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Research Article

Richardson Cascadic Multigrid Method for 2D Poisson Equation Based on a Fourth Order Compact Scheme

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Based on a fourth order compact difference scheme, a Richardson cascadic multigrid (RCMG) method for 2D Poisson equation is proposed, in which the an initial value on the each grid level is given by the Richardson extrapolation technique (Wang and Zhang (2009)) and a cubic interpolation operator. The numerical experiments show that the new method is of higher accuracy and less computation time.

1. Introduction

Poisson equation is a partial differential equation (PDE) with broad applications in theoretical physics, mechanical engineering and other fields, such as groundwater flow [1, 2], fluid pressure prediction [3], electromagnetics [4], semiconductor modeling [5], and electrical power network modeling [6].

We consider the following two-dimensional (2D) Poisson equation:

$$-\frac{\partial^2 u(x,y)}{\partial x^2} - \frac{\partial^2 u(x,y)}{\partial y^2} = f(x,y), \quad \text{in } \Omega, \qquad u(x,y) = 0, \quad \text{on } \partial\Omega, \tag{1}$$

where $\Omega \in R^2$ is a rectangular domain or union of rectangular domains with Dirichlet boundary $\partial\Omega$. The solution $u(x,y)$ and the forcing function $f(x,y)$ are assumed to be sufficiently smooth.

Multigrid (MG) method is one of the most effective algorithms to solve the large scale problem. In 1996, cascadic multigrid (CMG) method proposed by Bornemann and Deuflhard [7] and then analyzed by Shi et al. (see [8–11]) and Shaidurov (see [12]). In the recent years, there have been several theoretical analyses and the applications of these methods for the plate bending problems (see [13]), the parabolic problems (see [10]), the nonlinear problems (see [14, 15]), and the Stokes problems (see [16]). In order to improve the efficiency of the CMG, some new extrapolation formulas and extrapolation cascadic multigrid (EXCMG) methods are proposed by Chen et al. (see [17–20]). These new methods can provide a better initial value for smoothing operator on the refined grid level to accelerate their convergence rate.

Based on the Richardson extrapolation technique, Wang and Zhang [21] presented a multiscale multigrid algorithm. Numerical experiments show that the new method is of higher accuracy solution and higher efficiency.

In this paper, in order to develop a more efficient CMG method, we use the Richardson extrapolation technique presented in [21] and a new extrapolation formula; a new Richardson extrapolation cascadic multigrid (RCMG) method for 2D Poisson equation is proposed.

The sections are arranged as follows: the fourth order compact difference scheme and Richardson extrapolation technique are given in Section 2. Chen's new extrapolation formula and EXCMG method are introduced in Section 3. In Section 4, we present the RCMG method. In Section 5, the numerical experiments show the effectiveness of the new method.

2. Fourth Order Compact Difference Scheme and Richardson Extrapolation Technique

For convenience, we consider the rectangular domain $\Omega = [0, L_x] \times [0, L_y]$. We discretize Ω with uniform mesh sizes $h_x = L_x/N_x$ and $h_y = L_y/N_y$ in the x and y coordinate directions. The mesh points are (x_i, y_j) with $x_i = ih_x$ and $y_j = jh_y$, and $0 \leq i \leq N_x$, $0 \leq j \leq N_y$. Let's denote the mesh aspect ratio $\gamma = h_x/h_y$, and $u_{i,j}$ be the solution at the grid point (x_i, y_j), we can rewrite the fourth order compact difference scheme of (1) into the following form [22]:

$$\begin{aligned} &au_{ij} + b\left(u_{i+1,j} + u_{i-1,j}\right) + c\left(u_{i,j+1} + u_{i,j-1}\right) \\ &\quad + d\left(u_{i+1,j+1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i-1,j-1}\right) \\ &= \frac{h_x^2}{2}\left(8f_{i,j} + f_{i+1,j} + f_{i-1,j} + f_{i,j} + f_{i,j-1}\right). \end{aligned} \quad (2)$$

The coefficients in (2) are

$$\begin{aligned} a &= -10\left(1+\gamma^2\right), \qquad b = 5-\gamma^2, \\ c &= 5\gamma^2 - 1, \qquad d = \frac{\left(1+\gamma^2\right)}{2}. \end{aligned} \quad (3)$$

