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Research Article

On Positive Solutions and Mann Iterative Schemes of a Third Order Difference Equation

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The existence of uncountably many positive solutions and convergence of the Mann iterative schemes for a third order nonlinear neutral delay difference equation are proved. Six examples are given to illustrate the results presented in this paper.

1. Introduction and Preliminaries

Recently, many researchers studied the oscillation, nonoscillation, and existence of solutions for linear and nonlinear second and third order difference equations and systems see, for example, [1–23] and the references cited therein. By means of the Reccati transformation techniques, Saker [18] discussed the third order difference equation

$$\Delta^3 x_n + p_n x_{n+1} = 0, \quad \forall n \geq n_0, \tag{1}$$

and presented some sufficient conditions which ensure that all solutions are to be oscillatory or tend to zero. Utilizing the Schauder fixed point theorem, Yan and Liu [22] proved the existence of a bounded nonoscillatory solution for the third order difference equation

$$\Delta^3 x_n + f(n, x_n, x_{n-\tau}) = 0, \quad \forall n \geq n_0. \tag{2}$$

Agarwal [2] established the oscillatory and asymptotic properties for the third order nonlinear difference equation

$$\Delta^3 x_n + q_n f(x_{n+1}) = 0, \quad \forall n \geq 1. \tag{3}$$

Andruch-Sobiło and Migda [4] studied the third order linear difference equation of neutral type

$$\Delta^3 (x_n - p_n x_{\sigma n}) \pm q_n x_{\tau n} = 0, \quad \forall n \geq n_0, \tag{4}$$

and obtained sufficient conditions which ensure that all solutions of the equation are oscillatory. Grace and Hamedani [6] discussed the difference equation

$$\Delta^3 (x_n - x_{n-\tau}) \pm q_n |x_{n-\sigma}|^3 \operatorname{sgn} x_{n-\sigma} = 0, \quad \forall n \geq 0, \tag{5}$$

and gave some new criteria for the oscillation of all solutions and all bounded solutions.

Our goal is to discuss solvability and convergence of the Mann iterative schemes for the following third order nonlinear neutral delay difference equation:

$$\begin{aligned} &\Delta^3 (x_n + b_n x_{n-\tau}) + \Delta h\left(n, x_{h_{1n}}, x_{h_{2n}}, \ldots, x_{h_{kn}}\right) \\ &\quad + f\left(n, x_{f_{1n}}, x_{f_{2n}}, \ldots, x_{f_{kn}}\right) = c_n, \quad \forall n \geq n_0, \end{aligned} \tag{6}$$

where $\tau, k, n_0 \in \mathbb{N}$, $\{b_n\}_{n\in\mathbb{N}_{n_0}}, \{c_n\}_{n\in\mathbb{N}_{n_0}} \subset \mathbb{R}$, $h, f \in C(\mathbb{N}_{n_0} \times \mathbb{R}^k, \mathbb{R})$, $\{h_{ln}\}_{n\in\mathbb{N}_{n_0}}, \{f_{ln}\}_{n\in\mathbb{N}_{n_0}} \subseteq \mathbb{N}$, and

$$\lim_{n\to\infty} h_{ln} = \lim_{n\to\infty} f_{ln} = +\infty, \quad l \in \{1, 2, \ldots, k\}. \tag{7}$$

By employing the Banach fixed point theorem and some new techniques, we establish the existence of uncountably many positive solutions of (6), conceive a few Mann iterative schemes for approximating these positive solutions, and prove their convergence and the error estimates. Six nontrivial examples are included.

Throughout this paper, we assume that Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$ $\mathbb{N}_0$ and $\mathbb{N}$ denote the sets of nonnegative integers and positive integers, respectively,

$$\begin{aligned} \mathbb{N}_t &= \{n : n \in \mathbb{N} \text{ with } n \geq t\}, \quad \forall t \in \mathbb{N}, \\ \beta &= \min\left\{n_0 - \tau, \inf\left\{h_{ln}, f_{ln} : 1 \leq l \leq k, n \in \mathbb{N}_{n_0}\right\}\right\} \in \mathbb{N}, \\ H_n &= \max\left\{h_{ln}^2 : l \in \{1, 2, \ldots, k\}\right\}, \quad \forall n \in \mathbb{N}_{n_0}, \\ F_n &= \max\left\{f_{ln}^2 : l \in \{1, 2, \ldots, k\}\right\}, \quad \forall n \in \mathbb{N}_{n_0}, \end{aligned} \tag{8}$$

and l_β^∞ represents the Banach space of all real sequences on $\mathbb{N}_\beta$ with norm

$$\begin{aligned} \|x\| &= \sup_{n \in \mathbb{N}_\beta} \left|\frac{x_n}{n^2}\right| < +\infty \quad \text{for each } x = \{x_n\}_{n \in \mathbb{N}_\beta} \in l_\beta^\infty, \\ A(N, M) &= \left\{x = \{x_n\}_{n \in \mathbb{N}_\beta} \in l_\beta^\infty : N \leq \frac{x_n}{n^2} \leq M, n \in \mathbb{N}_\beta\right\} \\ &\qquad \text{for any } M > N > 0. \end{aligned} \tag{9}$$

It is easy to see that $A(N, M)$ is a closed and convex subset of l_β^∞. By a solution of (6), we mean a sequence $\{x_n\}_{n \in \mathbb{N}_\beta}$ with a positive integer $T \geq n_0 + \tau + \beta$ such that (6) holds for all $n \geq T$.

Lemma 1. *Let $\{p_t\}_{t \in \mathbb{N}}$ be a nonnegative sequence and $\tau \in \mathbb{N}$.*

(i) *If $\lim_{n \to \infty} (1/n^2) \sum_{t=n+\tau}^\infty t^2 p_t = 0$, then $\lim_{n \to \infty} (1/n^2) \sum_{i=1}^\infty \sum_{s=n+i\tau}^\infty \sum_{t=s}^\infty p_t = 0$.*

(ii) *If $\lim_{n \to \infty} (1/n^2) \sum_{t=n+\tau}^\infty t^3 p_t = 0$, then $\lim_{n \to \infty} (1/n^2) \sum_{i=1}^\infty \sum_{u=n+i\tau}^\infty \sum_{s=u}^\infty \sum_{t=s}^\infty p_t = 0$.*

