Research Article

Periodic Points of Weaker Meir-Keeler Contractive Mappings on Generalized Quasimetric Spaces

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By using the weaker Meir-Keeler function ϕ and the triangular α-admissible mapping α, we introduce the notion of $(\alpha - \phi)$-weaker Meir-Keeler contractive mappings and prove a theorem which assures the existence of a periodic point for these mappings on generalized quasimetric spaces.

1. Introduction and Preliminaries

Let X be a nonempty set and let $d : X \times X \rightarrow [0, \infty)$. Then d is called a distance function if for every $x, y, z \in X$, satisfies

(d_1) $d(x, x) = 0$;

(d_2) $d(x, y) = d(y, x) = 0 \Rightarrow x = y$;

(d_3) $d(x, y) = d(y, x)$;

(d_4) $d(x, z) \le d(x, y) + d(y, z)$.

If d satisfies conditions (d_1)–(d_4), then d is called a metric on X. If d satisfies conditions (d_1), (d_2), and (d_4), then d is called a quasimetric on X. If d satisfies conditions (d_2), (d_3), and (d_4), then d is called a dislocated metric on X. If d satisfies conditions (d_2) and (d_4), then d is called a dislocated quasimetric on X.

In 2000, Branciari [1] introduced the notion of generalized metric as a natural extension of the concept of a metric, where the triangle inequality condition of a metric had been replaced by a weaker condition, namely, quadrilateral inequality. At the first glance, both metric and generalized metric seem to have almost the same topological properties. Despite the first impression, the generalized metric does possess some fundamental topological feature, such as

(P1) generalized metric needs not to be continuous;

(P2) a convergent sequence in generalized metric space needs not to be Cauchy;

(P3) generalized metric space needs not to be Haussdorf and hence the uniqueness of limits cannot be guaranteed.

The question whether the analog of existing fixed point results in the literature are still valid in generalized metric space without assuming an extra conditions, such as, continuity of generalized metric function, and/or Hausdorffness of the corresponding space, and so forth. Several authors worked on this interesting questions and this space (e.g., [1–18]).

Definition 1 (see [1]). Let X be a nonempty set and let $d : X \times X \rightarrow [0, \infty)$ be a mapping such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y, one has

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \le d(x, u) + d(u, v) + d(v, y)$ (quadrilateral inequality).

Then (X, d) is called a generalized metric space (or shortly g.m.s).

We present an example to show that not every generalized metric on a set X is a metric on X.

Example 2. Let $X = \{t, 2t, 3t, 4t, 5t\}$ with $t > 0$ as a constant, and we define $d : X \times X \rightarrow [0, \infty)$ by

(1) $d(x, x) = 0$, for all $x \in X$;

(2) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(3) $d(t, 2t) = 3\gamma$;

(4) $d(t, 3t) = d(2t, 3t) = \gamma$;

(5) $d(t, 4t) = d(2t, 4t) = d(3t, 4t) = 2\gamma$;

(6) $d(t, 5t) = d(2t, 5t) = d(3t, 5t) = d(4t, 5t) = (3/2)\gamma$,

where $\gamma > 0$ is a constant. Then, let (X, d) be a generalized metric space, but it is not a metric space because

$$d(t, 2t) = 3\gamma > d(t, 3t) + d(3t, 2t) = 2\gamma. \tag{1}$$

We now introduce the new notion of generalized quasimetric space as follows.

Definition 3. Let X be a nonempty set and let $d : X \times X \rightarrow [0, \infty)$ be a mapping such that for all $x, y \in X$ and for all distinct point $u, v \in X$, each of them different from x and y, one has

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$.

Then (X, d) is called a generalized quasimetric space (or shortly g.q.m.s).

Remark 4. Any generalized metric space is a generalized quasimetric space, but the converse is not true in general.

We present an example to show that not every generalized quasimetric on a set X is a generalized metric on X.

