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Research Article

Linearizability of Nonlinear Third-Order Ordinary Differential Equations by Using a Generalized Linearizing Transformation

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We discuss the linearization problem of third-order ordinary differential equation under the generalized linearizing transformation. We identify the form of the linearizable equations and the conditions which allow the third-order ordinary differential equation to be transformed into the simplest linear equation. We also illustrate how to construct the generalized linearizing transformation. Some examples of linearizable equation are provided to demonstrate our procedure.

1. Introduction

There has been major interest in the nonlinear problems, since most equations are inherently nonlinear in nature. In general, the nonlinear problems are very difficult to solve explicitly. It is of interest to provide general criteria for the linearizability of nonlinear ordinary differential equations, as they can then be reduced to easily solvable equations. Therefore, the approach of investigating nonlinear ordinary differential equations via transforming to simpler ordinary differential equations becomes important and has been quite plentiful in analysis of physical problems. This includes the classical linearization problem of finding transformations that linearize a given ordinary differential equation. The linearization problem has been studied in many aspects. A short review can be found in [1, 2]. The tools commonly used for solving the linearization problem are the transformations such as point transformation, contact transformation, reduction of order, differential substitution, and generalized Sundman transformation. For this paper, we employ the extension of the generalized Sundman transformations.

The linearization problem for a second-order ordinary differential equation was investigated with respect to a generalized Sundman transformation

$$X = F(t, x), \qquad dT = G(t, x)\, dt \tag{1}$$

by Duarte et al. [3] earlier. They obtained the form of the linearizable equations and the conditions which allow the second-order ordinary differential equation to be transformed into the free particle equation. A characterization of these equations that can be linearized by means of generalized Sundman transformations in terms of first integral and procedure for construction of linearizing transformations has been given by Muriel and Romero [4]. In [5], Mustafa et al. gave a new characterization of linearizable equations in terms of the coefficients of ordinary differential equations and one auxiliary function. In [6], Nakpim and Meleshko pointed out that the solution of the linearization problem for a second-order ordinary differential equation via the generalized Sundman transformation considered earlier by Duarte et al. [3] using the Laguerre form is not complete.

The linearization problem for a third-order ordinary differential equation was also investigated with respect to a generalized Sundman transformation [7, 8]. Criteria for a third-order ordinary differential equation to be equivalent to the linear equation $X'''(T) = 0$ with respect to a Sundman transformation were presented in [8]. The generalized Sundman transformation was also applied for obtaining necessary and sufficient conditions for a third-order ordinary differential equation to be equivalent to a linear equation in the Laguerre form [6]. Some applications of the generalized Sundman transformation to ordinary differential equations were considered in [9] and earlier papers, summarized in the book [10].

The linearization problem of a fourth-order ordinary differential equation with respect to generalized Sundman transformations was studied in [11]. They found the necessary and sufficient conditions which allow the fourth-order ordinary differential equation to be transformed into the simplest linear equation.

In this work, we expose a more general transformation, that is, the extension of the generalized Sundman transformation

$$X = F(t,x), \qquad dT = G\left(t,x,x'\right)dt. \tag{2}$$

This transformation was studied in [12–14] where they designated the transformation as the *generalized linearizing transformation*. They showed that this transformation can be utilized to linearize a wider class of nonlinear ordinary differential equations and, in particular, certain equations which cannot be linearized by the nonpoint and invertible point transformations. If the function G in (2) is independent of the variable x', then it becomes a nonpoint transformation (vide (1)). On the other hand, if G is a differentiable function, then it becomes an invertible point transformation. So (2) is a unified transformation as it includes nonpoint and invertible point transformations as special cases. An example of an equation which can be linearized by a transformation of the form (2) is given in [13]. It is worth noting that any second-order equation $x'' = f(t,x,x')$ can be transformed by a transformation (2) into the free particle equation and that this is not so for third-order ordinary differential equations. Hence, the linearization problem using generalized linearizing transformations becomes interesting for ordinary differential equations of order greater than 2. In [12], the authors applied a particular class of transformations (2), where the function $G(t,x,x')$ is linear with respect to x'.

We are now paying attention to the case where G is a polynomial function in x' and in particular where it is linear in x' with coefficients which are arbitrary functions of t and x. To be specific, we focus here on the case

$$X = F(t,x), \qquad dT = \left(G_1(t,x)\,x' + G_2(t,x)\right)dt. \tag{3}$$

Notice that for the case $G_1 = 0$, the generalized linearizing transformation becomes a generalized Sundman transformation, so that we assume $G_1 \neq 0$.

The paper is organized as follows. In Section 2, the necessary conditions of linearization of a third-order ordinary differential equation are presented. In Section 3, we get the theorems that yield criteria for a third-order ordinary differential equation to be linearizable via generalized linearizing transformations. Examples which illustrate the procedure of using the linearization theorems are presented in Section 4.

2. Necessary Conditions of Linearization

Here we consider a nonlinear third-order ordinary differential equation

$$x''' = f\left(t,x,x',x''\right). \tag{4}$$

Our goal in this section is to describe all equations (4) which are equivalent with respect to generalized linearizing transformations

$$X = F(t,x), \qquad dT = \left(G_1(t,x)\,x' + G_2(t,x)\right)dt \tag{5}$$

to a linear equation

$$X'''(T) = 0. \tag{6}$$

We begin with investigating the necessary conditions for linearization, that is, the general form of third-order equation (4) that can be obtained from a linear equation (6) by any generalized linearizing transformation (5).

Applying a generalized linearizing transformation (5), one obtains the following transformation of the third-order derivatives:

$$\begin{aligned}
X'(T) &= \frac{D_tF}{G_1x'+G_2} = \frac{F_t + x'F_x}{G_1x'+G_2} = P\left(t,x,x'\right),\\
X''(T) &= \frac{D_tP}{G_1x'+G_2}\\
&= \frac{P_t + x'P_x + x''P_{x'}}{G_1x'+G_2}\\
&= -\Big(\left(G_{2x}x' + G_1x'' + G_{2t} + G_{1x}x'^2 + G_{1t}x'\right)F_t\\
&\qquad - \left(F_{tt} + F_{xx}x'^2 + 2F_{tx}x'\right)\left(G_1x'+G_2\right)\\
&\qquad + \left(G_{2x}x'^2 - G_2x'' + G_{2t}x' + G_{1x}x'^3 + G_{1t}x'^2\right)F_x\Big)\\
&\quad \times \left(G_1x'+G_2\right)^{-3}\\
&= Q\left(t,x,x',x''\right),\\
X'''(T) &= \Big(\Big(3G_{2x}^2x'^3 + G_2^2x''' + 3G_{2t}^2x' + 3G_{1x}^2x'^5\\
&\qquad + 3G_{1t}^2x'^3 + \left(x'x''' - 3x''^2\right)G_1G_2\\
&\qquad + 2\left(G_1x' - 2G_2\right)G_{2x}x'x''\\
&\qquad - \left(G_{2tt} + G_{2xx}x'^2 + 2G_{2tx}x' + G_{1xx}x'^3\right.\\
&\qquad\quad \left. + G_{1tt}x' + 2G_{1tx}x'^2\right)\left(G_1x'+G_2\right)x'\\
&\qquad + 3\left(\left(G_1x' - G_2\right)x'' + 2G_{2x}x'^2\right)G_{2t}\\
&\qquad + 6\left(G_{2x}x'^2 - G_2x'' + G_{2t}x'\right)G_{1x}x'^2
\end{aligned}$$