If the domain Ω is subdivided into a sequence of grids Z_{lh} (or Z_l), $l = 0, 1, 2, \ldots, L$ with step length $h_l = h/2^l = h_{l,x} = h_{l,y}$ (namely, $\gamma = 1$), by using the fourth order compact difference scheme (see (2)), a series of linear equations of the model problem (1) are given as follows

$$A^l u^l = F^l, \quad l = 0, 1, 2, \ldots, L. \quad (4)$$

Assume the fourth order accurate solutions $u_{i,j}^{2h}$ and $u_{i,j}^{h}$ on the Z_{2h} grid and the Z_h grid are given, respectively (Figure 1). In 2009, Wang and Zhang [21] applied the Richardson extrapolation (where $p = 4$)

$$\tilde{u}_{i,j}^{2h} = \frac{\left(2^p u_{2i,2j}^{h} - u_{i,j}^{2h}\right)}{2^p - 1} = \frac{\left(16 u_{2i,2j}^{h} - u_{i,j}^{2h}\right)}{15} \quad (5)$$

to get a sixth order accurate solution $\tilde{u}_{i,j}^{2h}$ on Z_{2h}.

The above extrapolation operator is rewritten as the following iterative operator RET.

Algorithm 1. Consider $\tilde{u}^{h,\text{new}} \leftarrow \text{RET}(\tilde{u}^h, \tilde{u}^{2h}, \varepsilon, k^{\max})$.

Step 1. Set $\tilde{u}^{h,\text{old}} := \tilde{u}^h$, $k := 0$.

Step 2. Update every (even, even) grid point on Z_h by Richardson extrapolation formula (see (5)); then use direct interpolation to get $\tilde{u}_{2i,2j}^{h,\text{new}} \in Z_h$. Consider

$$\tilde{u}_{2i,2j}^{h,\text{new}} := \frac{\left(16 u_{2i,2j}^{h,\text{old}} - u_{i,j}^{2h}\right)}{15}. \quad (6)$$

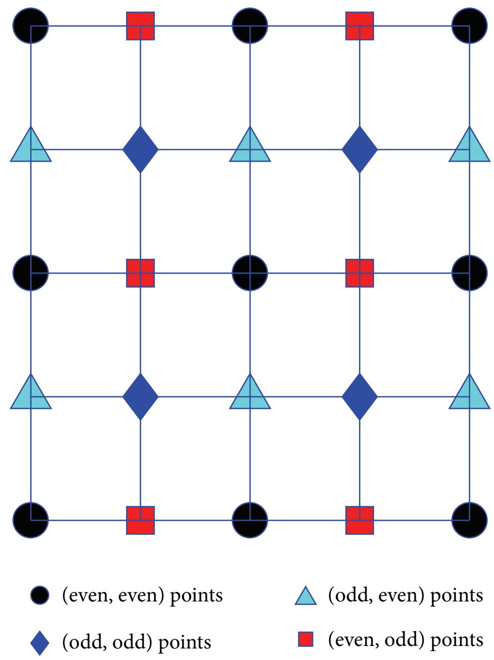

Figure 1: Four types of points on 4×4 grid.

Step 3. Update every (odd, odd) grid point on Z_h. From (2), for each (odd, odd) point (i, j), the updated solution is

$$\begin{aligned} \tilde{u}_{i,j}^{h,\text{new}} := \frac{1}{a}\Big[&F_{i,j} - b\left(\tilde{u}_{i+1,j}^{h,\text{old}} + \tilde{u}_{i-1,j}^{h,\text{old}}\right) \\ &- c\left(\tilde{u}_{i,j+1}^{h,\text{old}} + \tilde{u}_{i,j-1}^{h,\text{old}}\right) \\ &- d\left(\tilde{u}_{i+1,j+1}^{h,\text{new}} + \tilde{u}_{i+1,j-1}^{h,\text{new}} + \tilde{u}_{i-1,j+1}^{h,\text{new}} + \tilde{u}_{i-1,j-1}^{h,\text{new}}\right)\Big]. \end{aligned} \quad (7)$$

Here, $F_{i,j}$ represents the right-hand side part of (2).

