, \quad j = 1, 2, \ldots, 10,\end{aligned} \quad (63)$$

in system (62), where $|x(t)|_0 = |x_1(t)| + |x_2(t)|$. Then

$$\begin{aligned}e_A(t, t_0) &= e_{-0.5}(t, t_0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &+ e_{-0.5}(t, t_0)\int_{t_0}^{t} \frac{1}{1 - 2.5\mu(s)}\Delta s \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.\end{aligned} \quad (64)$$

Case 1. $\mathbb{T} = \mathbb{R}$, and $\omega = 0.5$. Let $t_0 = 0$; then $\delta_+^\omega(t) = t + 0.5$. It is easy to verify that $A(t)$, $b(t)$, and $f(t, x)$ satisfy

$$\begin{aligned}&A(\delta_+^\omega(t))\delta_+^{\Delta\omega}(t) = A(t), \qquad b(\delta_+^\omega(t))\delta_+^{\Delta\omega}(t) = b(t), \\ &f(\delta_+^\omega(t), x) = f(t, x), \quad \forall t \in \mathbb{T}^*,\end{aligned} \quad (65)$$

and $A \in \mathscr{R}$. By a direct calculation, we can get

$$\begin{aligned}&e_A(t, s) = e^{-0.5(t-s)}\begin{bmatrix} 1 - 2(t-s) & (t-s) \\ (t-s) & 1 - 2(t-s) \end{bmatrix}, \\ &G(t, s) \\ &= E(t, s) = (e_A(0, 0.5) - I)^{-1} e_A(t, s) \\ &= e^{-0.5(t-s)} \\ &\times \begin{bmatrix} 0.7662 - 1.2187(t-s) & 0.3137 + 0.1388(t-s) \\ 0.3137 + 0.1388(t-s) & 0.7662 - 1.2187(t-s) \end{bmatrix}.\end{aligned} \quad (66)$$

Since $s \in [t, \delta_+^\omega(t)]_{\mathbb{T}} = [t, t + 0.5]$, $t - s \in [-0.5, 0]$. Then

$$\begin{aligned}&A_3 = 1.0779, \qquad B_3 = 2.0800, \\ &\xi = 0.5192, \qquad B^m = 0.5.\end{aligned} \quad (67)$$

From the above, we can see that conditions (P_1) and (P_2) hold.

Let $b = 10$; then

(i) $f^0 + I^0 = 0.18 < 0.4808 = 1/B_3$, $f^\infty + I^\infty = 0.08 < 0.4808 = 1/B_3$;

(ii) $B^m f_b + I_b = 12.8474 > 4.8080 = \xi b/A_3$ for $5.1920 \leq |x|_0 \leq 10$, $t \in \mathbb{T}$.

According to Theorem 16, when $\mathbb{T} = \mathbb{R}$, system (62) has at least three positive ω-periodic solutions in shifts $\delta_\pm$.

Case 2. $\mathbb{T} = \mathbb{Z}$, and $\omega = 0.5$. Let $t_0 = 0$; then $\delta_+^\omega(t) = t + 0.5$. It is easy to verify that $A(t)$, $b(t)$, and $f(t, x)$ satisfy

$$\begin{aligned}&A(\delta_+^\omega(t))\delta_+^{\Delta\omega}(t) = A(t), \qquad b(\delta_+^\omega(t))\delta_+^{\Delta\omega}(t) = b(t), \\ &f(\delta_+^\omega(t), x) = f(t, x), \quad \forall t \in \mathbb{T}^*,\end{aligned} \quad (68)$$

and $A \in \mathscr{R}$. By a direct calculation, we can get

$$\begin{aligned}&e_A(t, s) \\ &= \left(\frac{1}{2}\right)^{(t-s)}\begin{bmatrix} 1 - \dfrac{4(t-s)}{3} & \dfrac{2(t-s)}{3} \\ \dfrac{2(t-s)}{3} & 1 - \dfrac{4(t-s)}{3} \end{bmatrix}, \\ &G(t, s) \\ &= (e_A(0, \omega) - I)^{-1} e_A(t, s)(I + A)^{-1}\end{aligned}$$

$$= \left(\frac{1}{2}\right)^{(t-s)} \times \begin{bmatrix} 0.9468 - 0.3882(t-s) & 1.3114 - 1.1174(t-s) \\ 1.3114 - 1.1174(t-s) & 0.9468 - 0.3882(t-s) \end{bmatrix},$$

$$E(t,s) = (e_A(0,\omega) - I)^{-1} e_A(t,s) = \left(\frac{1}{2}\right)^{(t-s)} \times \begin{bmatrix} 0.8380 - 0.9233(t-s) & 0.2911 + 0.1705(t-s) \\ 0.2911 + 0.1705(t-s) & 0.8380 - 0.9233(t-s) \end{bmatrix}. \quad (69)$$

Since $s \in [t, \delta_+^\omega(t)]_{\mathbb{T}} = [t, t+0.5]$, $t - s \in [-0.5, 0]$. Then

$$A_3 = 1.1291, \quad B_3 = 4.2582, \quad \xi = 0.2652, \quad B^m = 0.5. \quad (70)$$

From the above, we can see that conditions (P_1) and (P_2) hold.