Example 5. Let $X = \{t, 2t, 3t, 4t, 5t\}$ with $t > 0$ as a constant, and we define $d : X \times X \rightarrow [0, \infty)$ by

(1) $d(x, x) = 0$, for all $x \in X$;

(2) $d(t, 2t) = d(2t, t) = 3\gamma$;

(3) $d(t, 3t) = d(2t, 3t) = d(3t, t) = d(3t, 2t) = \gamma$;

(4) $d(t, 4t) = d(2t, 4t) = d(3t, 4t) = d(4t, t) = d(4t, 2t) = d(4t, 3t) = 2\gamma$;

(5) $d(t, 5t) = d(2t, 5t) = d(3t, 5t) = d(4t, 5t) = (3/2)\gamma$;

(6) $d(5t, t) = d(5t, 2t) = d(5t, 3t) = d(5t, 4t) = (5/4)\gamma$,

where $\gamma > 0$ is a constant. Then, let (X, d) be a generalized quasimetric space, but it is not a generalized metric space because

$$d(t, 5t) = \frac{3}{2}\gamma \neq d(5t, t) = \frac{5}{4}\gamma. \tag{2}$$

We next give the definitions of convergence and completeness on generalized quasimetric spaces.

Definition 6. Let (X, d) be a g.q.m.s, let $\{x_n\}$ be a sequence in X, and let $x \in X$. We say that $\{x_n\}$ is g.q.m.s convergent to x if and only if

$$\lim_{n \to \infty} d(x_n, x) = \lim_{n \to \infty} d(x, x_n) = 0. \tag{3}$$

Definition 7. Let (X, d) be a g.q.m.s and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is left-Cauchy if and only if, for every $\varepsilon > 0$, there exits $k \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n \geq m > k$.

Definition 8. Let (X, d) be a g.q.m.s and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is right-Cauchy if and only if, for every $\varepsilon > 0$, there exits $k \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $m \geq n > k$.

Definition 9. Let (X, d) be a g.q.m.s and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is Cauchy if and only if, for every $\varepsilon > 0$, there exits $k \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $m, n > k$.

Remark 10. A sequence $\{x_n\}$ in a g.q.m.s is Cauchy if and only if it is left-Cauchy and right-Cauchy.

Definition 11. Let (X, d) be a g.q.m.s. We say that

(1) (X, d) is left-complete if and only if each left-Cauchy sequence in X is convergent;

(2) (X, d) is right-complete if and only if each right-Cauchy sequence in X is convergent;

(3) (X, d) is complete if and only if each Cauchy sequence in X is convergent.

In the sequel, we let the function $\phi : [0, \infty) \rightarrow [0, \infty)$ satisfy the following conditions:

(ϕ_1) $\phi : [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing weaker Meir-Keeler function;

(ϕ_2) $\phi(t) > 0$ for $t > 0$ and $\phi(0) = 0$;

(ϕ_3) for all $t \in (0, \infty)$, $\{\phi^n(t)\}_{n \in \mathbb{N}}$ is decreasing;

(ϕ_4) for $t > 0$, if $\lim_{n \to \infty} \phi^n(t) = 0$, then $\lim_{n \to \infty} \sum_{i=n}^{m} \phi^i(t) = 0$, where $m > n$.

Now, we recall the notion of α-admissible mappings. The following definition was introduced in [3].

Definition 12. Let $f : X \rightarrow X$ be a self-mapping of a set X and $\alpha : X \times X \rightarrow \mathbb{R}^+$. Then f is called a α-admissible if

$$x, y \in X, \qquad \alpha(x, y) \geq 1 \Longrightarrow \alpha(fx, fy) \geq 1. \tag{4}$$

In the sequel, we use the notion of *triangular α-admissible* which was defined in [4] as follows.

Definition 13. Let $f : X \rightarrow X$ and $\alpha : X \times X \rightarrow [0, \infty)$. The mapping f is said to be a triangular α-admissible if, for all $x, y, z \in X$, we have

(1) for all $x, y \in X$, $\alpha(x, y) \geq 1$ implies $\alpha(fx, fy) \geq 1$;

(2) for all $x, y, z \in X$, $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$ imply $\alpha(x, z) \geq 1$.