$$
\begin{aligned}
&+\left(\left(G_1x'-5G_2\right)x''+6G_{2x}x'^2+6G_{2t}x'\right.\\
&\quad\left.+\,6G_{1x}x'^3\right)G_{1t}x'\Big)F_x\\
&-\Big(3\left(G_{2t}+G_{2x}x'+G_{1x}x'^2+G_{1t}x'+G_1x''\right)F_{tt}\\
&\quad-\left(\left(3F_{ttx}+F_{xxx}x'^2\right)x'+F_{ttt}+3F_{txx}x'^2\right)\\
&\quad\times\left(G_1x'+G_2\right)\\
&\quad+3\left(\left(G_{2t}+G_{2x}x'+G_{1x}x'^2+G_{1t}x'\right)x'-G_2x''\right)\\
&\quad\times F_{xx}x'+3\left(2\left(G_{2t}+G_{2x}x'+G_{1x}x'^2+G_{1t}x'\right)x'\right.\\
&\qquad\left.+\left(G_1x'-G_2\right)x''\right)F_{tx}\Big)\left(G_1x'+G_2\right)\\
&-\Big(\Big(G_{2tt}+G_{2xx}x'^2+2G_{2tx}x'+G_{1xx}x'^3\\
&\quad+G_{1tt}x'+2G_{1tx}x'^2\Big)\\
&\quad\times\left(G_1x'+G_2\right)\\
&\quad-\left(3G_{2x}^2x'^2-G_1G_2x'''+3G_{2t}^2+3G_{1x}^2x'^4\right.\\
&\qquad\left.+\,3G_{1t}^2x'^2-\left(x'x'''-3x''^2\right)G_1^2\right)\\
&\quad-\left(5G_1x'-G_2\right)G_{2x}x''\\
&\quad-6\left(G_{2x}x'+G_1x''\right)G_{2t}\\
&\quad-3\left(\left(G_1x'-G_2\right)x''+2G_{2x}x'^2+2G_{2t}x'\right)\\
&\quad\times G_{1x}x'-2\Big(\left(2G_1x'-G_2\right)x''+3G_{2x}x'^2\\
&\qquad+3G_{2t}x'+3G_{1x}x'^3\Big)G_{1t}\Big)F_t\Big)\\
&\times\left(G_1x'+G_2\right)^{-5}\\
&=R\left(t,x,x',x'',x'''\right),
\end{aligned}
\tag{7}
$$

where $D_t=\partial/\partial t+x'(\partial/\partial x)+x''(\partial/\partial x')+x'''(\partial/\partial x'')+\cdots$ is a total of derivatives. Substituting the resulting expression in linear equation (6) and setting $r=G_2/G_1$, $K=F_t-F_xr$, we arrive at the following equation:

$$
\begin{aligned}
&x'''+\frac{1}{x'+r}\\
&\quad\times\left[-3x''^2+\left(A_2x'^2+A_1x'+A_0\right)x''\right.\\
&\qquad\left.+B_5x'^5+B_4x'^4+B_3x'^3+B_2x'^2+B_1x'+B_0\right]\\
&=0,
\end{aligned}
\tag{8}
$$

where $A_i\ (i=0,1,2)$ and $B_j\ (j=0,1,\ldots,5)$ are functions of t and x determined as follows:

$$
\begin{aligned}
A_2&=\left(3\left(\left(F_{tx}-F_{xx}r\right)G_1-F_tG_{1x}\right)\right.\\
&\qquad\left.+\left(2\left(2G_{1x}r-r_xG_1\right)-G_{1t}\right)F_x\right)/\left(KG_1\right),\\
A_1&=-\Big(\left(2G_{1x}r+5r_xG_1+4G_{1t}\right)F_t\\
&\quad-3\left(F_{tt}-F_{xx}r^2\right)G_1\\
&\quad+\Big(\left(3r_t-4r_xr\right)G_1\\
&\qquad-4G_{1x}r^2-2G_{1t}r\Big)F_x\Big)/\left(KG_1\right),
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
A_0=-\Big(&3F_{tx}G_1r^2-3F_{tt}G_1r+4F_tG_{1t}r\\
&-F_tG_{1x}r^2+6F_tr_tG_1-F_tr_xG_1r\\
&-3F_xG_{1t}r^2-3F_xr_tG_1r\Big)/\left(KG_1\right),
\end{aligned}
\tag{10}
$$

$$
B_5=-\frac{\left(\left(F_{xxx}G_1-3F_{xx}G_{1x}\right)G_1-\left(G_{1xx}G_1-3G_{1x}^2\right)F_x\right)}{\left(KG_1^2\right)},
\tag{11}
$$

$$
\begin{aligned}
B_4=\ &\left(3\left(G_{1t}+2G_{1x}r+r_xG_1\right)F_{xx}G_1\right.\\
&+\left(G_{1xx}G_1-3G_{1x}^2\right)F_t\\
&+\Big(2G_{1tx}G_1-6G_{1t}G_{1x}+2G_{1xx}G_1r-6G_{1x}^2r\\
&\qquad-4G_{1x}r_xG_1+r_{xx}G_1^2\Big)F_x\\
&\left.+\left(2\left(3F_{tx}G_{1x}-F_{xxx}G_1r\right)-3F_{txx}G_1\right)G_1\right)\\
&\times\left(KG_1^2\right)^{-1},
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
B_3=\ &\Big(\Big(3F_{tt}G_{1x}-F_{xxx}G_1r^2-3F_{ttx}G_1-6F_{txx}G_1r\\
&\quad+6\left(G_{1t}+2G_{1x}r+r_xG_1\right)F_{tx}\Big)G_1\\
&+\Big(2G_{1tx}G_1-6G_{1t}G_{1x}+2G_{1xx}G_1r-6G_{1x}^2r\\
&\quad-4G_{1x}r_xG_1+r_{xx}G_1^2\Big)F_t\\
&+3\left(\left(G_{1x}r+r_xG_1+G_{1t}\right)r+G_{1t}r+r_tG_1\right)F_{xx}G_1\\
&+\Big(\left(2G_{1tx}r+G_{1tt}\right)G_1-3\left(G_{1x}r+r_xG_1\right)^2\\
&\quad-6\left(G_{1t}r+r_tG_1\right)G_{1x}\\
&\quad-3\left(2\left(G_{1x}r+r_xG_1\right)+G_{1t}\right)G_{1t}\\
&\quad+\left(2G_{1x}r_x+r_{xx}G_1+G_{1xx}r\right)G_1r
\end{aligned}
$$

$$+\ 2\left(G_{1x}r_t + r_{tx}G_1 + G_{1t}r_x + G_{1tx}r\right)G_1\Big)F_x\Big)$$
$$\times\left(KG_1^2\right)^{-1}, \tag{13}$$