Step 4. Update every (odd, even) grid point on Z_h. From (2), for each (odd, even) grid point, the updated value is

$$\begin{aligned} \tilde{u}_{i,j}^{h,\text{new}} := \frac{1}{a}\Big[&F_{i,j} - b\left(\tilde{u}_{i+1,j}^{h,\text{new}} + \tilde{u}_{i-1,j}^{h,\text{new}}\right) \\ &- c\left(\tilde{u}_{i,j+1}^{h,\text{new}} + \tilde{u}_{i,j-1}^{h,\text{new}}\right) \\ &- d\left(\tilde{u}_{i+1,j+1}^{h,\text{old}} + \tilde{u}_{i+1,j-1}^{h,\text{old}} + \tilde{u}_{i-1,j+1}^{h,\text{old}} + \tilde{u}_{i-1,j-1}^{h,\text{old}}\right)\Big]. \end{aligned} \quad (8)$$

Step 5. Update every (even, odd) grid point on Z_h. From (2), the idea is similar to the (odd, even) grid point. Let $k := k+1$.

Step 6. If $\|\tilde{u}^{h,\text{new}} - \tilde{u}^{h,\text{old}}\| \leq \varepsilon$ or $k = k^{\max}$, stop. Else, let $\tilde{u}^{h,\text{old}} := \tilde{u}^{h,\text{new}}$ and return to Step 3.

3. New Extrapolation Formula and EXCMG Method

Based on an asymptotic expansion of finite element method, a new extrapolation formula and an extrapolation cascadic multigrid (EXCMG) method are proposed by Chen et al. (see [17–20]). The numerical experiments show that the EXCMG method is of high accuracy and efficiency. Now we rewrite the new extrapolation formula as follows.

$$\begin{aligned}
\mathrm{Ex}\widetilde{u}^{h}_{2i,2j} &:= \frac{\left(5u^{h}_{2i,2j} - u^{2h}_{i,j}\right)}{4},\\
\mathrm{Ex}\widetilde{u}^{h}_{2i+1,j} &:= u^{h}_{2i+1,j} + \frac{\left[\left(u^{h}_{2i,2j} - u^{2h}_{i,j}\right) + \left(u^{h}_{2i+2,2j} - u^{2h}_{i+1,j}\right)\right]}{8},\\
\mathrm{Ex}\widetilde{u}^{h}_{2i,2j+1} &:= u^{h}_{2i,2j+1} + \frac{\left[\left(u^{h}_{2i,2j} - u^{2h}_{i,j}\right) + \left(u^{h}_{2i,2j+2} - u^{2h}_{i,j+1}\right)\right]}{8},\\
\mathrm{Ex}\widetilde{u}^{h}_{2i+1,2j+1} &:= u^{h}_{2i+1,2j+1} + \left[\left(u^{h}_{2i,2j} - u^{2h}_{i,j}\right) + \left(u^{h}_{2i+2,2j} - u^{2h}_{i+1,j}\right)\right.\\
&\quad + \left(u^{h}_{2i,2j+2} - u^{2h}_{i,j+1}\right)\\
&\quad \left. + \left(u^{h}_{2i+2,2j+2} - u^{2h}_{i+1,j+1}\right)\right]\\
&\quad \times 16^{-1}.
\end{aligned} \tag{9}$$

Let us denote the above new extrapolation formula by operator

$$\mathrm{Ex}\widetilde{u}^{h} := F\left(u^{2h}, u^{h}\right). \tag{10}$$

Now let $\overline{u}^{i}$, on Z_i, $i = 0, 1$ denote the exact solutions, the EXCMG method is as following:

Algorithm 2 (EXCMG). For $l = 2, \ldots, L$, consider the following

Step 1. Extrapolate by using the new extrapolation formula (see (10))

$$\mathrm{Ex}\widetilde{u}^{l-1} := F\left(\overline{u}^{l-2}, \overline{u}^{l-1}\right). \tag{11}$$

Step 2. Compute the initial value

$$u^{l,0} := I_2 \mathrm{Ex}\widetilde{u}^{l-1} \tag{12}$$

on Z_l by using quadratic interpolation operator I_2.
Step 3. Smooth m_l times to get the iterative solution

$$\overline{u}^{l} := S_l^{m_l} u^{l,0} \tag{13}$$

on Z_l by using some classical iterative operator S_l.

Step 4. Return to Step 1 if $l < L$, until you get the final iterative solution $\overline{u}^{L}$ on the finest grid Z_L.