Proof. Note that

$$\begin{aligned} 0 &\leq \frac{1}{n^2} \sum_{i=1}^\infty \sum_{s=n+i\tau}^\infty \sum_{t=s}^\infty p_t \\ &= \frac{1}{n^2} \sum_{i=1}^\infty \left(\sum_{t=n+i\tau}^\infty p_t + \sum_{t=n+1+i\tau}^\infty p_t + \sum_{t=n+2+i\tau}^\infty p_t + \cdots \right) \\ &= \frac{1}{n^2} \sum_{i=1}^\infty \sum_{t=n+i\tau}^\infty (1 + t - n - i\tau) p_t \leq \frac{1}{n^2} \sum_{i=1}^\infty \sum_{t=n+i\tau}^\infty t p_t \\ &= \frac{1}{n^2} \left(\sum_{t=n+\tau}^\infty t p_t + \sum_{t=n+2\tau}^\infty t p_t + \sum_{t=n+3\tau}^\infty t p_t + \cdots \right) \\ &\leq \frac{1}{n^2} \sum_{t=n+\tau}^\infty \left(1 + \frac{t - n - \tau}{\tau}\right) t p_t = \frac{1}{n^2} \sum_{t=n+\tau}^\infty \frac{t - n}{\tau} t p_t \\ &\leq \frac{1}{n^2 \tau} \sum_{t=n+\tau}^\infty t^2 p_t \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \end{aligned} \tag{10}$$

that is,

$$\lim_{n \to \infty} \frac{1}{n^2} \sum_{i=1}^\infty \sum_{s=n+i\tau}^\infty \sum_{t=s}^\infty p_t = 0. \tag{11}$$

As in the proof of (10), we infer that

$$\begin{aligned} 0 &\leq \frac{1}{n^2} \sum_{i=1}^\infty \sum_{u=n+i\tau}^\infty \sum_{s=u}^\infty \sum_{t=s}^\infty p_t \\ &= \frac{1}{n^2} \sum_{i=1}^\infty \sum_{u=n+i\tau}^\infty \sum_{t=u}^\infty (1 + t - u) p_t \\ &\leq \frac{1}{n^2} \sum_{i=1}^\infty \sum_{u=n+i\tau}^\infty \sum_{t=u}^\infty t p_t \leq \frac{1}{n^2 \tau} \sum_{t=n+\tau}^\infty t^3 p_t \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \end{aligned} \tag{12}$$

which implies that

$$\lim_{n \to \infty} \frac{1}{n^2} \sum_{i=1}^\infty \sum_{u=n+i\tau}^\infty \sum_{s=u}^\infty \sum_{t=s}^\infty p_t = 0. \tag{13}$$

This completes the proof. □

2. Uncountably Many Positive Solutions and Mann Iterative Schemes

In this section, using the Banach fixed point theorem and Mann iterative schemes, we establish the existence of uncountably many positive solutions of (6), prove convergence of the Mann iterative schemes relative to these positive solutions, and compute the error estimates between the Mann iterative schemes and the positive solutions.

Theorem 2. *Assume that there exist two constants M and N with $M > N > 0$ and four nonnegative sequences $\{P_n\}_{n \in \mathbb{N}_{n_0}}$, $\{Q_n\}_{n \in \mathbb{N}_{n_0}}$, $\{R_n\}_{n \in \mathbb{N}_{n_0}}$ and $\{W_n\}_{n \in \mathbb{N}_{n_0}}$ satisfying*

$$\begin{aligned} &\left|f(n, u_1, u_2, \ldots, u_k) - f(n, \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k)\right| \\ &\quad \leq P_n \max\{|u_l - \bar{u}_l| : 1 \leq l \leq k\}, \\ &\left|h(n, u_1, u_2, \ldots, u_k) - h(n, \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k)\right| \\ &\quad \leq R_n \max\{|u_l - \bar{u}_l| : 1 \leq l \leq k\}, \\ &\forall (n, u_l, \bar{u}_l) \in \mathbb{N}_{n_0} \times (\mathbb{R}^+ \setminus \{0\})^2, \quad 1 \leq l \leq k; \end{aligned} \tag{14}$$

$$\begin{aligned} &\left|f(n, u_1, u_2, \ldots, u_k)\right| \leq Q_n, \qquad \left|h(n, u_1, u_2, \ldots, u_k)\right| \leq W_n, \\ &\qquad \forall (n, u_l) \in \mathbb{N}_{n_0} \times (\mathbb{R}^+ \setminus \{0\}), \quad 1 \leq l \leq k; \end{aligned} \tag{15}$$

$$\lim_{n \to \infty} \frac{1}{n^2} \sum_{i=1}^\infty \sum_{u=n+i\tau}^\infty \sum_{s=u}^\infty \max\{R_s H_s, W_s\} = 0; \tag{16}$$

$$\lim_{n \to \infty} \frac{1}{n^2} \sum_{i=1}^\infty \sum_{u=n+i\tau}^\infty \sum_{s=u}^\infty \sum_{t=s}^\infty \max\{P_t F_t, Q_t, |c_t|\} = 0; \tag{17}$$

$$b_n = -1 \quad \textit{eventually}. \tag{18}$$

Then one has the following.

(a) *For any $L \in (N, M)$, there exist $\theta \in (0,1)$ and $T \geq n_0 + \tau + \beta$ such that, for each $x_0 = \{x_{0n}\}_{n\in\mathbb{N}_\beta} \in A(N,M)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0} = \{\{x_{mn}\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0}$ generated by the scheme*

$$x_{m+1n} = \begin{cases} (1-\alpha_m) x_{mn} + \alpha_m \Big\{ n^2 L + \sum_{i=1}^{\infty} \sum_{u=n+i\tau}^{\infty} \sum_{s=u}^{\infty} \Big[h\left(s, x_{mh_{1s}}, x_{mh_{2s}}, \ldots, x_{mh_{ks}}\right) - \sum_{t=s}^{\infty} \left(f\left(t, x_{mf_{1t}}, x_{mf_{2t}}, \ldots, x_{mf_{kt}}\right) - c_t \right) \Big] \Big\}, & n \geq T, \quad m \geq 0, \\ (1-\alpha_m) x_{mT} + \alpha_m \Big\{ T^2 L + \sum_{i=1}^{\infty} \sum_{u=T+i\tau}^{\infty} \sum_{s=u}^{\infty} \Big[h\left(s, x_{mh_{1s}}, x_{mh_{2s}}, \ldots, x_{mh_{ks}}\right) - \sum_{t=s}^{\infty} \left(f\left(t, x_{mf_{1t}}, x_{mf_{2t}}, \ldots, x_{mf_{kt}}\right) - c_t \right) \Big] \Big\}, & \beta \leq n < T, \quad m \geq 0, \end{cases} \tag{19}$$

converges to a positive solution $z = \{z_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$ of (6) with $\lim_{n\to\infty} z_n = +\infty$ and has the following error estimate:

$$\|x_{m+1} - z\| \leq e^{-(1-\theta)\sum_{i=0}^{m} \alpha_i} \|x_m - z\|, \quad \forall m \in \mathbb{N}_0, \tag{20}$$

where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $[0,1]$ such that

$$\sum_{m=0}^{\infty} \alpha_m = +\infty. \tag{21}$$

(b) *Equation (6) possesses uncountably many positive solutions in $A(N,M)$.*

Proof. Firstly, we show that (a) holds. Put $L \in (N,M)$. It follows from (16)~(18) that there exist $\theta \in (0,1)$ and $T \geq n_0 + \tau + \beta$ satisfying

$$\theta = \frac{1}{T^2} \sum_{i=1}^{\infty} \sum_{u=T+i\tau}^{\infty} \sum_{s=u}^{\infty} \left(R_s H_s + \sum_{t=s}^{\infty} P_t F_t \right); \tag{22}$$