2. Main Results

In this section we state our main result. First we introduce the notion of $(\alpha - \phi)$-weaker Meir-Keeler contractive mappings via the weaker Meir-Keeler function ϕ and the triangular α-admissible mapping α.

Definition 14. Let (X, d) be a g.q.m.s, let $\alpha : X \times X \to \mathbb{R}^+$, and let $f : X \to X$ be a function satisfying

$$\alpha(x, y) d(fx, fy) \leq \phi(d(x, y)), \tag{5}$$

for all $x, y \in X$. Then f is said to be a $(\alpha - \phi)$-weaker Meir-Keeler contractive mapping.

Now, we state our main periodic point theorems as follows.

Theorem 15. *Let (X, d) be a Hausdorff and complete g.q.m.s, and let $f : X \to X$ be a $(\alpha - \phi)$-weaker Meir-Keeler contractive mapping. Suppose that*

(i) *f is triangular α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$ and $\alpha(fx_0, x_0) \geq 1$;*

(iii) *f is continuous.*

Then f has a periodic point in X; that is, there exists a $u \in X$ such that $u = f^p(u)$ for some $p \in \mathbb{N}$.

Proof. Regarding the assumption (ii) of the theorem, we let $x_0 \in X$ be an arbitrary point such that $\alpha(x_0, fx_0) \geq 1$ and $\alpha(x_0, fx_0) \geq 1$. We will construct a sequence $\{x_n\}$ in X by $x_{n+1} = fx_n = f^{n+1}x_0$ for all $n \geq 0$. If we have $x_{n_0} = x_{n_0+1}$, for some n_0, then $u = x_{n_0}$ is a fixed point of f; that is, $u = x_{n_0}$ is a periodic point in X. Hence, for the rest of the proof, we presume that

$$x_n \neq x_{n+1} \quad \forall n. \tag{6}$$

Since f is triangular α-admissible, we also have

$$\alpha(x_0, x_1) = \alpha(x_0, fx_0) \geq 1$$
$$\implies \alpha(fx_0, fx_1) = \alpha(x_1, x_2) \geq 1. \tag{7}$$

Utilizing the expression above, we obtain that

$$\alpha(x_n, x_{n+1}) \geq 1, \quad \forall n = 0, 1, \ldots. \tag{8}$$

By repeating the same steps with starting with the assumption $\alpha(x_1, x_0) = \alpha(fx_0, x_0) \geq 1$, we conclude that

$$\alpha(x_{n+1}, x_n) \geq 1, \quad \forall n = 0, 1, \ldots. \tag{9}$$

Since, f is triangular α-admissible, we derive that

$$\alpha(x_0, x_1) \geq 1, \quad \alpha(x_1, x_2) \geq 1 \implies \alpha(x_0, x_2) \geq 1. \tag{10}$$

Recursively, we get that

$$\alpha(x_n, x_{n+k}) \geq 1, \quad \forall n = 0, 1, \ldots, \ k = 1, 2, \ldots. \tag{11}$$

Analogously, we can easily derive that

$$\alpha(x_{n+k}, x_n) \geq 1, \quad \forall n = 0, 1, \ldots, \ k = 1, 2, \ldots. \tag{12}$$

In the sequel, we prove that the sequence $\{x_n\}$ is Cauchy; that is, $\{x_n\}$ is both right-Cauchy and left-Cauchy.