$$B_2 = \Big(\Big(\left(2G_{1tx}r + G_{1tt}\right)G_1 - 3\left(G_{1x}r + r_xG_1\right)^2$$
$$- 6\left(G_{1t}r + r_tG_1\right)G_{1x}$$
$$- 3\left(2\left(G_{1x}r + r_xG_1\right) + G_{1t}\right)G_{1t}$$
$$+ \left(2G_{1x}r_x + r_{xx}G_1 + G_{1xx}r\right)G_1r$$
$$+\ 2\left(G_{1x}r_t + r_{tx}G_1 + G_{1t}r_x + G_{1tx}r\right)G_1\Big)F_t$$
$$-\Big(\Big(6\left(G_{1x}r + r_xG_1 + G_{1t}\right)$$
$$\times\left(G_{1t}r + r_tG_1\right) - G_{1tt}G_1r$$
$$-\left(2G_{1t}r_t + r_{tt}G_1 + G_{1tt}r\right)G_1$$
$$-\ 2\left(G_{1x}r_t + r_{tx}G_1 + G_{1t}r_x + G_{1tx}r\right)G_1r\Big)F_x$$
$$+\Big(\left(3F_{txx}r^2 + F_{ttt}\right)G_1$$
$$- 3\left(G_{1t} + 2G_{1x}r + r_xG_1\right)F_{tt}$$
$$- 3\left(\left(G_{1t}r + r_tG_1\right)F_{xx} - 2F_{ttx}G_1\right)r$$
$$- 6\left(\left(G_{1x}r + r_xG_1 + G_{1t}\right)r + G_{1t}r + r_tG_1\right)$$
$$\times F_{tx}\Big)G_1\Big)\Big)$$
$$\times\left(KG_1^2\right)^{-1}, \tag{14}$$

$$B_1 = -\Big(\left(6\left(G_{1x}r + r_xG_1 + G_{1t}\right)\left(G_{1t}r + r_tG_1\right)\right.$$
$$- G_{1tt}G_1r - \left(2G_{1t}r_t + r_{tt}G_1 + G_{1tt}r\right)G_1$$
$$\left.-\ 2\left(G_{1x}r_t + r_{tx}G_1 + G_{1t}r_x + G_{1tx}r\right)G_1r\right)F_t$$
$$-\Big(\left(3\left(\left(G_{1x}r + r_xG_1 + G_{1t}\right)r + G_{1t}r + r_tG_1\right)F_{tt}\right.$$
$$- \left(\left(2F_{ttt} + 3F_{ttx}r\right)G_1\right.$$
$$\left.\left.-6\left(G_{1t}r + r_tG_1\right)F_{tx}\right)r\right)G_1$$
$$+\Big(\left(2G_{1t}r_t + r_{tt}G_1 + G_{1tt}r\right)G_1r$$
$$-3\left(G_{1t}r + r_tG_1\right)^2\Big)F_x\Big)\Big)$$
$$\times\left(KG_1^2\right)^{-1},$$

$$B_0 = \Big(\Big(\left(2G_{1t}r_t + r_{tt}G_1 + G_{1tt}r\right)G_1r$$
$$-3\left(G_{1t}r + r_tG_1\right)^2\Big)F_t$$
$$+\ \left(3\left(G_{1t}r + r_tG_1\right)F_{tt} - F_{ttt}G_1r\right)G_1r\Big)/\left(KG_1^2\right). \tag{15}$$

Thus, we proved the theorem.

Theorem 1. *Any third-order ordinary differential equation* (4) *obtained from a linear equation* (6) *by a generalized linearizing transformation* (5) *has to be in the form* (8).

3. Formulation of the Linearization Theorem

We have shown in the previous section that every linearizable third-order ordinary differential equation belongs to the class of equations (8). In this section, we formulate the main theorems containing necessary and sufficient conditions for linearization as well as the methods for constructing the linearizing transformations.

For obtaining sufficient conditions, one has to solve the compatibility problem. Consider the representations of the coefficients A_i and B_i through the unknown functions F and G_1. According to our notation $K = F_t - F_xr$, we define the derivative F_t as

$$F_t = F_xr + K. \tag{16}$$

From (9), one can find the derivatives

$$K_x = \frac{\left(F_xG_{1t} - F_xG_{1x}r - F_xr_xG_1 + 3G_{1x}K + A_2G_1K\right)}{\left(3G_1\right)},$$
$$K_t = \Big(F_xG_{1t}r - F_xG_{1x}r^2 - F_xr_xG_1r + 4G_{1t}K - G_{1x}Kr$$
$$+\ G_1K\left(5r_x + A_1 - A_2r\right)\Big)/\left(3G_1\right). \tag{17}$$

From (10), one obtains the condition

$$r_t = \frac{\left(6r_xr - A_0 + A_1r - A_2r^2\right)}{6}. \tag{18}$$

Equation (11) defines the derivative

$$F_{xxx} = \frac{\left(3F_{xx}G_{1x}G_1 + F_xG_{1xx}G_1 - 3F_xG_{1x}^2 - B_5G_1^2K\right)}{G_1^2}. \tag{19}$$

So that equation (12) becomes

$$6F_{xx}G_{1t}G_1 - 6F_{xx}G_{1x}G_1r - 6F_{xx}r_xG_1^2$$
$$+\ 3F_xG_{1tx}G_1 - 12F_xG_{1t}G_{1x}$$
$$-\ F_xG_{1t}A_2G_1 - 3F_xG_{1xx}G_1r + 12F_xG_{1x}^2r$$

$$+ F_xG_{1x}G_1\left(6r_x + A_2r\right) + F_xG_1^2\left(-3r_{xx} + r_xA_2\right) - 6G_{1xx}G_1K + 9G_{1x}^2K + G_1^2K\left(-3A_{2x} - A_2^2 - 3B_4 + 15B_5r\right) = 0. \tag{20}$$

The compatibility analysis depends on the value of F_x. A complete study of all cases is given here.

3.1. Case F_x=0. In this case, the forms of derivatives F_t, K_x, and K_t become

$$F_t = K,\quad K_x = \frac{\left(3G_{1x} + A_2G_1\right)K}{\left(3G_1\right)},\quad K_t = \frac{\left(4G_{1t} - G_{1x}r + G_1\left(5r_x + A_1 - A_2r\right)\right)K}{\left(3G_1\right)}. \tag{21}$$

Substituting F_x into F_{xxx}, one arrives at the condition

$$B_5 = 0. \tag{22}$$

Comparing the mixed derivatives $(F_x)_t = (F_t)_x$, one gets the derivative

$$G_{1x} = \frac{-\left(A_2G_1\right)}{3}. \tag{23}$$

In this case, $(F_{xxx})_t = (F_t)_{xxx}$ is satisfied. Equations (12) and (13) give the conditions

$$A_{2x} = \frac{\left(-2A_2^2 - 9B_4\right)}{3},\quad r_{xx} = \frac{\left(-9A_{1x} + 6A_{2t} + 3r_xA_2 - 3A_1A_2 - 2A_2^2r - 9B_3\right)}{36}. \tag{24}$$

Comparing the mixed derivative $(K_t)_x = (K_x)_t$, one obtains the condition

$$A_{1x} = \frac{\left(-6A_{2t} - 3r_xA_2 - 5A_1A_2 + 2A_2^2r - 15B_3 + 24B_4r\right)}{3}. \tag{25}$$

Equation (14) provides the derivative

$$G_{1tt} = \frac{\left(2250G_{1t}^2 + 150G_{1t}G_1h_1 + G_1^2h_2\right)}{\left(1350G_1\right)}, \tag{26}$$

where

$$h_1 = 15r_x + 3A_1 - 2A_2r,$$
$$h_2 = -225A_{0x} - 1350A_{1t} - 1350A_{2t}r - 1050A_0A_2 - 477A_1^2 + 516A_1A_2r + 33A_1h_1 - 432A_2^2r^2 - 57A_2h_1r - 4050B_2 + 4275B_3r - 4275B_4r^2 - 8h_1^2. \tag{27}$$