$$\frac{1}{T^2} \sum_{i=1}^{\infty} \sum_{u=T+i\tau}^{\infty} \sum_{s=u}^{\infty} \left(W_s + \sum_{t=s}^{\infty} (Q_t + |c_t|) \right) < \min\{M - L, L - N\}; \tag{23}$$

$$b_n = -1, \quad \forall n \geq T. \tag{24}$$

Define a mapping $S_L : A(N,M) \to l_\beta^\infty$ by

$$S_L x_n = \begin{cases} n^2 L + \sum_{i=1}^{\infty} \sum_{u=n+i\tau}^{\infty} \sum_{s=u}^{\infty} \Big\{ h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right) - \sum_{t=s}^{\infty} \left[f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right) - c_t \right] \Big\}, & n \geq T, \\ S_L x_T, & \beta \leq n < T, \end{cases} \tag{25}$$

for each $x = \{x_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$. In light of (14), (15), (22), (23), and (25), we obtain that for each $x = \{x_n\}_{n\in\mathbb{N}_\beta}, y = \{y_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$

$$\begin{aligned} \left| \frac{S_L x_n}{n^2} - \frac{S_L y_n}{n^2} \right| &\leq \frac{1}{n^2} \sum_{i=1}^{\infty} \sum_{u=n+i\tau}^{\infty} \sum_{s=u}^{\infty} \Big[\left| h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right) - h\left(s, y_{h_{1s}}, y_{h_{2s}}, \ldots, y_{h_{ks}}\right) \right| \\ &\quad + \sum_{t=s}^{\infty} \left| f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right) - f\left(t, y_{f_{1t}}, y_{f_{2t}}, \ldots, y_{f_{kt}}\right) \right| \Big] \\ &\leq \frac{1}{n^2} \sum_{i=1}^{\infty} \sum_{u=n+i\tau}^{\infty} \sum_{s=u}^{\infty} \Big[R_s \max\left\{ \left| x_{h_{ls}} - y_{h_{ls}} \right| : 1 \leq l \leq k \right\} + \sum_{t=s}^{\infty} P_t \max\left\{ \left| x_{f_{lt}} - y_{f_{lt}} \right| : 1 \leq l \leq k \right\} \Big] \\ &\leq \frac{\|x - y\|}{n^2} \sum_{i=1}^{\infty} \sum_{u=n+i\tau}^{\infty} \sum_{s=u}^{\infty} \Big[R_s \max\left\{ h_{ls}^2 : 1 \leq l \leq k \right\} + \sum_{t=s}^{\infty} P_t \max\left\{ f_{lt}^2 : 1 \leq l \leq k \right\} \Big] \\ &\leq \frac{\|x - y\|}{T^2} \sum_{i=1}^{\infty} \sum_{u=T+i\tau}^{\infty} \sum_{s=u}^{\infty} \left(R_s H_s + \sum_{t=s}^{\infty} P_t F_t \right) \\ &= \theta \|x - y\|, \end{aligned}$$

$$\begin{aligned} \left| \frac{S_L x_n}{n^2} - L \right| &= \left| \frac{1}{n^2} \sum_{i=1}^{\infty} \sum_{u=n+i\tau}^{\infty} \sum_{s=u}^{\infty} \Big\{ h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right) - \sum_{t=s}^{\infty} \left[f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right) - c_t \right] \Big\} \right| \\ &\leq \frac{1}{n^2} \sum_{i=1}^{\infty} \sum_{u=n+i\tau}^{\infty} \sum_{s=u}^{\infty} \Big\{ \left| h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right) \right| \end{aligned}$$

$$+\sum_{t=s}^{\infty}\left[\left|f\left(t,x_{f_{1t}},x_{f_{2t}},\dots,x_{f_{kt}}\right)\right|+\left|c_t\right|\right]\Bigg\}$$

$$\le \frac{1}{T^2}\sum_{i=1}^{\infty}\sum_{u=T+i\tau}^{\infty}\sum_{s=u}^{\infty}\left[W_s+\sum_{t=s}^{\infty}\left(Q_t+\left|c_t\right|\right)\right]$$

$$< \min\{M-L,L-N\}, \tag{26}$$

which yield that

$$S_L(A(N,M))\subseteq A(N,M),$$
$$\left\|S_Lx-S_Ly\right\|\le\theta\left\|x-y\right\|,\quad \forall x,\ y\in A(N,M), \tag{27}$$

which implies that S_L is a contraction in $A(N,M)$. The Banach fixed point theorem and (27) ensure that S_L has a unique fixed point $z=\{z_n\}_{n\in\mathbb{N}_\beta}\in A(N,M)$; that is,

$$z_n = n^2L+\sum_{i=1}^{\infty}\sum_{u=n+i\tau}^{\infty}\sum_{s=u}^{\infty}\Bigg\{h\left(s,z_{h_{1s}},z_{h_{2s}},\dots,z_{h_{ks}}\right)-\sum_{t=s}^{\infty}\left[f\left(t,z_{f_{1t}},z_{f_{2t}},\dots,z_{f_{kt}}\right)-c_t\right]\Bigg\},\quad \forall n\ge T,$$

$$z_{n-\tau} = (n-\tau)^2L+\sum_{i=1}^{\infty}\sum_{u=n+(i-1)\tau}^{\infty}\sum_{s=u}^{\infty}\Bigg\{h\left(s,z_{h_{1s}},z_{h_{2s}},\dots,z_{h_{ks}}\right)-\sum_{t=s}^{\infty}\left[f\left(t,z_{f_{1t}},z_{f_{2t}},\dots,z_{f_{kt}}\right)-c_t\right]\Bigg\},\quad \forall n\ge T+\tau, \tag{28}$$

which mean that

$$z_n-z_{n-\tau}=\left(2n\tau-\tau^2\right)L-\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\Bigg\{h\left(s,z_{h_{1s}},z_{h_{2s}},\dots,z_{h_{ks}}\right)-\sum_{t=s}^{\infty}\left[f\left(t,z_{f_{1t}},z_{f_{2t}},\dots,z_{f_{kt}}\right)-c_t\right]\Bigg\},\quad \forall n\ge T+\tau, \tag{29}$$

which yields that

$$\Delta\left(z_n-z_{n-\tau}\right)=2\tau L+\sum_{s=n}^{\infty}\Bigg\{h\left(s,z_{h_{1s}},z_{h_{2s}},\dots,z_{h_{ks}}\right)-\sum_{t=s}^{\infty}\left[f\left(t,z_{f_{1t}},z_{f_{2t}},\dots,z_{f_{kt}}\right)-c_t\right]\Bigg\},\quad \forall n\ge T+\tau,$$

$$\Delta^2\left(z_n-z_{n-\tau}\right)=-h\left(n,z_{h_{1n}},z_{h_{2n}},\dots,z_{h_{kn}}\right)+\sum_{t=n}^{\infty}\left[f\left(t,z_{f_{1t}},z_{f_{2t}},\dots,z_{f_{kt}}\right)-c_t\right],\quad \forall n\ge T+\tau, \tag{30}$$