Step 1. We first assert that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \tag{13}$$

Regarding (5) and (8), we deduce that

$$d(x_n, x_{n+1}) = d(fx_{n-1}, fx_n)$$
$$\leq \alpha(x_{n-1}, x_n) d(fx_{n-1}, fx_n) \tag{14}$$
$$\leq \phi(d(x_{n-1}, x_n)),$$

for all $n \geq 1$. Since ϕ is nondecreasing, by iteration, we derive the following inequality:

$$d(x_n, x_{n+1}) \leq \phi(d(x_{n-1}, x_n))$$
$$\leq \phi(\phi(d(x_{n-2}, x_{n-1})))$$
$$= \phi^2(d(x_{n-3}, x_{n-2})) \leq \cdots \tag{15}$$
$$\leq \phi^n(d(x_0, x_1)).$$

Since $\{\phi^n(d(x_0, x_1))\}_{n \in \mathbb{N}}$ is decreasing, it must converge to some $\eta \geq 0$. We claim that $\eta = 0$. Suppose that, on the contrary, $\eta > 0$. Then by the definition of weaker Meir-Keeler function ϕ, corresponding to the given η, there exists $\delta > 0$ such that, for $x_0, x_1 \in X$ with $\eta \leq d(x_0, x_1) < \delta + \eta$, and $n_0 \in \mathbb{N}$ such that $\phi^{n_0}(d(x_0, x_1)) < \eta$. Since $\lim_{n \to \infty} \phi^n(d(x_0, x_1)) = \eta$, there exists $p_0 \in \mathbb{N}$ such that $\eta \leq \phi^p(d(x_0, x_1)) < \delta + \eta$, for all $p \geq p_0$. Thus, we conclude that $\phi^{p_0+n_0}(d(x_0, x_1)) < \eta$, which is a contradiction. Therefore $\lim_{n \to \infty} \phi^n(d(x_0, x_1)) = 0$; that is,

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \tag{16}$$

Step 2. We will show that

$$\lim_{n \to \infty} d(x_n, x_{n+2}) = 0. \tag{17}$$

We repeat the same argument that was used in Step 1. On account of (5) and (11), we observe that

$$d(x_n, x_{n+2})$$
$$= d(fx_{n-1}, fx_{n+1})$$
$$\leq \alpha(x_{n+1}, x_{n-1}) d(fx_{n-1}, fx_{n+1}) \tag{18}$$
$$\leq \phi(d(x_{n-1}, x_{n+1})),$$

for all $n \geq 1$. Since ϕ is nondecreasing, by iteration, we derive the following inequality:

$$d(x_n, x_{n+2}) \leq \phi(d(x_{n-1}, x_{n+1}))$$
$$\leq \phi(\phi(d(x_{n-2}, x_n)))$$
$$= \phi^2(d(x_{n-3}, x_{n-1})) \leq \cdots \tag{19}$$
$$\leq \phi^n(d(x_0, x_2)).$$

Since $\{\phi^n(d(x_0, x_2))\}_{n \in \mathbb{N}}$ is decreasing, it must converge to some $\eta \geq 0$. We claim that $\eta = 0$. Suppose that, on the contrary, $\eta > 0$. Then by the definition of weaker Meir-Keeler function ϕ, corresponding to the given η, there exists $\delta > 0$ such that, for $x_0, x_2 \in X$ with $\eta \leq d(x_0, x_2) < \delta + \eta$, and $n_0 \in \mathbb{N}$ such that $\phi^{n_0}(d(x_0, x_2)) < \eta$. Since $\lim_{n \to \infty} \phi^n(\varphi(d(x_0, x_2))) = \eta$, there exists $p_0 \in \mathbb{N}$ such that $\eta \leq \phi^p(d(x_0, x_2)) < \delta + \eta$, for all $p \geq p_0$. Thus, we conclude that $\phi^{p_0 + n_0}(d(x_0, x_2)) < \eta$, which is a contradiction. Therefore $\lim_{n \to \infty} \phi^n(d(x_0, x_2)) = 0$; that is,

$$\lim_{n \to \infty} d(x_n, x_{n+2}) = 0. \tag{20}$$

Step 3. We next will prove that the sequence $\{x_n\}$ is right-Cauchy by standard technique. For this purpose, it is sufficient to examine two cases.