The relation $(r_x)_x = r_{xx}$ gives the condition

$$h_{1x} = 4A_{2t}. \tag{28}$$

Comparing the mixed derivative $(G_{1tt})_x = (G_{1x})_{tt}$, one arrives at the condition

$$A_{2tt} = \frac{\left(50A_{2t}h_1 - h_{2x}\right)}{450}. \tag{29}$$

Solving (15), one finds the conditions

$$A_{0t} = \left(15930A_{1t}r + 15930A_{2t}r^2 - 1260h_{1x}r - 1575A_0A_1 + 11970A_0A_2r + 5517A_1^2r - 5697A_1A_2r^2 - 558A_1h_1r + 4986A_2^2r^3 + 504A_2h_1r^2 - 8100B_1 + 48600B_2r - 48600B_3r^2 + 48600B_4r^3 + 148h_1^2r + 8h_2r\right)/1350, \tag{30}$$

$$B_0 = \left(-3240A_{1t}r^2 - 3240A_{2t}r^3 + 180h_{1x}r^2 - 135A_0^2 + 270A_0A_1r - 2430A_0A_2r^2 - 1107A_1^2r^2 + 1134A_1A_2r^3 + 108A_1h_1r^2 - 999A_2^2r^4 - 108A_2h_1r^3 + 1620B_1r - 9720B_2r^2 + 9720B_3r^3 - 9720B_4r^4 - 28h_1^2r^2 - 2h_2r^2\right)/1620. \tag{31}$$

3.2. Case $F_x \neq 0$. From (20) and (13), one obtains the derivatives

$$G_{1tx} = \left(-6F_{xx}G_{1t}G_1 + 6F_{xx}G_{1x}G_1r + 6F_{xx}r_xG_1^2 + 12F_xG_{1t}G_{1x} + F_xG_{1t}A_2G_1 + 3F_xG_{1xx}G_1r - 12F_xG_{1x}^2r + F_xG_{1x}G_1\left(-6r_x - A_2r\right) + F_xG_1^2\left(3r_{xx} - r_xA_2\right) + 6G_{1xx}G_1K - 9G_{1x}^2K + G_1^2K\left(3A_{2x} + A_2^2 + 3B_4 - 15B_5r\right)\right) \times \left(3F_xG_1\right)^{-1}, \tag{32}$$

$$G_{1tt} = \left(-24F_{xx}F_xG_{1t}G_1r + 24F_{xx}F_xG_{1x}G_1r^2 + 24F_{xx}F_xr_xG_1^2r - 24F_{xx}G_{1t}G_1K + 24F_{xx}G_{1x}G_1Kr + 24F_{xx}r_xG_1^2K + 14F_x^2G_{1t}^2 + 20F_x^2G_{1t}G_{1x}r\right.$$

$$\begin{aligned}
&+ 2F_x^2G_{1t}G_1\left(r_x + A_1\right) + 6F_x^2G_{1xx}G_1r^2 - 34F_x^2G_{1x}^2r^2 \\
&+ F_x^2G_{1x}G_1\left(-26r_xr - A_0 - A_1r - A_2r^2\right) \\
&+ F_x^2G_1^2\left(-A_{0x} + A_{1x}r - A_{2x}r^2 + 12r_{xx}r \right.\\
&\qquad \left. - 4r_x^2 - r_xA_1 - 2r_xA_2r\right) \\
&+ 24F_xG_{1t}G_{1x}K + 24F_xG_{1xx}G_1Kr - 60F_xG_{1x}^2Kr \\
&- 24F_xG_{1x}r_xG_1K \\
&+ 2F_xG_1^2K\left(3A_{1x} + 18r_{xx} - 3r_xA_2 \right.\\
&\qquad \left. + A_1A_2 + 3B_3 - 6B_4r\right) + 24G_{1xx}G_1K^2 \\
&- 36G_{1x}^2K^2 + 4G_1^2K^2\left(3A_{2x} + A_2^2 + 3B_4 - 15B_5r\right)\Big) \\
&\times\left(6F_x^2G_1\right)^{-1}.
\end{aligned} \tag{33}$$

Comparing the mixed derivative $(K_t)_x = (K_x)_t$, one obtains

$$\begin{aligned}
G_{1xx} = &\left(6F_{xx}G_{1t}G_1K - 6F_{xx}G_{1x}G_1Kr\right.\\
&- 6F_{xx}r_xG_1^2K - F_x^2G_{1t}^2 + 2F_x^2G_{1t}G_{1x}r \\
&+ 2F_x^2G_{1t}r_xG_1 - F_x^2G_{1x}^2r^2 - 2F_x^2G_{1x}r_xG_1r \\
&- F_x^2r_x^2G_1^2 - 6F_xG_{1t}G_{1x}K + 6F_xG_{1x}^2Kr \\
&+ 6F_xG_{1x}r_xG_1K \\
&+ F_xG_1^2K\left(- 3A_{2t} + 3A_{2x}r - A_1A_2 + 2A_2^2r \right.\\
&\qquad \left. - 3B_3 + 12B_4r - 30B_5r^2\right) \\
&+ 9G_{1x}^2K^2 + G_1^2K^2\left(-3A_{2x} - A_2^2 - 3B_4 + 15B_5r\right)\Big) \\
&\times\left(6G_1K^2\right)^{-1}.
\end{aligned} \tag{34}$$

Equation (14) becomes

$$F_xs_1 + 2Ks_2 = 0, \tag{35}$$

where

$$\begin{aligned}
s_1 = &- 6A_{1t} + 6A_{1x}r + 12A_{2t}r - 12A_{2x}r^2 \\
&- 5A_0A_2 - 2A_1^2 + 13A_1A_2r - 13A_2^2r^2 \\
&- 18B_2 + 54B_3r - 108B_4r^2 + 180B_5r^3,
\end{aligned} \tag{36}$$

$$\begin{aligned}
s_2 = &- 3A_{1x} + 6A_{2t} - 18r_{xx} + 3r_xA_2 \\
&+ A_1A_2 - 2A_2^2r + 3B_3 - 12B_4r + 30B_5r^2.
\end{aligned}$$

Further analysis of the compatibility depends on value of s_1 in (35): it is separated into two cases; that is, $s_1 = 0$ and $s_1 \neq 0$.

3.2.1. Case $s_1 \neq 0$. From (35), one finds

$$F_x = -\frac{(2Ks_2)}{s_1}. \tag{37}$$

Since this case $F_x \neq 0$, then $s_2 \neq 0$ too. Comparing the mixed derivatives $(F_x)_t = (F_t)_x$, one gets the derivative

$$G_{1t} = \frac{\left(3G_{1x}s_1\left(2rs_2 - s_1\right) + G_1s_3\right)}{\left(6s_1s_2\right)}, \tag{38}$$

where

$$\begin{aligned}
s_3 = &- 6r_xs_1s_2 + 6s_{1t}s_2 - 6s_{1x}rs_2 - 6s_{2t}s_1 \\
&+ 6s_{2x}rs_1 - 2A_1s_1s_2 + 4A_2rs_1s_2 - A_2s_1^2.
\end{aligned} \tag{39}$$