4. Richardson Cascadic Multigrid Method

One of the main tasks in cascadic multigrid method is constructing a suitable interpolation. Based on a new extrapolation-interpolation formula, Chen [17–20] proposed the following extrapolation cascadic multigrid (EXCMG) method, in which the new extrapolation and quadratic interpolation are used to provide a better initial value on refined grid.

In this section, we use RET operator and a cubic interpolation to interpolate the initial guess $\widetilde{u}^{l,0}$ on the refined grid Z_{lh}. Then a classical iterative operator (such as conjugate gradient method) is used as a smoothing operator to compute the high accuracy solution on the fine grid Z_{lh}. Similar to the standard CMG method, we propose the following Richardson cascadic multigrid (RCMG) method.

Algorithm 3 (RCMG).

Step 1. Exactly solve the equation $A^l u^l = F^l$ on coarsest grid Z_l, $l = 1, 2$.

Step 2. Run Algorithm 1; we have

$$\overline{u}^{l} = \mathrm{RET}\left(u^{l}, u^{l-1}, \varepsilon, k_l^{\max}\right). \tag{14}$$

Step 3. Use a cubic interpolation operator I_3 to have the initial value

$$u^{l+1,0} := I_3 \overline{u}^{l} \tag{15}$$

on the gird level Z_{l+1}.

Step 4. Smoothing w_l times by using the classical iterative operator S_l,

$$u^{l+1} := S_l^{w_l} u^{l+l,0} \tag{16}$$

on the level Z_{l+1}. Set $l := l + 1$;

Step 5. Return to Step 2, if $l < L$.
The difference between RCMG method and EXCMG method is that

$$\begin{aligned}
\mathrm{RCMG} &= \mathrm{RET} + \text{cubic interpolation}\\
&\quad + \text{classical iterative operator} + \mathrm{CMG},\\
\mathrm{EXCMG} &= \text{new extrapolation} + \text{quadratic interpolation}\\
&\quad + \text{classical iterative operator} + \mathrm{CMG}.
\end{aligned} \tag{17}$$

5. Numerical Experiment and Comparison

Numerical experiments are conducted to solve a 2D Poisson equation (1) on the unit square domain $[0, 1] \times [0, 1]$.

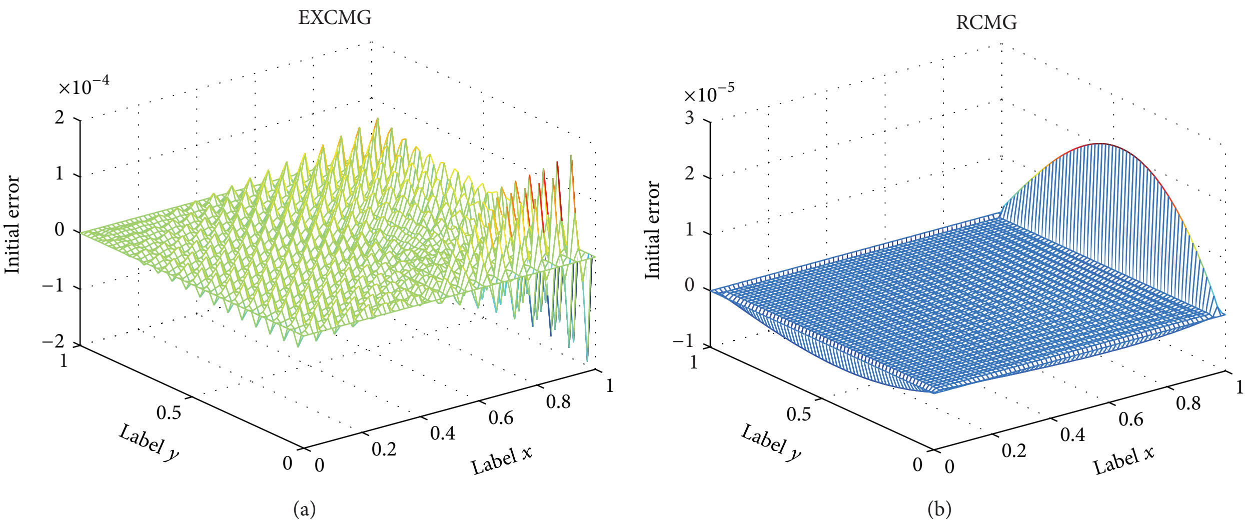

FIGURE 2: Example 4, grid 64×64, initial error of EXCMG ((a) scale 10^{-4}) and RCMG ((b) scale 10^{-5}).