Case (I). Suppose that $k > 2$ and k is odd. Let $k = 2m + 1$, $k \geq 1$. Then, by using the quadrilateral inequality, we have

$$\begin{aligned}
d(x_n, x_{n+k}) &= d(x_n, x_{n+2m+1}) \\
&\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \\
&\quad + d(x_{n+2}, x_{n+2m+1}) \\
&\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \\
&\quad + d(x_{n+2}, x_{n+3}) + \cdots \\
&\quad + d(x_{n+2m}, x_{n+2m+1}) \\
&= d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1}) \\
&\quad + d(fx_{n+1}, fx_{n+2}) + \cdots \\
&\quad + d(fx_{n+2m-1}, x_{n+2m}) \\
&\leq \alpha(x_{n-1}, x_n) d(fx_{n-1}, fx_n) \\
&\quad + \alpha(x_n, x_{n+1}) d(fx_n, fx_{n+1}) \\
&\quad + \alpha(x_{n+1}, x_{n+2}) d(fx_{n+1}, fx_{n+2}) + \cdots \\
&\quad + \alpha(x_{n+2m-1}, x_{n+2m}) d(fx_{n+2m-1}, fx_{n+2m}) \\
&\leq \cdots \leq \phi^n(d(x_0, x_1)) \\
&\quad + \phi^{n+1}(d(x_0, x_1)) \\
&\quad + \phi^{n+2}(d(x_0, x_1)) + \cdots + \phi^{n+2m}(d(x_0, x_1)) \\
&\leq \sum_{i=0}^{2m} \phi^{n+i}(d(x_0, x_1)).
\end{aligned} \tag{21}$$

Let $n \to \infty$. Then, by using condition ϕ_4, we have

$$\lim_{n \to \infty} d(x_n, x_{n+k}) = 0. \tag{22}$$

Case (II). Suppose that $k > 2$ and k is even. Let $k = 2m$, $k \geq 1$. Then, by using the quadrilateral inequality, we also have

$$\begin{aligned}
d(x_n, x_{n+k}) &= d(x_n, x_{n+2m}) \\
&\leq d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+4}) \\
&\quad + d(x_{n+4}, x_{n+2m}) \\
&\leq d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+4}) \\
&\quad + d(x_{n+4}, x_{n+6}) + \cdots + d(x_{n+2m-2}, x_{n+2m}) \\
&= d(fx_{n-1}, fx_{n+1}) + d(fx_{n+1}, fx_{n+3}) \\
&\quad + d(fx_{n+3}, fx_{n+5}) + \cdots \\
&\quad + d(fx_{n+2m-3}, fx_{n+2m-1}) \\
&\leq \alpha(x_{n-1}, x_{n+1}) d(fx_{n-1}, fx_{n+1}) \\
&\quad + \alpha(x_{n+1}, x_{n+3}) d(fx_{n+1}, fx_{n+3}) \\
&\quad + \alpha(x_{n+3}, x_{n+5}) d(fx_{n+3}, fx_{n+5}) + \cdots \\
&\quad + \alpha(x_{n+2m-3}, x_{n+2m-1}) \\
&\quad \times d(fx_{n+2m-3}, fx_{n+2m-1}) \leq \cdots \\
&\leq \phi^n(d(x_0, x_2)) \\
&\quad + \phi^{n+2}(d(x_0, x_2)) \\
&\quad + \phi^{n+4}(d(x_0, x_2)) + \cdots + \phi^{n+2m}(d(x_0, x_2)) \\
&\leq \sum_{i=0}^{m} \phi^{n+2i}(d(x_0, x_2)).
\end{aligned} \tag{23}$$

Let $n \to \infty$. Then, by using condition ϕ_4, we have

$$\lim_{n \to \infty} d(x_n, x_{n+k}) = 0. \tag{24}$$

By above argument, we get that $\{x_n\}$ is a right-Cauchy sequence.