Let $b = 10$; then

(i) $f^0 + I^0 = 0.18 < 0.2348 = 1/B_3$, $f^\infty + I^\infty = 0.08 < 0.2348 = 1/B_3$;

(ii) $B^m f_b + I_b = 3.4510 > 2.3488 = \xi b / A_3$ for $2.6520 \le |x|_0 \le 10$, $t \in \mathbb{T}$.

According to Theorem 16, when $\mathbb{T} = \mathbb{Z}$, system (62) has at least three positive ω-periodic solutions in shifts $\delta_\pm$.

Example 2. Let

$$A(t) = \begin{bmatrix} -\frac{1}{5t} & 0 \\ 0 & -\frac{1}{6t} \end{bmatrix}, \quad b(t) = \frac{1}{2t},$$

$$f(t, x(g(t))) = \begin{bmatrix} |x(t)|_0 (0.15 - 0.05|\sin 2\pi t|) \\ (|x(t)|_0)^2 e^{-0.01|x(t)|_0} \end{bmatrix}, \quad (71)$$

$$I_j^i(x(t_j)) = 0.01 \left|\sin\left(\left|x(t_j)\right|_0\right)\right|, \quad i = 1,2, \ j = 1,2,\ldots,10,$$

in system (62), where $|x(t)|_0 = |x_1(t)| + |x_2(t)|$.

Let $\mathbb{T} = 2^{\mathbb{N}_0}$, $t_0 = 1$, and $\omega = 4$; then $\delta_+^\omega(t) = 4t$. It is easy to verify that $A(t)$, $b(t)$, and $f(t,x)$ satisfy

$$A(\delta_+^\omega(t))\delta_+^{\Delta\omega}(t) = A(t), \quad b(\delta_+^\omega(t))\delta_+^{\Delta\omega}(t) = b(t),$$
$$f(\delta_+^\omega(t), x) = f(t,x), \quad \forall t \in \mathbb{T}^*, \quad (72)$$

and $A \in \mathscr{R}^+$. By a direct calculation, we can get

$$e_A(t,s) = \begin{bmatrix} e_{a_{11}}(t,s) & 0 \\ 0 & e_{a_{22}}(t,s) \end{bmatrix},$$
$$a_{11}(t) = -\frac{1}{5t}, \quad a_{22}(t) = -\frac{1}{6t},$$
$$G(t,s) = (e_A(1,4) - I)^{-1} e_A(t,s)(I + \mu(t)A)^{-1} = \begin{bmatrix} \frac{15}{13}e_{a_{11}}(t,s) & 0 \\ 0 & \frac{3}{2}e_{a_{22}}(t,s) \end{bmatrix}, \quad (73)$$
$$E(t,s) = (e_A(1,4) - I)^{-1} e_A(t,s) = \begin{bmatrix} \frac{12}{13}e_{a_{11}}(t,s) & 0 \\ 0 & \frac{5}{4}e_{a_{22}}(t,s) \end{bmatrix}.$$

Since $1 + \mu(t)a_{11}(t) = 4/5 > 0$, $1 + \mu(t)a_{22}(t) = 5/6 > 0$, then $e_{a_{11}}(t,s) > 0$, $e_{a_{22}}(t,s) > 0$, $\forall s \in [t, \delta_+^\omega(t)]_{\mathbb{T}}$. Moreover, we have

$$A_3 = 1.9230, \quad B_3 = 2.7, \quad \xi = 0.7122, \quad B^m = 1. \quad (74)$$

From the above, we can see that conditions (P_1) and (P_2) hold.

Let $b = 10$; then

(i) $f^0 + I^0 = 0.3 < 0.3704 = 1/B_3$, $f^\infty + I^\infty = 0.2 < 0.3704 = 1/B_3$;

(ii) $B^m f_b + I_b = 47.9482 > 3.7036 = \xi b / A_3$ for $7.1220 \le |x|_0 \le 10$, $t \in \mathbb{T}$.

According to Theorem 16, when $\mathbb{T} = 2^{\mathbb{N}_0}$, system (62) has at least three positive ω-periodic solutions in shifts $\delta_\pm$.

Remark 3. From Examples 1 and 2, we can see that the results obtained in this paper can be applied to systems on more general time scales, and not only time scales are unbounded above and below.

Remark 4. In system (62), if $A(t)$ is a diagonal matrix, a similar calculation in Example 2 shows that $G_{ij} = G_{ji} = 0$, $E_{ij} = E_{ji} = 0$, $i \ne j$, and the condition (P_4) in [14] cannot be satisfied. So the main results in [14] cannot ensure the existence of positive periodic solution of system (62), while $A(t)$ is a diagonal matrix. Therefore, our main results improve and generalize the results in [14].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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