which gives that

$$\Delta^3\left(z_n-z_{n-\tau}\right)=-\Delta h\left(n,z_{h_{1n}},z_{h_{2n}},\dots,z_{h_{kn}}\right)-f\left(t,z_{f_{1n}},z_{f_{2n}},\dots,z_{f_{kn}}\right)+c_n,\quad \forall n\ge T+\tau, \tag{31}$$

which together with (24) implies that $z=\{z_n\}_{n\in\mathbb{N}_\beta}$ is a positive solution of (6) in $A(N,M)$. Note that

$$N\le\frac{z_n}{n^2}\le M,\quad \forall n\in\mathbb{N}_\beta, \tag{32}$$

which guarantees that $\lim_{n\to\infty}z_n=+\infty$. It follows from (19), (22), (24), (25), and (27) that for any $m\in\mathbb{N}_0$ and $n\ge T$

$$\left|\frac{x_{m+1n}}{n^2}-\frac{z_n}{n^2}\right|=\frac{1}{n^2}\Bigg|\left(1-\alpha_m\right)x_{mn}+\alpha_m\Bigg\{n^2L+\sum_{i=1}^{\infty}\sum_{u=n+i\tau}^{\infty}\sum_{s=u}^{\infty}\Bigg[h\left(s,x_{mh_{1s}},x_{mh_{2s}},\dots,x_{mh_{ks}}\right)-\sum_{t=s}^{\infty}\left(f\left(t,x_{mf_{1t}},x_{mf_{2t}},\dots,x_{mf_{kt}}\right)-c_t\right)\Bigg]\Bigg\}-z_n\Bigg|$$

$$\le\left(1-\alpha_m\right)\frac{\left|x_{mn}-z_n\right|}{n^2}+\alpha_m\frac{\left|S_Lx_{mn}-S_Lz_n\right|}{n^2}$$

$$\le\left(1-\alpha_m\right)\left\|x_m-z\right\|+\theta\alpha_m\left\|x_m-z\right\|$$

$$\le\left[1-(1-\theta)\alpha_m\right]\left\|x_m-z\right\|,\quad \forall m\in\mathbb{N}_0,n\ge T, \tag{33}$$

which implies that

$$\left\|x_{m+1}-z\right\|\le e^{-(1-\theta)\sum_{i=0}^{m}\alpha_i}\left\|x_m-z\right\|,\quad \forall m\in\mathbb{N}_0. \tag{34}$$

That is, (20) holds. Thus Lemma 1, (20), and (21) guarantee that $\lim_{m\to\infty}x_m=z$.

Next we show that (b) holds. Let $L_1,\ L_2\in(N,M)$ and $L_1\ne L_2$. As in the proof of (a), we deduce similarly that, for each $c\in\{1,2\}$, there exist constants $\theta_c\in(0,1)$ and $T_c\ge n_0+\tau+\beta$ and a mapping S_{L_c} satisfying

(22)~(27), where θ, L, and T are replaced by θ_c, L_c, and T_c, respectively, and the mapping S_{L_c} has a fixed point $z^c = \{z_n^c\}_{n\in\mathbb{N}_\beta} \in A(N,M)$, which is a positive solution of (6) in $A(N,M)$ with $\lim_{n\to\infty} z_n^c = +\infty$. It follows that

$$z_n^c = n^2 L_c + \sum_{i=1}^{\infty}\sum_{u=n+i\tau}^{\infty}\sum_{s=u}^{\infty}\left\{h\left(s, z_{h_{1s}}^c, z_{h_{2s}}^c, \dots, z_{h_{ks}}^c\right) - \sum_{t=s}^{\infty}\left[f\left(t, z_{f_{1t}}^c, z_{f_{2t}}^c, \dots, z_{f_{kt}}^c\right) - c_t\right]\right\}, \quad \forall n \ge T_c, \tag{35}$$

which together with (14) and (20) means that for $n \ge \max\{T_1, T_2\}$

$$\begin{aligned}
\left|\frac{z_n^1}{n^2} - \frac{z_n^2}{n^2}\right|
&\ge |L_1 - L_2| - \frac{1}{n^2}\sum_{i=1}^{\infty}\sum_{u=n+i\tau}^{\infty}\sum_{s=u}^{\infty}\left|h\left(s, z_{h_{1s}}^1, z_{h_{2s}}^1, \dots, z_{h_{ks}}^1\right) - h\left(s, z_{h_{1s}}^2, z_{h_{2s}}^2, \dots, z_{h_{ks}}^2\right)\right| \\
&\quad + \sum_{t=s}^{\infty}\left|f\left(t, z_{f_{1t}}^1, z_{f_{2t}}^1, \dots, z_{f_{kt}}^1\right) - f\left(t, z_{f_{1t}}^2, z_{f_{2t}}^2, \dots, z_{f_{kt}}^2\right)\right| \\
&\ge |L_1 - L_2| - \frac{1}{n^2}\sum_{i=1}^{\infty}\sum_{u=n+i\tau}^{\infty}\sum_{s=u}^{\infty}\left[R_s \max\left\{\left|z_{h_{ls}}^1 - z_{h_{ls}}^2\right| : 1 \le l \le k\right\} + \sum_{t=s}^{\infty} P_t \max\left\{\left|z_{f_{lt}}^1 - z_{f_{lt}}^2\right| : 1 \le l \le k\right\}\right] \\
&\ge |L_1 - L_2| - \frac{\left\|z^1 - z^2\right\|}{n^2}\sum_{i=1}^{\infty}\sum_{u=n+i\tau}^{\infty}\sum_{s=u}^{\infty}\left(R_s H_s + \sum_{t=s}^{\infty} P_t F_t\right) \\
&\ge |L_1 - L_2| - \frac{\left\|z^1 - z^2\right\|}{\max\{T_1^2, T_2^2\}}\sum_{i=1}^{\infty}\sum_{u=\max\{T_1,T_2\}+i\tau}^{\infty}\sum_{s=u}^{\infty}\left(R_s H_s + \sum_{t=s}^{\infty} P_t F_t\right) \\
&\ge |L_1 - L_2| - \max\{\theta_1, \theta_2\}\left\|z^1 - z^2\right\|,
\end{aligned} \tag{36}$$

which yields that

$$\left\|z^1 - z^2\right\| \ge \frac{|L_1 - L_2|}{1 + \max\{\theta_1, \theta_2\}} > 0; \tag{37}$$

that is, $z^1 \ne z^2$. This completes the proof. □

Theorem 3. *Assume that there exist two constants M and N with $M > N > 0$ and four nonnegative sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, $\{R_n\}_{n\in\mathbb{N}_{n_0}}$, and $\{W_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (14), (15), *and*

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\max\{R_s H_s, W_s\} = 0; \tag{38}$$

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\max\{P_t F_t, Q_t, |c_t|\} = 0; \tag{39}$$

$$b_n = 1 \quad \text{eventually}. \tag{40}$$

Then one has the following.