Example 4. The exact solution $u = \sin(y)(1-e^x)(1-x^2)(1-y^2)$; the forcing function

$$\begin{aligned} f &= 2\sin(y)(e^x-1)(x^2-1) + 2\sin(y)(e^x-1)(y^2-1) \\ &\quad - \sin(y)(e^x-1)(x^2-1)(y^2-1) \\ &\quad + 4y\cos(y)(e^x-1)(x^2-1) \\ &\quad + 4xe^x\sin(y)(y^2-1) + e^x\sin(y)(x^2-1)(y^2-1). \end{aligned} \tag{18}$$

Example 5. The exact solution $u = \ln(1+\sin(\pi x^2))(\cos(\sin(x))-1)\sin(\pi y)$; the forcing function

$$\begin{aligned} f &= \pi^2\sin(\pi y)\log(\sin(\pi x^2)+1)(\cos(\sin(x))-1) \\ &\quad - \sin(\sin(x))\sin(\pi y)\log(\sin(\pi x^2)+1)\sin(x) \\ &\quad + \cos(\sin(x))\sin(\pi y)\log(\sin(\pi x^2)+1)\cos^2(x) \\ &\quad - \frac{2\pi\sin(\pi y)\cos(\pi x^2)(\cos(\sin(x))-1)}{\sin(\pi x^2)+1} \\ &\quad + \frac{(4\pi^2x^2\sin(\pi y)\cos^2(\pi x^2)(\cos(\sin(x))-1))}{(\sin(\pi x^2)+1)^2} \\ &\quad + \frac{4\pi^2x^2\sin(\pi y)\sin(\pi x^2)(\cos(\sin(x))-1)}{\sin(\pi x^2)+1} \\ &\quad + \frac{4\pi x\sin(\sin(x))\sin(\pi y)\cos(x)\cos(\pi x^2)}{\sin(\pi x^2)+1}. \end{aligned} \tag{19}$$

We use the conjugate gradient (CG) method as a smoothing iterative operator S in EXCMG method and RCMG method. In EXCMG method, the number of iterations $\widehat{m}_l$ on each grid level has to increase from finer to coarser grids; in this paper let $\widehat{m}_l = 8 \times 2^{L-l+1}$. And in RCMG, we set the number of iteration $k_l^{\max}$ (Step 2) and w_l (Step 4) be $8 \times 2^{L-l}$. We set $\varepsilon = 10^{-8}$ of RET in the RCMG method (on Step 2).

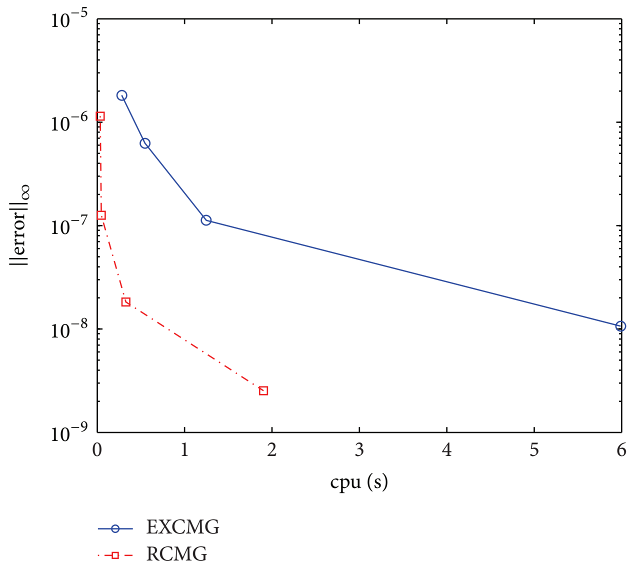

FIGURE 3: Comparison of the maximum error $\|\overline{u}^L - u\|_\infty$ and cpu time for Example 4 with $L = 3$, taking step lengths $h_L = 1/m$, $m = 64$, 128, 256, and 512, respectively.

5.1. Comparison of the Initial Errors. Assume that the exact solutions of the difference equation on grids 16×16 and 32×32 are given. We compare EXCMG method with RCMG method for the initial error $\|\text{Err}_{64}^0\| = \|u_{64}^0 - u_{64}\|$ on grid 64×64.