Substituting F_x into F_{xxx}, G_{1t} into G_{1tx} and G_{1tt}, one arrives at the conditions

$$\begin{aligned}
s_{2xx} = &\left(-12A_{2t}s_1^2s_2^2 + 12A_{2x}rs_1^2s_2^2 - 6A_{2x}s_1^3s_2\right.\\
&+ 36r_xs_{1x}s_1s_2^2 - 36r_xs_{2x}s_1^2s_2 - 12r_xA_2s_1^2s_2^2 \\
&+ 18s_{1xx}s_1^2s_2 - 36s_{1x}^2s_1s_2 + 36s_{1x}s_{2x}s_1^2 \\
&+ 12s_{1x}A_2s_1^2s_2 - 6s_{1x}s_2s_3 - 12s_{2x}A_2s_1^3 \\
&+ 6s_{2x}s_1s_3 - 4A_1A_2s_1^2s_2^2 + 8A_2^2rs_1^2s_2^2 \\
&- 2A_2^2s_1^3s_2 + 2A_2s_1s_2s_3 - 12B_3s_1^2s_2^2 \\
&\left.+ 48B_4rs_1^2s_2^2 - 120B_5r^2s_1^2s_2^2 + 9B_5s_1^4\right) \\
&\times\left(18s_1^3\right)^{-1},
\end{aligned}$$

$$\begin{aligned}
s_{3x} = &\left(-6A_{1x}s_1^3s_2 - 6A_{2t}s_1^3s_2 + 18A_{2x}rs_1^3s_2\right.\\
&- 9A_{2x}s_1^4 + 36r_x^2s_1^2s_2^2 - 36r_xs_{1x}s_1^2s_2 + 36r_xs_{2x}s_1^3 \\
&+ 6r_xA_2s_1^3s_2 - 12r_xs_1s_2s_3 + 12s_{1x}s_1s_3 \\
&- 4A_1A_2s_1^3s_2 + 8A_2^2rs_1^3s_2 - 3A_2^2s_1^4 - 12B_3s_1^3s_2 \\
&+ 48B_4rs_1^3s_2 - 9B_4s_1^4 - 120B_5r^2s_1^3s_2 \\
&\left.+ 45B_5rs_1^4 - 2s_1^3s_2^2 + s_3^2\right) / \left(6s_1^2\right),
\end{aligned}$$

$$\begin{aligned}
s_{3t} = &\left(-6A_{0x}s_1^3s_2 - 3A_{1x}s_1^4 - 6A_{2t}rs_1^3s_2 - 12A_{2t}s_1^4\right.\\
&+ 12A_{2x}r^2s_1^3s_2 + 9A_{2x}rs_1^4 + 36r_x^2rs_1^2s_2^2
\end{aligned}$$

$$\begin{aligned}
&- 36r_x^2 s_1^3 s_2 - 36r_x s_{1x} r s_1^2 s_2 + 36r_x s_{2x} r s_1^3 \\
&- 6r_x A_1 s_1^3 s_2 + 18r_x A_2 r s_1^3 s_2 - 3r_x A_2 s_1^4 \\
&- 12r_x r s_1 s_2 s_3 + 12 s_{1t} s_1 s_3 - 4A_1 A_2 r s_1^3 s_2 \\
&- 5A_1 A_2 s_1^4 + 8A_2^2 r^2 s_1^3 s_2 + 7A_2^2 r s_1^4 - 12B_3 r s_1^3 s_2 \\
&- 15B_3 s_1^4 + 48B_4 r^2 s_1^3 s_2 + 51B_4 r s_1^4 \\
&- 120B_5 r^3 s_1^3 s_2 - 105B_5 r^2 s_1^4 - 2r s_1^3 s_2^2 \\
&+ r s_3^2 + 5 s_1^4 s_2 \big) / \left(6 s_1^2\right).
\end{aligned} \tag{40}$$

Equation (15) provides the conditions

$$\begin{aligned}
A_{0t} = {} & \big(6A_{0x} r + 6A_{2t} r^2 - 6A_{2x} r^3 - 7A_0 A_1 \\
&+ 9A_0 A_2 r + 5A_1^2 r - 8A_1 A_2 r^2 + A_2^2 r^3 - 36B_1 \\
&+ 54B_2 r - 54B_3 r^2 + 36B_4 r^3 - r s_1\big)/6, \\
B_0 = {} & \big(-A_0^2 + 2A_0 A_1 r - 2A_0 A_2 r^2 - A_1^2 r^2 \\
&+ 2A_1 A_2 r^3 - A_2^2 r^4 + 12B_1 r - 12B_2 r^2 + 12B_3 r^3 \\
&- 12B_4 r^4 + 12B_5 r^5\big)/12.
\end{aligned} \tag{41}$$

Comparing the mixed derivatives $(G_{1tt})_x = (G_{1tx})_t$, $(G_{1xx})_t = (G_{1tx})_x$, and $(F_{xxx})_t = (F_t)_{xxx}$, one gets the conditions