(a) *For any $L \in (N,M)$, there exist $\theta \in (0,1)$ and $T \ge n_0 + \tau + \beta$ such that, for each $x_0 = \{x_{0n}\}_{n\in\mathbb{N}_\beta} \in A(N,M)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0} = \{\{x_{mn}\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0}$ generated by the scheme*

$$x_{m+1n} = \begin{cases}
(1-\alpha_m)x_{mn} + \alpha_m\left\{n^2 L - \sum_{i=1}^{\infty}\sum_{u=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{s=u}^{\infty}\left[h\left(s, x_{mh_{1s}}, x_{mh_{2s}}, \dots, x_{mh_{ks}}\right) - \sum_{t=s}^{\infty}\left(f\left(t, x_{mf_{1t}}, x_{mf_{2t}}, \dots, x_{mf_{kt}}\right) - c_t\right)\right]\right\}, & n \ge T, \quad m \ge 0, \\
(1-\alpha_m)x_{mT} + \alpha_m\left\{T^2 L - \sum_{i=1}^{\infty}\sum_{u=T+(2i-1)\tau}^{T+2i\tau-1}\sum_{s=u}^{\infty}\left[h\left(s, x_{mh_{1s}}, x_{mh_{2s}}, \dots, x_{mh_{ks}}\right) - \sum_{t=s}^{\infty}\left(f\left(t, x_{mf_{1t}}, x_{mf_{2t}}, \dots, x_{mf_{kt}}\right) - c_t\right)\right]\right\}, & \beta \le n < T, \quad m \ge 0,
\end{cases} \tag{41}$$

converges to a positive solution $z = \{z_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$ of (6) *with $\lim_{n\to\infty} z_n = +\infty$ and has the error estimate* (20), *where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $[0,1]$ satisfying* (21).

(b) *Equation* (6) *possesses uncountably many positive solutions in $A(N,M)$.*

Proof. Let $L \in (N, M)$. It follows from (38)~(40) that there exist $\theta \in (0,1)$ and $T \geq n_0 + \tau + \beta$ satisfying

$$\theta = \frac{1}{T^2}\sum_{u=T}^{\infty}\sum_{s=u}^{\infty}\left(R_s H_s + \sum_{t=s}^{\infty} P_t F_t\right); \tag{42}$$

$$\frac{1}{T^2}\sum_{u=T}^{\infty}\sum_{s=u}^{\infty}\left(W_s + \sum_{t=s}^{\infty}\left(Q_t + |c_t|\right)\right) < \min\{M-L, L-N\}; \tag{43}$$

$$b_n = 1, \quad \forall n \geq T. \tag{44}$$

Define a mapping $S_L : A(N,M) \to l_\beta^\infty$ by

$$S_L x_n = \begin{cases} n^2 L - \sum_{i=1}^{\infty}\sum_{u=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{s=u}^{\infty}\left\{h\left(s, x_{h_{1s}}, x_{h_{2s}}, \dots, x_{h_{ks}}\right) - \sum_{t=s}^{\infty}\left[f\left(t, x_{f_{1t}}, x_{f_{2t}}, \dots, x_{f_{kt}}\right) - c_t\right]\right\}, & n \geq T, \\ S_L x_T, & \beta \leq n < T, \end{cases} \tag{45}$$

for each $x = \{x_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$. Using (14), (15), (42), (43), and (45), we get that for each $x = \{x_n\}_{n\in\mathbb{N}_\beta}$, $y = \{y_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$ and $n \geq T$

$$\begin{aligned} &\left|\frac{S_L x_n}{n^2} - \frac{S_L y_n}{n^2}\right| \\ &\leq \frac{1}{n^2}\sum_{i=1}^{\infty}\sum_{u=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{s=u}^{\infty}\Big[\Big|h\left(s, x_{h_{1s}}, x_{h_{2s}}, \dots, x_{h_{ks}}\right) - h\left(s, y_{h_{1s}}, y_{h_{2s}}, \dots, y_{h_{ks}}\right)\Big| \\ &\quad + \sum_{t=s}^{\infty}\Big|f\left(t, x_{f_{1t}}, x_{f_{2t}}, \dots, x_{f_{kt}}\right) - f\left(t, y_{f_{1t}}, y_{f_{2t}}, \dots, y_{f_{kt}}\right)\Big|\Big] \\ &\leq \frac{\|x-y\|}{n^2}\sum_{i=1}^{\infty}\sum_{u=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{s=u}^{\infty}\left(R_s H_s + \sum_{t=s}^{\infty} P_t F_t\right) \\ &\leq \frac{\|x-y\|}{T^2}\sum_{u=T}^{\infty}\sum_{s=u}^{\infty}\left(R_s H_s + \sum_{t=s}^{\infty} P_t F_t\right) = \theta\|x-y\|, \\ &\left|\frac{S_L x_n}{n^2} - L\right| \\ &\leq \frac{1}{n^2}\sum_{i=1}^{\infty}\sum_{u=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{s=u}^{\infty}\Big\{\Big|h\left(s, x_{h_{1s}}, x_{h_{2s}}, \dots, x_{h_{ks}}\right)\Big| + \sum_{t=s}^{\infty}\Big[\Big|f\left(t, x_{f_{1t}}, x_{f_{2t}}, \dots, x_{f_{kt}}\right)\Big| + |c_t|\Big]\Big\} \\ &\leq \frac{1}{T^2}\sum_{u=T}^{\infty}\sum_{s=u}^{\infty}\left(W_s + \sum_{t=s}^{\infty}\left[W_s + \sum_{t=s}^{\infty}\left(Q_t + |c_t|\right)\right]\right) \\ &< \min\{M-L, L-N\}, \end{aligned} \tag{46}$$

which imply (27). The rest of the proof is similar to the proof of Theorem 2 and is omitted. This completes the proof. □

Theorem 4. *Assume that there exist three constants b, M, and N with $(1-b)M > N > 0$ and four nonnegative sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, $\{R_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{W_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (14), (15), (38), (39) *and*

$$0 \leq b_n \leq b < 1 \quad \textit{eventually}. \tag{47}$$

Then one has the following.

(a) *For any $L \in (bM + N, M)$, there exist $\theta \in (0,1)$ and $T \geq n_0 + \tau + \beta$ such that, for any $x_0 = \{x_{0n}\}_{n\in\mathbb{N}_\beta} \in A(N,M)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0} = \{\{x_{mn}\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0}$ generated by the scheme*

$$x_{m+1n} = \begin{cases} (1-\alpha_m)x_{mn} + \alpha_m\Big\{n^2 L - b_n x_{mn-\tau} - \sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\Big[h\left(s, x_{mh_{1s}}, x_{mh_{2s}}, \dots, x_{mh_{ks}}\right) - \sum_{t=s}^{\infty}\left(f\left(t, x_{mf_{1t}}, x_{mf_{2t}}, \dots, x_{mf_{kt}}\right) - c_t\right)\Big]\Big\}, & n \geq T, \ m \geq 0, \\ (1-\alpha_m)x_{mT} + \alpha_m\Big\{T^2 L - b_T x_{mT-\tau} - \sum_{u=T}^{\infty}\sum_{s=u}^{\infty}\Big[h\left(s, x_{mh_{1s}}, x_{mh_{2s}}, \dots, x_{mh_{ks}}\right) - \sum_{t=s}^{\infty}\left(f\left(t, x_{mf_{1t}}, x_{mf_{2t}}, \dots, x_{mf_{kt}}\right) - c_t\right)\Big]\Big\}, & \beta \leq n < T, \ m \geq 0, \end{cases} \tag{48}$$

converges to a positive solution $z = \{z_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$ of (6) *with $\lim_{n\to\infty} z_n = +\infty$ and has the* *error estimate* (20)*, where* $\{\alpha_m\}_{m\in\mathbb{N}_0}$ *is an arbitrary sequence in* $[0,1]$ *satisfying* (21).