Analogously, we derive that the sequence $\{x_n\}$ is left-Cauchy. Consequently, the sequence $\{x_n\}$ is Cauchy. Since X is a complete g.q.m.s, there exists $u \in X$ such that

$$\lim_{n \to \infty} d(x_n, u) = \lim_{n \to \infty} d(u, x_n) = 0. \tag{25}$$

Step 4. We claim that f has a periodic point in X. Suppose that, on the contrary, f has no periodic point. Since f is continuous, we obtain from (25) that

$$\begin{aligned}
\lim_{n \to \infty} d(x_{n+1}, fu) &= \lim_{n \to \infty} d(fx_n, fu) = 0, \\
\lim_{n \to \infty} d(fu, x_{n+1}) &= \lim_{n \to \infty} d(fu, fx_n) = 0.
\end{aligned} \tag{26}$$

From (25) and (26), we get immediately that $\lim_{n \to \infty} f^n x_0 = \lim_{n \to \infty} fx_n = fu$. As (X, d) is Hausdorff, we conclude that

$u = fu$ which contradicts the assumption that f has no periodic point. Therefore, there exists $u \in X$ such that $u = f^p(u)$ for some $p \in \mathbb{N}$. So f has a periodic point in X. $\square$

Following the proof of Theorem 15, we can easily get the following periodic point theorem.

Theorem 16. *Let (X, d) be a Hausdorff and complete g.q.m.s, and let $f : X \to X$ be a $(\alpha - \phi - \varphi)$-weaker Meir-Keeler contractive mapping. Suppose that*

(i) *f is triangular α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$ and $\alpha(fx_0, x_0) \geq 1$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $\alpha(x_{n+1}, x_n) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ and $\alpha(x, x_n) \geq 1$ for all n.*

Then f has a periodic point in X.

Proof. Following the proof of Theorem 15, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = fx_n$, for all $n \geq 0$, converges for some $u \in X$. From (25) and condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all k. Applying (5), for all k, we get that

$$
\begin{aligned}
d\left(x_{n(k)+1}, fu\right) &= d\left(fx_{n(k)}, fu\right) \\
&\leq \alpha\left(x_{n(k)}, u\right) d\left(fx_{n(k)}, fu\right) \\
&\leq \phi\left(d\left(x_{n(k)}, u\right)\right), \\
d\left(fu, x_{n(k)+1}\right) &= d\left(fu, fx_{n(k)}\right) \\
&\leq \alpha\left(u, x_{n(k)}\right) d\left(fu, fx_{n(k)}\right) \\
&\leq \phi\left(u, d\left(x_{n(k)}\right)\right).
\end{aligned}
\tag{27}
$$

Letting $k \to \infty$ in the above equality, we find that

$$
\begin{aligned}
d\left(x_{n(k)+1}, fu\right) &= \lim_{k \to \infty} d\left(fx_{n(k)}, fu\right) = 0, \\
d\left(fu, x_{n(k)+1}\right) &= \lim_{k \to \infty} d\left(fu, fx_{n(k)}\right) = 0.
\end{aligned}
\tag{28}
$$

Therefore, we have $\lim_{k \to \infty} f^{n(k)} x_0 = \lim_{k \to \infty} fx_{n(k)} = fu$. As (X, d) is Hausdorff, we conclude that $u = fu$ which contradicts the assumption that f has no periodic point. Therefore, there exists $u \in X$ such that $u = f^p(u)$ for some $p \in \mathbb{N}$. So f has a periodic point in X. $\square$

3. Consequences

Using the weaker Meir-Keeler function ϕ, we introduce the notion of ϕ-weaker Meir-Keeler contractive mappings and prove a theorem which assures the existence of a periodic point for these mappings on generalized quasimetric spaces.

Definition 17. Let (X, d) be a g.q.m.s, and let $f : X \to X$ be a function satisfying

$$
d(fx, fy) \leq \phi(d(x, y)),
\tag{29}
$$

for all $x, y \in X$. Then f is said to be a ϕ-weaker Meir-Keeler contractive mapping.