$$\begin{aligned}
A_{1xx} = {} & \big(-33A_{1x} A_2 s_1^2 - 18A_{2tx} s_1^2 - 108A_{2t} r_x s_1 s_2 \\
&+ 24A_{2t} A_2 s_1^2 + 18A_{2t} s_3 + 54A_{2xx} r s_1^2 \\
&+ 108A_{2x} r_x r s_1 s_2 + 18A_{2x} r_x s_1^2 - 30A_{2x} A_1 s_1^2 \\
&+ 102A_{2x} A_2 r s_1^2 - 18A_{2x} r s_3 - 90B_{3x} s_1^2 \\
&+ 54B_{4t} s_1^2 + 306B_{4x} r s_1^2 - 270B_{5t} r s_1^2 - 630B_{5x} r^2 s_1^2 \\
&- 36r_x A_1 A_2 s_1 s_2 + 72r_x A_2^2 r s_1 s_2 + 27r_x A_2^2 s_1^2 \\
&- 108r_x B_3 s_1 s_2 + 432r_x B_4 r s_1 s_2 + 252r_x B_4 s_1^2 \\
&- 1080r_x B_5 r^2 s_1 s_2 - 1260r_x B_5 r s_1^2 + 36r_x s_1 s_2^2 \\
&- 18s_{1x} s_1 s_2 + 30s_{2x} s_1^2 + 45A_0 B_5 s_1^2 - 5A_1 A_2^2 s_1^2 \\
&+ 6A_1 A_2 s_3 - 45A_1 B_5 r s_1^2 + 10A_2^3 r s_1^2 - 12A_2^2 r s_3 \\
&- 15A_2 B_3 s_1^2 + 60A_2 B_4 r s_1^2 - 105A_2 B_5 r^2 s_1^2 \\
&+ 5A_2 s_1^2 s_2 + 18B_3 s_3 - 72B_4 r s_3 + 180B_5 r^2 s_3 \\
&- 6s_2 s_3\big) / \left(18 s_1^2\right), \\
A_{2tt} = {} & \big(36A_{2tx} r s_1 + 72A_{2t} r_x s_1 - 18A_{2xx} r^2 s_1 \\
&- 72A_{2x} r_x r s_1 - 3A_{2x} A_0 s_1 + 3A_{2x} A_1 r s_1 \\
&- 3A_{2x} A_2 r^2 s_1 - 18B_{3t} s_1 + 18B_{3x} r s_1 \\
&+ 72B_{4t} r s_1 - 72B_{4x} r^2 s_1 - 180B_{5t} r^2 s_1 \\
&+ 180B_{5x} r^3 s_1 + 24r_x A_1 A_2 s_1 - 48r_x A_2^2 r s_1 \\
&+ 72r_x B_3 s_1 - 288r_x B_4 r s_1 + 720r_x B_5 r^2 s_1 - 6r_x s_1 s_2 \\
&- 3s_{1x} s_1 + 3A_0 A_2^2 s_1 - 12A_0 B_4 s_1 + 60A_0 B_5 r s_1 \\
&+ 4A_1^2 A_2 s_1 - 19A_1 A_2^2 r s_1 + 6A_1 B_3 s_1 - 12A_1 B_4 r s_1 \\
&+ 19A_2^3 r^2 s_1 + 18A_2 B_2 s_1 \\
&- 66A_2 B_3 r s_1 + 144A_2 B_4 r^2 s_1 \\
&- 240A_2 B_5 r^3 s_1 + 2A_2 s_1^2 + s_3\big) / \left(18 s_1\right), \\
A_{2tx} = {} & \big(-6A_{1x} A_2 s_1^2 s_2 - 72A_{2t} r_x s_1 s_2^2 + 90A_{2t} s_{1x} s_1 s_2 \\
&- 90A_{2t} s_{2x} s_1^2 - 24A_{2t} A_2 s_1^2 s_2 + 12A_{2t} s_2 s_3 \\
&+ 18A_{2xx} r s_1^2 s_2 - 9A_{2xx} s_1^3 + 72A_{2x} r_x r s_1 s_2^2 \\
&+ 18A_{2x} r_x s_1^2 s_2 - 90A_{2x} s_{1x} r s_1 s_2 + 90A_{2x} s_{2x} r s_1^2 \\
&- 6A_{2x} A_1 s_1^2 s_2 + 48A_{2x} A_2 r s_1^2 s_2 - 18A_{2x} A_2 s_1^3 \\
&- 12A_{2x} r s_2 s_3 - 18B_{3x} s_1^2 s_2 + 72B_{4x} r s_1^2 s_2 \\
&- 27B_{4x} s_1^3 + 54B_{5t} s_1^3 - 180B_{5x} r^2 s_1^2 s_2 \\
&+ 81B_{5x} r s_1^3 - 24r_x A_1 A_2 s_1 s_2^2 + 48r_x A_2^2 r s_1 s_2^2 \\
&+ 12r_x A_2^2 s_1^2 s_2 - 72r_x B_3 s_1 s_2^2 + 288r_x B_4 r s_1 s_2^2 \\
&+ 72r_x B_4 s_1^2 s_2 - 720r_x B_5 r^2 s_1 s_2^2 - 360r_x B_5 r s_1^2 s_2 \\
&+ 135r_x B_5 s_1^3 + 30s_{1x} A_1 A_2 s_1 s_2 - 60s_{1x} A_2^2 r s_1 s_2 \\
&+ 90s_{1x} B_3 s_1 s_2 - 360s_{1x} B_4 r s_1 s_2 \\
&+ 900s_{1x} B_5 r^2 s_1 s_2 - 30s_{2x} A_1 A_2 s_1^2 + 60s_{2x} A_2^2 r s_1^2 \\
&- 90s_{2x} B_3 s_1^2 + 360s_{2x} B_4 r s_1^2 - 900s_{2x} B_5 r^2 s_1^2 \\
&- 8A_1 A_2^2 s_1^2 s_2 + 4A_1 A_2 s_2 s_3 + 18A_1 B_5 s_1^3 \\
&+ 16A_2^3 r s_1^2 s_2 - 4A_2^3 s_1^3 - 8A_2^2 r s_2 s_3 \\
&- 24A_2 B_3 s_1^2 s_2 + 96A_2 B_4 r s_1^2 s_2 - 18A_2 B_4 s_1^3 \\
&- 240A_2 B_5 r^2 s_1^2 s_2 + 54A_2 B_5 r s_1^3 + 12B_3 s_2 s_3 \\
&- 48B_4 r s_2 s_3 + 120B_5 r^2 s_2 s_3\big) / \left(18 s_1^2 s_2\right).
\end{aligned} \tag{42}$$

3.2.2. Case s_1=0. From (35), one finds the condition

$$s_2 = 0. \tag{43}$$

Equation (15) gives the conditions

$$\begin{aligned} A_{0t} = {} & \left(6A_{0x}r + 6A_{2t}r^2 - 6A_{2x}r^3 - 7A_0A_1 \right.\\ & + 9A_0A_2r + 5A_1^2r - 8A_1A_2r^2 + A_2^2r^3 \\ & \left. - 36B_1 + 54B_2r - 54B_3r^2 + 36B_4r^3\right)/6, \\ B_0 = {} & \left(-A_0^2 + 2A_0A_1r - 2A_0A_2r^2 - A_1^2r^2 \right.\\ & + 2A_1A_2r^3 - A_2^2r^4 + 12B_1r - 12B_2r^2 \\ & \left. + 12B_3r^3 - 12B_4r^4 + 12B_5r^5\right)/12. \end{aligned} \tag{44}$$

From the mixed derivative $(G_{1xx})_t = (G_{1tx})_x$, one finds the condition

$$\begin{aligned} A_{2tt} = {} & \Big(36A_{2tx}r + 72A_{2t}r_x - 18A_{2xx}r^2 \\ & - 72A_{2x}r_xr - 3A_{2x}A_0 + 3A_{2x}A_1r \\ & - 3A_{2x}A_2r^2 - 18B_{3t} + 18B_{3x}r + 72B_{4t}r \\ & - 72B_{4x}r^2 - 180B_{5t}r^2 + 180B_{5x}r^3 \\ & + 24r_xA_1A_2 - 48r_xA_2^2r + 72r_xB_3 - 288r_xB_4r \\ & + 720r_xB_5r^2 + 3A_0A_2^2 - 12A_0B_4 + 60A_0B_5r \\ & + 4A_1^2A_2 - 19A_1A_2^2r + 6A_1B_3 - 12A_1B_4r \\ & + 19A_2^3r^2 + 18A_2B_2 - 66A_2B_3r \\ & + 144A_2B_4r^2 - 240A_2B_5r^3\Big)/18. \end{aligned} \tag{45}$$

The relation $(G_{1tt})_x = (G_{1tx})_t$ becomes

$$18\left(F_xG_{1t} - F_xG_{1x}r - F_xr_xG_1 - G_{1x}K\right)s_4 + G_1Ks_5 = 0, \tag{46}$$

where

$$\begin{aligned} s_4 = {} & 3A_{2t} - 3A_{2x}r + A_1A_2 - 2A_2^2r + 3B_3 \\ & - 12B_4r + 30B_5r^2, \\ s_5 = {} & 18A_{1xx} + 27A_{1x}A_2 - 36A_{2xx}r + 24A_{2x}A_1 \\ & - 102A_{2x}A_2r + 72B_{3x} - 54B_{4t} - 234B_{4x}r \\ & + 270B_{5t}r + 450B_{5x}r^2 - 15r_xA_2^2 - 180r_xB_4 \\ & + 900r_xB_5r + 6s_{4x} - 45A_0B_5 + 13A_1A_2^2 \\ & + 45A_1B_5r - 26A_2^3r + 39A_2B_3 - 156A_2B_4r \\ & + 345A_2B_5r^2 - 8A_2s_4. \end{aligned} \tag{47}$$

The relation $(A_{2t})_t - A_{2tt} = 0$ provides the condition

$$s_{4t} = \frac{\left(12r_xs_4 + 3s_{4x}r + A_1s_4 - 2A_2rs_4\right)}{3}. \tag{48}$$

Further study depends on s_4.