(b) *Equation* (6) *possesses uncountably many positive solutions in* $A(N,M)$.

Proof. Put $L \in (bM+N, M)$. It follows from (38), (39), and (47) that there exist $\theta \in (0,1)$ and $T \geq n_0+\tau+\beta$ satisfying

$$\theta = b + \frac{1}{T^2}\sum_{u=T}^{\infty}\sum_{s=u}^{\infty}\left(R_sH_s + \sum_{t=s}^{\infty}P_tF_t\right);$$
$$\frac{1}{T^2}\sum_{u=T}^{\infty}\sum_{s=u}^{\infty}\left(W_s + \sum_{t=s}^{\infty}(Q_t+|c_t|)\right) < \min\{M-L, L-bM-N\}; \quad (49)$$
$$0 \leq b_n \leq b < 1, \quad \forall n \geq T.$$

Define a mapping $S_L : A(N,M) \to l_\beta^\infty$ by

$$S_Lx_n = \begin{cases} n^2L - b_nx_{n-\tau} - \displaystyle\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\left\{h\left(s, x_{h_{1s}}, x_{h_{2s}}, \dots, x_{h_{ks}}\right) - \sum_{t=s}^{\infty}\left[f\left(t, x_{f_{1t}}, x_{f_{2t}}, \dots, x_{f_{kt}}\right) - c_t\right]\right\}, & n \geq T, \\ S_Lx_T, & \beta \leq n < T, \end{cases} \quad (50)$$

for each $x = \{x_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$. In view of (14), (15), and (49) and (50), we obtain that for each $x = \{x_n\}_{n\in\mathbb{N}_\beta}, y = \{y_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$ and $n \geq T$

$$\begin{aligned}
\left|\frac{S_Lx_n}{n^2} - \frac{S_Ly_n}{n^2}\right| &\leq b_n\left|\frac{x_{n-\tau}-y_{n-\tau}}{n^2}\right| + \frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\Big[\left|h\left(s, x_{h_{1s}}, x_{h_{2s}}, \dots, x_{h_{ks}}\right) - h\left(s, y_{h_{1s}}, y_{h_{2s}}, \dots, y_{h_{ks}}\right)\right| \\
&\quad + \sum_{t=s}^{\infty}\left|f\left(t, x_{f_{1t}}, x_{f_{2t}}, \dots, x_{f_{kt}}\right) - f\left(t, y_{f_{1t}}, y_{f_{2t}}, \dots, y_{f_{kt}}\right)\right|\Big] \\
&\leq b_n\left|\frac{x_{n-\tau}-y_{n-\tau}}{(n-\tau)^2}\right|\frac{(n-\tau)^2}{n^2} + \frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\Big[R_s\max\left\{\left|x_{h_{ls}}-y_{h_{ls}}\right| : 1\leq l\leq k\right\} \\
&\quad + \sum_{t=s}^{\infty}P_t\max\left\{\left|x_{f_{lt}}-y_{f_{lt}}\right| : 1\leq l\leq k\right\}\Big] \\
&\leq b\|x-y\| + \frac{\|x-y\|}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\Big[R_s\max\left\{h_{ls}^2 : 1\leq l\leq k\right\} + \sum_{t=s}^{\infty}P_t\max\left\{f_{lt}^2 : 1\leq l\leq k\right\}\Big] \\
&\leq \left[b + \frac{1}{T^2}\sum_{u=T}^{\infty}\sum_{s=u}^{\infty}\left(R_sH_s + \sum_{t=s}^{\infty}P_tF_t\right)\right]\|x-y\| = \theta\|x-y\|, \\
\frac{S_Lx_n}{n^2} &\leq L + \frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\left\{\left|h\left(s, x_{h_{1s}}, x_{h_{2s}}, \dots, x_{h_{ks}}\right)\right| + \sum_{t=s}^{\infty}\left[\left|f\left(t, x_{f_{1t}}, x_{f_{2t}}, \dots, x_{f_{kt}}\right)\right| + |c_t|\right]\right\} \\
&\leq L + \frac{1}{T^2}\sum_{u=T}^{\infty}\sum_{s=u}^{\infty}\left(W_s + \sum_{t=s}^{\infty}(Q_t+|c_t|)\right) \\
&< L + \min\{M-L, L-bM-N\} \leq M, \\
\frac{S_Lx_n}{n^2} &\geq L - bM - \frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\left\{\left|h\left(s, x_{h_{1s}}, x_{h_{2s}}, \dots, x_{h_{ks}}\right)\right| + \sum_{t=s}^{\infty}\left[\left|f\left(t, x_{f_{1t}}, x_{f_{2t}}, \dots, x_{f_{kt}}\right)\right| + |c_t|\right]\right\} \\
&\geq L - bM - \frac{1}{T^2}\sum_{u=T}^{\infty}\sum_{s=u}^{\infty}\left[W_s + \sum_{t=s}^{\infty}(Q_t+|c_t|)\right] \\
&> L - bM - \min\{M-L, L-bM-N\} \geq N,
\end{aligned} \quad (51)$$

which imply (27). The rest of the proof is similar to that of Theorem 2 and is omitted. This completes the proof. □

Theorem 5. *Assume that there exist constants b, M, and N with* $(1+b)M > N > 0$ *and four nonnegative sequences* $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, $\{R_n\}_{n\in\mathbb{N}_{n_0}}$, *and* $\{W_n\}_{n\in\mathbb{N}_{n_0}}$ *satisfying* (14), (15), (38), (39), *and*

$$-1 < b \leq b_n \leq 0 \quad \text{eventually}. \quad (52)$$

Then one has the following.

(a) *For any* $L \in (N,(1+b)M)$, *there exist* $\theta \in (0,1)$ *and* $T \geq n_0+\tau+\beta$ *such that, for any* $x_0 = \{x_{0n}\}_{n\in\mathbb{N}_\beta} \in A(N,M)$, *the Mann iterative sequence* $\{x_m\}_{m\in\mathbb{N}_0} = \{\{x_{mn}\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0}$ *generated by* (48) *converges to a positive solution* $z = \{z_n\}_{n\in\mathbb{N}_\beta} \in$

$A(N,M)$ *of* (6) *with* $\lim_{n\to\infty} z_n = +\infty$ *and has the error estimate* (20), *where* $\{\alpha_m\}_{m\in\mathbb{N}_0}$ *is an arbitrary sequence in* $[0,1]$ *satisfying* (21).