From Figure 2, the accuracy of the initial error on the next grid of RCMG method is higher than EXCMG method.

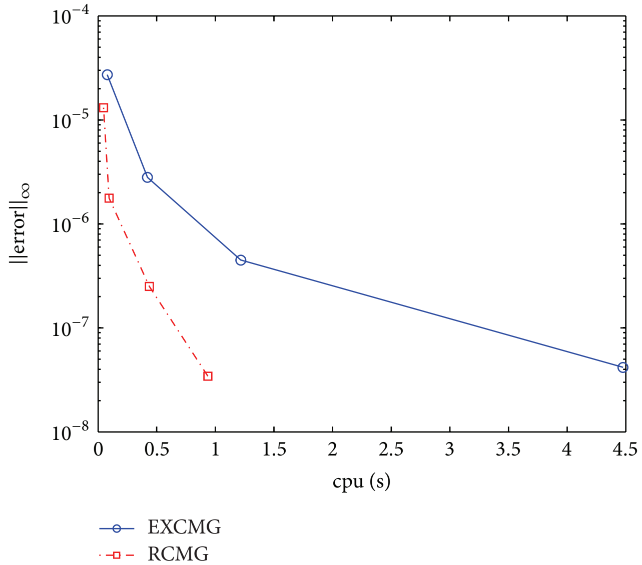

Figure 4: Comparison of the maximum error $\|\overline{u}^L - u\|_\infty$ and cpu time for Example 5 with $L = 3$, taking step lengths $h_L = 1/m$, $m = 64, 128, 256$, and 512, respectively.

Table 1: Numerical results of EXCMG and RCMG for Example 4.

L	$1/h_L$	EXCMG		RCMG	
		$\|\|\overline{u}^L - u\|\|_\infty$	cpu	$\|\|\overline{u}^L - u\|\|_\infty$	cpu
3	128	$6.28E-07$	0.37	$1.27E-07$	0.14
	256	$1.12E-07$	1.12	$1.85E-08$	0.31
	512	$1.07E-08$	4.10	$2.58E-09$	1.12
4	128	$4.76E-07$	0.34	$2.61E-07$	0.08
	256	$1.27E-07$	1.19	$6.11E-08$	0.42
	512	$3.58E-08$	4.38	$6.07E-09$	1.29
5	128	$9.21E-07$	0.51	$3.05E-07$	0.08
	256	$1.62E-07$	1.06	$7.82E-08$	0.31
	512	$2.72E-08$	4.15	$1.72E-08$	1.15

Namely, a better initial value on the fine grid can be got by using RCMG method. Based on the results of the literature [17–20], the RCMG method can obtain good convergence rate.

5.2. Comparison between EXCMG Method and RCMG Method. Let $\|\text{Error}\|_\infty = \|\overline{u}^L - u\|_\infty$ denote the maximum absolute error between the computed solution $\overline{u}^L$ and the exact solution u on the finest grid points. The "cpu" denotes the computing time (unit: second) of EXCMG method and RCMG method.

From Figures 3 and 4 and Tables 1 and 2, we see that, under the same conditions, the RCMG method can obtain higher computational precision and spend less computing time than EXCMG method.

Table 2: Numerical results of EXCMG and RCMG for Example 5.

L	$1/h_L$	EXCMG		RCMG	
		$\|\|\overline{u}^L - u\|\|_\infty$	cpu	$\|\|\overline{u}^L - u\|\|_\infty$	cpu
3	128	$2.80E-06$	0.39	$1.77E-06$	0.16
	256	$4.50E-07$	1.22	$2.50E-07$	0.50
	512	$4.18E-08$	4.17	$3.16E-08$	1.23
4	128	$1.07E-05$	0.25	$4.29E-06$	0.11
	256	$1.92E-06$	1.00	$9.28E-07$	0.30
	512	$1.71E-07$	4.23	$1.39E-07$	1.17
5	128	$1.74E-05$	0.28	$2.56E-06$	0.09
	256	$5.29E-06$	1.01	$2.00E-06$	0.31
	512	$7.94E-07$	4.07	$3.35E-07$	1.06

6. Conclusion

In this paper, based on a fourth order compact scheme, we present a Richardson cascadic multigrid method for 2D Poisson problem by using Richardson technique presented by [21]. The numerical results show that RCMG method has higher computational accuracy and higher efficiency.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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