(i) Case $s_4 \neq 0$

From (46), one gets the derivative

$$g_{1t} = \frac{\left(18\left(F_xG_{1x}r + F_xr_xG_1 + G_{1x}K\right)s_4 - G_1Ks_5\right)}{\left(18F_xs_4\right)}. \tag{49}$$

Differentiating g_{1t} with respect to x, one obtains the derivative

$$F_x = \frac{\left(Ks_6\right)}{\left(108s_4^3\right)}, \tag{50}$$

where

$$\begin{aligned} s_6 = {} & 324A_{2x}s_4^2 - 36s_{4x}s_5 + 36s_{5x}s_4 + 108A_2^2s_4^2 \\ & + 324B_4s_4^2 - 1620B_5rs_4^2 - s_5^2. \end{aligned} \tag{51}$$

The relations $(F_x)_t = (F_t)_x, (G_{1t})_t = G_{1tt}, (F_{xxx})_t = (F_t)_{xxx}$, and $(F_x)_{xx} = F_{xxx}$ provide the conditions

$$\begin{aligned} s_{6t} = {} & \left(30r_xs_6 + 3s_{6x}r + 2A_1s_6 - 4A_2rs_6 + 108A_2s_4^3 \right.\\ & \left. + 18s_4^2s_5\right)/3, \\ s_{5t} = {} & \left(-108A_{1x}s_4^2 - 108A_{2x}rs_4^2 + 108r_xA_2s_4^2 \right.\\ & + 180r_xs_4s_5 + 36s_{4x}rs_5 - 36A_1A_2s_4^2 \\ & + 12A_1s_4s_5 - 36A_2^2rs_4^2 - 24A_2rs_4s_5 \\ & - 108B_3s_4^2 + 108B_4rs_4^2 + 540B_5r^2s_4^2 + rs_5^2 \\ & \left. + rs_6 - 144s_4^3\right)/\left(36s_4\right), \\ A_{2xx} = {} & \left(-5832A_{2x}A_2s_4^3 - 8748B_{4x}s_4^3 + 17496B_{5t}s_4^3 \right.\\ & + 26244B_{5x}rs_4^3 + 43740r_xB_5s_4^3 - 126s_{4x}s_6 \\ & + 45s_{6x}s_4 + 5832A_1B_5s_4^3 - 1296A_2^3s_4^3 \\ & - 5832A_2B_4s_4^3 + 17496A_2B_5rs_4^3 \\ & \left. + 12A_2s_4s_6 - s_5s_6\right)/\left(2916s_4^3\right), \\ s_{6xx} = {} & \left(-324A_{2x}s_4^2s_6 + 2916s_{4xx}s_4s_6 - 11664s_{4x}^2s_6 \right.\\ & + 5832s_{4x}s_{6x}s_4 + 1944s_{4x}A_2s_4s_6 - 162s_{4x}s_5s_6 \\ & - 648s_{6x}A_2s_4^2 + 54s_{6x}s_4s_5 - 108A_2^2s_4^2s_6 \\ & \left. + 18A_2s_4s_5s_6 - 104976B_5s_4^5 + s_6^2\right)/\left(972s_4^2\right). \end{aligned} \tag{52}$$

(ii) Case $s_4 = 0$

From (46), one gets the condition

$$s_5 = 0. \quad (53)$$

Comparing the mixed derivative $(F_{xxx})_t = (F_t)_{xxx}$, one arrives at the condition

$$A_{2xx} = \left(-18A_{2x}A_2 - 27B_{4x} + 54B_{5t} + 81B_{5x}r + 135r_xB_5 + 18A_1B_5 - 4A_2^3 - 18A_2B_4 + 54A_2B_5r\right)/9. \quad (54)$$

All obtained results can be summarized in the following theorems.

Theorem 2. *Sufficient conditions for* (8) *to be linearizable via the generalized linearizing transformation* (5) *with* $F_x = 0$ *are equations* (18), (22), (24), (25), (28), (29), (30), *and* (31).

Corollary 3. *Provided that the sufficient conditions in Theorem 2 are satisfied, the transformation* (5) *mapping equation* (8) *to a linear equation* (6) *is obtained by solving the compatible system of equations* (21), (23), *and* (26) *for the functions* $F(t), G_1(t,x)$, *and* $G_2(t,x)$.

Theorem 4. *Sufficient conditions for equation* (8) *to be linearizable via the generalized linearizing transformation* (5) *with* $F_x \neq 0$ *are as follows.*

(a) *If* $s_1 \neq 0$, *then the conditions are* (18), (40), (41), *and* (42).

(b) *If* $s_1 = 0, s_4 \neq 0$, *then the conditions are* (18), (43), (44), (48), *and* (52).

(c) *If* $s_1 = 0, s_4 = 0$, *then the conditions are* (18), (43), (44), (53), *and* (54).

Corollary 5. *Provided that the sufficient conditions in Theorem 4 are satisfied, the transformation* (5) *mapping equation* (8) *to a linear equation* (6) *is obtained by solving the following compatible system of equations for the functions* $F(t,x), G_1(t,x)$, *and* $G_2(t,x)$:

(a) (16), (17), (34), (37), *and* (38);

(b) (16), (17), (34), (49), *and* (50);

(c) (16), (17), (19), (32), (33), *and* (34).

4. Examples

For understanding the procedure of using the linearization theorems, we consider the following examples.

Example 1. Consider the nonlinear third-order ordinary differential equation

$$\begin{aligned} &3x'^4t^2 + 2x'^3t(3t+2x) + 3x'^2x''t^2x \\ &\quad + x'^2\left(3t^2 + 8tx + 3x^2\right) + 2x'x''tx(t+2x) \\ &\quad - x'x'''t^2x^2 + 2x'x(2t+3x) + 3x''^2t^2x^2 \\ &\quad + x''tx(-t+4x) - x'''t^2x^2 + 3x^2 = 0. \end{aligned} \quad (55)$$

It is an equation of the form (8) in Theorem 1 with the coefficients

$$\begin{gathered} A_2 = -\frac{3}{x}, \qquad A_1 = -\frac{2(t+2x)}{tx}, \\ A_0 = \frac{t-4x}{tx}, \qquad B_5 = 0, \qquad B_4 = -\frac{3}{x^2}, \\ B_3 = -\frac{2(3t+2x)}{tx^2}, \qquad B_2 = -\frac{3t^2+8tx+3x^2}{t^2x^2}, \\ B_1 = -\frac{2(2t+3x)}{t^2x}, \qquad B_0 = -\frac{3}{t^2}, \\ r = 1, \qquad h_1 = -\frac{12}{t}, \qquad h_2 = -\frac{450}{t^2}. \end{gathered} \quad (56)$$

One can check that these coefficients obey the conditions in Theorem 2. Thus, (55) is linearizable via a generalized linearizing transformation. For finding the functions F, G_1, and G_2, we have to solve equations in Corollary 3, which become

$$F_x = 0, \qquad F_t = K, \quad (57)$$

$$G_{1x} = \frac{G_1}{x}, \qquad G_{1tt} = \frac{\left(5G_{1t}^2t^2 - 4G_{1t}G_1t - G_1^2\right)}{\left(3G_1t^2\right)}, \quad (58)$$

$$K_x = 0, \qquad K_t = \frac{\left(4KG_{1t}t - G_1\right)}{\left(3G_1t\right)}. \quad (59)$$

From the first equation of system (58), we get $G_1 = xf(t)$, and choosing $f(t) = t$, we have

$$G_1 = xt \quad (60)$$

and this solution satisfies the second equation. Since $r = 1$, then we obtain

$$G_2 = xt. \quad (61)$$

System (59) becomes

$$K_x = 0, \qquad K_t = 0, \quad (62)$$

and one can take the simplest solution

$$K = 1. \quad (63)$$

System (90) becomes

$$F_x = 0, \qquad F_t = 1, \quad (64)$$

so that we get the particular solution

$$F = t. \tag{65}$$

Thus, one obtains the linearizing transformation

$$X = t, \qquad dT = tx\left(x' + 1\right)dt. \tag{66}$$

Hence, (55) is mapped by the transformation of (66) into the linear equation (6).