(b) *Equation* (6) *possesses uncountably many positive solutions in* $A(N,M)$.

Proof. Put $L \in (N,(1+b)M)$. It follows from (38), (39), and (52) that there exist $\theta \in (0,1)$ and $T \geq n_0 + \tau + \beta$ satisfying

$$\theta = -b + \frac{1}{T^2}\sum_{u=T}^{\infty}\sum_{s=u}^{\infty}\left(R_s H_s + \sum_{t=s}^{\infty} P_t F_t\right); \tag{53}$$

$$\frac{1}{T^2}\sum_{u=T}^{\infty}\sum_{s=u}^{\infty}\left(W_s + \sum_{t=s}^{\infty}(Q_t + |c_t|)\right) < \min\{(1+b)M - L, L - N\}; \tag{54}$$

$$-1 < b \leq b_n \leq 0, \quad \forall n \geq T. \tag{55}$$

Define a mapping $S_L : A(N,M) \to l_\beta^\infty$ by (50). By virtue of (15), (50), (53), and (55), we infer that for all $x = \{x_n\}_{n\in\mathbb{N}_\beta}$, $y = \{y_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$ and $n \geq T$

$$\begin{aligned}
&\left|\frac{S_L x_n}{n^2} - \frac{S_L y_n}{n^2}\right| \\
&\leq b_n \left|\frac{x_{n-\tau} - y_{n-\tau}}{n^2}\right| \\
&\quad + \frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\Big[\left|h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right) - h\left(s, y_{h_{1s}}, y_{h_{2s}}, \ldots, y_{h_{ks}}\right)\right| \\
&\quad + \sum_{t=s}^{\infty}\left|f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right) - f\left(t, y_{f_{1t}}, y_{f_{2t}}, \ldots, y_{f_{kt}}\right)\right|\Big] \\
&\leq \left[-b + \frac{1}{T^2}\sum_{u=T}^{\infty}\sum_{s=u}^{\infty}\left(R_s H_s + \sum_{t=s}^{\infty} P_t F_t\right)\right]\|x - y\| \\
&= \theta\|x - y\|, \\
&\frac{S_L x_n}{n^2} \\
&\leq L - bM \\
&\quad + \frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\Big\{\left|h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right)\right| \\
&\quad + \sum_{t=s}^{\infty}\left[\left|f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right)\right| + |c_t|\right]\Big\} \\
&\leq L - bM + \frac{1}{T^2}\sum_{u=T}^{\infty}\sum_{s=u}^{\infty}\left(W_s + \sum_{t=s}^{\infty}(Q_t + |c_t|)\right) \\
&< L - bM + \min\{(1+b)M - L, L - N\} \leq M, \\
&\frac{S_L x_n}{n^2} \\
&\geq L - \frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\Big\{\left|h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right)\right| \\
&\quad + \sum_{t=s}^{\infty}\left[\left|f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right)\right| + |c_t|\right]\Big\} \\
&\geq L - \frac{1}{T^2}\sum_{u=T}^{\infty}\sum_{s=u}^{\infty}\left[W_s + \sum_{t=s}^{\infty}(Q_t + |c_t|)\right] \\
&> L - \min\{(1+b)M - L, L - N\} \geq N.
\end{aligned} \tag{56}$$

That is, (27) holds. The rest of the proof is similar to that of Theorem 2 and is omitted. This completes the proof. □

Theorem 6. *Assume that there exist constants* b, M, *and* N *with* $(1 - 1/b)M > N > 0$ *and four nonnegative sequences* $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, $\{R_n\}_{n\in\mathbb{N}_{n_0}}$, *and* $\{W_n\}_{n\in\mathbb{N}_{n_0}}$ *satisfying* (14), (15), (38), (39), *and*

$$b_n \geq b > 1 \quad \text{eventually}. \tag{57}$$

Then one has the following.

(a) *For any* $L \in ((1/b)M + N, M)$, *there exist* $\theta \in (0,1)$ *and* $T \geq n_0 + \tau + \beta$ *such that, for any* $x_0 = \{x_{0n}\}_{n\in\mathbb{N}_\beta} \in A(N,M)$, *the Mann iterative sequence* $\{x_m\}_{m\in\mathbb{N}_0} = \{\{x_{mn}\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0}$ *generated by the scheme*

$$x_{m+1\,n} = \begin{cases}
(1-\alpha_m)x_{mn} + \alpha_m\Big\{n^2 L - \dfrac{x_{mn+\tau}}{b_{n+\tau}} - \dfrac{1}{b_{n+\tau}} \\
\quad\times\displaystyle\sum_{u=n+\tau}^{\infty}\sum_{s=u}^{\infty}\Big[h\left(s, x_{mh_{1s}}, x_{mh_{2s}}, \ldots, x_{mh_{ks}}\right) \\
\quad - \displaystyle\sum_{t=s}^{\infty}\left(f\left(t, x_{mf_{1t}}, x_{mf_{2t}}, \ldots, x_{mf_{kt}}\right) - c_t\right)\Big]\Big\}, \\
\qquad n \geq T, \quad m \geq 0, \\
(1-\alpha_m)x_{mT} + \alpha_m\Big\{T^2 L - \dfrac{x_{mT+\tau}}{b_{T+\tau}} \\
\quad - \displaystyle\sum_{u=T+\tau}^{\infty}\sum_{s=u}^{\infty}\Big[h\left(s, x_{mh_{1s}}, x_{mh_{2s}}, \ldots, x_{mh_{ks}}\right) \\
\quad - \displaystyle\sum_{t=s}^{\infty}\left(f\left(t, x_{mf_{1t}}, x_{mf_{2t}}, \ldots, x_{mf_{kt}}\right) - c_t\right)\Big]\Big\}, \\
\qquad \beta \leq n < T, \quad m \geq 0,
\end{cases} \tag{58}$$

converges to a positive solution $z = \{z_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$ *of* (6) *with* $\lim_{n\to\infty} z_n = +\infty$ *and has the error estimate* (20)*, where* $\{\alpha_m\}_{m\in\mathbb{N}_0}$ *is an arbitrary sequence in* $[0,1]$ *satisfying* (21).

(b) *Equation* (6) *possesses uncountably many positive solutions in* $A(N,M)$.