Theorem 18. *Let (X, d) be a Hausdorff and complete g.q.m.s, and let f be a ϕ-weaker Meir-Keeler contractive mapping. Suppose that f is continuous. Then f has a periodic point u in X; that is, there exists $u \in X$ such that $u = f^p u$ for some $p \in \mathbb{N}$.*

Proof. It is sufficient to take $\alpha(x, y) = 1$, for all $x, y \in X$, in Theorem 15. $\square$

Theorem 19. *Let (X, d) be a Hausdorff and complete g.q.m.s. Suppose that f is continuous and there exists $k \in [0, 1)$ such that*

$$
d(fx, fy) \leq kd(x, y),
\tag{30}
$$

for all $x, y \in X$. Then f has a periodic point u in X; that is, there exists $u \in X$ such that $u = f^p u$ for some $p \in \mathbb{N}$.

Proof. It is sufficient to take $\phi(t) = kt$, for all $x, y \in X$, in Theorem 18, where $k \in [0, 1)$. $\square$

Definition 20. Let $(X, \preceq)$ be a partially ordered set and let $T : X \to X$ be a given mapping. We say that T is nondecreasing with respect to $\preceq$ if

$$
x, y \in X, \quad x \preceq y \implies Tx \preceq Ty.
\tag{31}
$$

Definition 21. Let $(X, \preceq)$ be a partially ordered set. A sequence $\{x_n\} \subset X$ is said to be nondecreasing with respect to $\preceq$ if $x_n \preceq x_{n+1}$ for all n.

Definition 22. Let $(X, \preceq)$ be a partially ordered set and let d be a metric on X. We say that $(X, \preceq, d)$ is regular if, for every nondecreasing sequence $\{x_n\} \subset X$ such that $x_n \to x \in X$ as $n \to \infty$, we have $x_n \preceq x$ for all n.

We have the following result.

Corollary 23. *Let $(X, \preceq)$ be a partially ordered set and let d be a metric on X such that (X, d) is complete. Let $f : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a function ϕ such that*

$$
d(fx, fy) \leq \phi(d(x, y)),
\tag{32}
$$

for all $x, y \in X$ with $x \geq y$. Suppose also that the following conditions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq fx_0$;*

(ii) *f is continuous or $(X, \preceq, d)$ is regular.*

Then f has a periodic point.

Proof. Define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$
\alpha(x, y) = \begin{cases} 1 & \text{if } x \preceq y \text{ or } x \succeq y, \\ 0 & \text{otherwise.} \end{cases}
\tag{33}
$$

It is evident that f is a $(\alpha - \phi)$ contractive mapping; that is,

$$\alpha(x, y) d(fx, fy) \le \psi(d(x, y)), \qquad (34)$$

for all $x, y \in X$. From condition (i), we have that $\alpha(x_0, fx_0) \ge 1$ and $\alpha(fx_0, x_0) \ge 1$. Moreover, for all $x, y \in X$, from the monotone property of f, we have

$$\alpha(x, y) \ge 1 \implies x \succeq y \text{ or } x \preceq y$$
$$\implies fx \succeq fy \text{ or } fx \preceq fy \qquad (35)$$
$$\implies \alpha(fx, fy) \ge 1.$$

Thus f is α-admissible. Now, if f is continuous, the existence of a fixed point follows from Theorem 15. Suppose now that $(X, \preceq, d)$ is regular. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all n and $x_n \to x \in X$ as $n \to \infty$. From the regularity hypothesis, we have $x_n \preceq x$ for all n. This implies from the definition of α that $\alpha(x_n, x) \ge 1$ for all n. In this case, the existence of a fixed point follows from Theorem 16. □

Corollary 24. *Let $(X, \preceq)$ be a partially ordered set and let d be a metric on X such that (X, d) is complete. Let $f : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a function $k \in (0, 1)$ such that*

$$d(fx, fy) \le kd(x, y), \qquad (36)$$

for all $x, y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$;

(ii) f is continuous or $(X, \preceq, d)$ is regular.

Then f has a periodic point.

Proof. It is sufficient to take $\phi(t) = kt$, for all $x, y \in X$, in the corollary above, where $k \in [0, 1)$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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