Example 2. Consider the nonlinear third-order ordinary differential equation

$$\begin{aligned} &3x'^5t^2 + x'^4t\,(3t + 4x) + x'^3x\,(4t + 3x) \\ &\quad + x'^2x''tx\,(3t + x) + 3x'^2x^2 + 4x'x''tx^2 \\ &\quad - x'x'''t^2x^2 + 3x''^2t^2x^2 = 0. \end{aligned} \tag{67}$$

It is an equation of the form (8) in Theorem 1 with the coefficients

$$\begin{gathered} A_2 = -\frac{(3t + x)}{tx}, \qquad A_1 = -\frac{4}{t}, \qquad A_0 = 0, \\ B_5 = -\frac{3}{x^2}, \qquad B_4 = -\frac{(3t + 4x)}{tx^2}, \\ B_3 = -\frac{(4t + 3x)}{t^2x}, \qquad B_2 = -\frac{3}{t^2}, \\ B_1 = 0, \qquad B_0 = 0, \qquad r = 0, \\ s_1 = -\frac{2}{t^2}, \qquad s_2 = \frac{1}{t^2}, \qquad s_3 = \frac{12\,(t - x)}{t^5x}. \end{gathered} \tag{68}$$

One can check that these coefficients obey the conditions in Theorem 4(a). Thus, (67) is linearizable via a generalized linearizing transformation. For finding the functions F, G_1, and G_2, we have to solve equations in Corollary 5(a), which become

$$F_x = K, \qquad F_t = K, \tag{69}$$

$$\begin{aligned} G_{1t} &= \frac{(G_{1x}tx - G_1t + G_1x)}{(tx)}, \\ G_{1xx} &= \left(G_{1x}^2tx^2 + 4G_{1x}G_1tx - 4G_{1x}G_1x^2 \right. \\ &\quad \left. - 5G_1^2t + 4G_1^2x\right) / \left(3G_1tx^2\right), \end{aligned} \tag{70}$$

$$\begin{aligned} K_x &= \frac{(4K\,(G_{1x}x - G_1))}{(3G_1x)}, \\ K_t &= \frac{(4K\,(G_{1x}x - G_1))}{(3G_1x)}. \end{aligned} \tag{71}$$

From the first equation of system (70), one can take the particular solution

$$G_1 = tx \tag{72}$$

and this solution satisfies the second equation. Since $r = 0$, then we obtain

$$G_2 = 0. \tag{73}$$

System (71) becomes

$$K_x = 0, \qquad K_t = 0, \tag{74}$$

and one can take the simplest solution

$$K = 1. \tag{75}$$

System (69) becomes

$$F_x = 1, \qquad F_t = 1, \tag{76}$$

so that we get the particular solution

$$F = t + x. \tag{77}$$

Thus, one obtains the linearizing transformation

$$X = t + x, \qquad dT = txx'dt. \tag{78}$$

Hence, (67) is mapped by the transformation of (78) into the linear equation (6).

Example 3. Consider the nonlinear third-order ordinary differential equation

$$3x''^2x^2 - 3x'^4 - 3x'^2x''x - x'x'''x^2 = 0. \tag{79}$$

Note that this equation can be reduced to an autonomous equation by the substitution

$$x = tv\,(s)\,, \qquad s = \ln(t)\,, \tag{80}$$

and then to the second-order ordinary differential equation

$$\begin{aligned} &y''z^2y^2\,(z + y) \\ &\quad = y'^2z^2y\,(-2z + y) - 3y'zy\left(z^2 + y^2\right) \\ &\qquad - 3z^4 - 14z^3y - 20z^2y^2 - 15zy^3 - 3y^4, \end{aligned} \tag{81}$$

where $y = y(z)$. However, the latter equation is not linearizable by point transformations.

Equation (79) is an equation of the form (8) in Theorem 1 with the coefficients

$$\begin{gathered} A_2 = \frac{3}{x}, \qquad A_1 = 0, \qquad A_0 = 0, \\ B_5 = 0, \qquad B_4 = \frac{3}{x^2}, \qquad B_3 = 0, \qquad B_2 = 0, \\ B_1 = 0, \qquad B_0 = 0, \qquad r = 0, \\ s_1 = 0, \qquad s_2 = 0, \qquad s_4 = 0, \qquad s_5 = 0. \end{gathered} \tag{82}$$

One can check that these coefficients obey the conditions in Theorem 4(c). Thus, (79) is linearizable via a generalized linearizing transformation. For finding the functions F, G_1, and G_2, we have to solve equations in Corollary 5(c), which become

$$F_t = K,$$

$$F_{xxx} = \left(6F_{xx}F_xG_{1t}G_1Kx^2 + 18F_{xx}G_{1x}G_1K^2x^2 - F_x^3G_{1t}^2x^2 - 6F_x^2G_{1t}G_{1x}Kx^2 - 9F_xG_{1x}^2K^2x^2 - 9F_xG_1^2K^2\right) / \left(6G_1^2K^2x^2\right), \tag{83}$$

$$G_{1tt} = \frac{\left(5G_{1t}^2\right)}{\left(3G_1\right)},$$

$$G_{1tx} = \frac{\left(G_{1t}\left(-F_xG_{1t}x + 6G_{1x}Kx + 3G_1K\right)\right)}{\left(3G_1Kx\right)}, \tag{84}$$

$$G_{1xx} = \left(6F_{xx}G_{1t}G_1Kx^2 - F_x^2G_{1t}^2x^2 - 6F_xG_{1t}G_{1x}Kx^2 + 9G_{1x}^2K^2x^2 - 9G_1^2K^2\right) / \left(6G_1K^2x^2\right),$$

$$K_x = \frac{\left(F_xG_{1t}x + 3G_{1x}Kx + 3G_1K\right)}{\left(3G_1x\right)}, \quad K_t = \frac{\left(4G_{1t}K\right)}{\left(3G_1\right)}. \tag{85}$$

From the first equation of system (84), one can take the particular solution

$$G_1 = x \tag{86}$$

and this solution satisfies the second and third equations. Since $r = 0$, then we obtain

$$G_2 = 0. \tag{87}$$

System (85) becomes

$$K_x = \frac{(2K)}{x}, \qquad K_t = 0, \tag{88}$$

and one can take the particular solution

$$K = x^2. \tag{89}$$

System (83) becomes

$$F_x = x^2, \qquad F_{xxx} = \frac{\left(3\left(F_{xx}x - F_x\right)\right)}{x^2}, \tag{90}$$

so that one obtains the particular solution of the first equation as

$$F = tx^2 \tag{91}$$

and this solution satisfies the second equation. Then we get the linearizing transformation

$$X = tx^2, \qquad dT = xx'dt. \tag{92}$$

Hence, equation (79) is mapped by the transformation of (92) into the linear equation (6).

5. Conclusion

This paper is devoted to find the conditions which allow the third-order ordinary differential equation to be transformed into the simplest linear equation. Necessary conditions which guarantee that the third-order ordinary differential equation can be linearized are found in Theorem 1. Theorems 2 and 4 are sufficient conditions for the linearization problem. The linearizing transformation can be found by solving the compatible system in Corollaries 3 and 5. Finally, some examples are provided to demonstrate our procedure.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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