Proof. Put $L \in ((1/b)M + N, M)$. It follows from (38), (39), and (57) that there exist $\theta \in (0,1)$ and $T \geq n_0 + \tau + \beta$ satisfying

$$\theta = \frac{1}{b}\left[\left(1+\frac{\tau}{T}\right)^2 + \frac{1}{T^2}\sum_{u=T}^{\infty}\sum_{s=u}^{\infty}\left(R_s H_s + \sum_{t=s}^{\infty} P_t F_t\right)\right]; \quad (59)$$

$$\frac{1}{bT^2}\sum_{u=T}^{\infty}\sum_{s=u}^{\infty}\left(W_s + \sum_{t=s}^{\infty}(Q_t + |c_t|)\right) < \min\left\{M-L, L-\frac{1}{b}M - N\right\}; \quad (60)$$

$$b_n \geq b > 1, \quad \forall n \geq T. \quad (61)$$

Define a mapping $S_L : A(N,M) \to l_\beta^\infty$ by

$$S_L x_n = \begin{cases} n^2 L - \dfrac{x_{n+\tau}}{b_{n+\tau}} - \dfrac{1}{b_{n+\tau}} \times \displaystyle\sum_{u=n+\tau}^{\infty}\sum_{s=u}^{\infty}\Big\{h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right) - \displaystyle\sum_{t=s}^{\infty}\Big[\left(f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right) - c_t\right)\Big]\Big\}, & n \geq T, \\ S_L x_T, & \beta \leq n < T, \end{cases} \quad (62)$$

for each $x = \{x_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$. In view of (14), (15), and (59)–(62), we obtain that for each $x = \{x_n\}_{n\in\mathbb{N}_\beta}$, $y = \{y_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$ and $n \geq T$

$$\begin{aligned}
&\left|\frac{S_L x_n}{n^2} - \frac{S_L y_n}{n^2}\right| \\
&\leq \frac{1}{b_{n+\tau}}\left|\frac{x_{n+\tau} - y_{n+\tau}}{n^2}\right| + \frac{1}{b_{n+\tau} n^2}\sum_{u=n+\tau}^{\infty}\sum_{s=u}^{\infty}\Big[\left|h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right) - h\left(s, y_{h_{1s}}, y_{h_{2s}}, \ldots, y_{h_{ks}}\right)\right| \\
&\quad + \sum_{t=s}^{\infty}\left|f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right) - f\left(t, y_{f_{1t}}, y_{f_{2t}}, \ldots, y_{f_{kt}}\right)\right|\Big] \\
&\leq \frac{1}{b_{n+\tau}}\left|\frac{x_{n+\tau} - y_{n+\tau}}{(n+\tau)^2}\right|\frac{(n+\tau)^2}{n^2} + \frac{1}{b_{n+\tau} n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\Big[R_s \max\left\{\left|x_{h_{ls}} - y_{h_{ls}}\right| : 1 \leq l \leq k\right\} \\
&\quad + \sum_{t=s}^{\infty} P_t \max\left\{\left|x_{f_{lt}} - y_{f_{lt}}\right| : 1 \leq l \leq k\right\}\Big] \\
&\leq \frac{1}{b}\left[\left(1+\frac{\tau}{T}\right)^2 + \frac{1}{T^2}\sum_{u=T}^{\infty}\sum_{s=u}^{\infty}\left(R_s H_s + \sum_{t=s}^{\infty} P_t F_t\right)\right]\|x-y\| \\
&= \theta\|x-y\|, \\
&\frac{S_L x_n}{n^2} \\
&\leq L + \frac{1}{bn^2}\sum_{u=n+\tau}^{\infty}\sum_{s=u}^{\infty}\Big\{\left|h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right)\right| + \sum_{t=s}^{\infty}\Big[\left|f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right)\right| + |c_t|\Big]\Big\} \\
&\leq L + \frac{1}{bT^2}\sum_{u=T}^{\infty}\sum_{s=u}^{\infty}\left(W_s + \sum_{t=s}^{\infty}(Q_t + |c_t|)\right) \\
&< L + \min\left\{M-L, L-\frac{1}{b}M-N\right\} \leq M, \\
&\frac{S_L x_n}{n^2} \\
&\geq L - \frac{1}{b}M - \frac{1}{bn^2}\sum_{u=n+\tau}^{\infty}\sum_{s=u}^{\infty}\Big\{\left|h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right)\right| + \sum_{t=s}^{\infty}\Big[\left|f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right)\right| + |c_t|\Big]\Big\} \\
&\geq L - \frac{1}{b}M - \frac{1}{bT^2}\sum_{u=T}^{\infty}\sum_{s=u}^{\infty}\left(W_s + \sum_{t=s}^{\infty}(Q_t + |c_t|)\right) \\
&> L - \frac{1}{b}M - \min\left\{M-L, L-\frac{1}{b}M-N\right\} \geq N,
\end{aligned} \quad (63)$$

which imply (27). The rest of the proof is similar to that of Theorem 2 and is omitted. This completes the proof. □

Theorem 7. *Assume that there exist constants* b, M, *and* N *with* $(1 + 1/b)M > N > 0$ *and four nonnegative sequences* $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, $\{R_n\}_{n\in\mathbb{N}_{n_0}}$, *and* $\{W_n\}_{n\in\mathbb{N}_{n_0}}$ *satisfying* (14), (15), (38), (39) *and*

$$b_n \leq b < -1 \quad \text{eventually}. \quad (64)$$

Then one has the following.

(a) *For any $L \in (-(1+1/b)M, -N)$, there exist $\theta \in (0,1)$ and $T \geq n_0 + \tau + \beta$ such that, for any $x_0 = \{x_{0n}\}_{n\in\mathbb{N}_\beta} \in A(N,M)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0} = \{\{x_{mn}\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0}$ generated by the scheme*

$$x_{m+1n} = \begin{cases} (1-\alpha_m)x_{mn} + \alpha_m\Big\{-n^2L - \dfrac{x_{mn+\tau}}{b_{n+\tau}} - \dfrac{1}{b_{n+\tau}} \times \displaystyle\sum_{u=n+\tau}^{\infty}\sum_{s=u}^{\infty}\Big[h\left(s, x_{mh_{1s}}, x_{mh_{2s}}, \ldots, x_{mh_{ks}}\right) - \displaystyle\sum_{t=s}^{\infty}\left(f\left(t, x_{mf_{1t}}, x_{mf_{2t}}, \ldots, x_{mf_{kt}}\right) - c_t\right)\Big]\Big\}, & n \geq T,\quad m \geq 0, \\ (1-\alpha_m)x_{mT} + \alpha_m\Big\{-T^2L - \dfrac{x_{mT+\tau}}{b_{T+\tau}} - \displaystyle\sum_{u=T+\tau}^{\infty}\sum_{s=u}^{\infty}\Big[h\left(s, x_{mh_{1s}}, x_{mh_{2s}}, \ldots, x_{mh_{ks}}\right) - \displaystyle\sum_{t=s}^{\infty}\left(f\left(t, x_{mf_{1t}}, x_{mf_{2t}}, \ldots, x_{mf_{kt}}\right) - c_t\right)\Big]\Big\}, & \beta \leq n < T,\quad m \geq 0, \end{cases} \quad (65)$$

converges to a positive solution $z = \{z_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$ of (6) with $\lim_{n\to\infty} z_n = +\infty$ and has the error estimate (20), where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $[0,1]$ satisfying (21).

(b) *Equation (6) possesses uncountably many positive solutions in $A(N,M)$.*

Proof. Put $L \in (-(1+1/b)M, -N)$. It follows from (38), (39), and (64) that there exist $\theta \in (0,1)$ and $T \geq n_0 + \tau + \beta$ satisfying

$$\theta = -\frac{1}{b}\left[\left(1+\frac{\tau}{T}\right)^2 + \frac{1}{T^2}\sum_{u=T}^{\infty}\sum_{s=u}^{\infty}\left(R_sH_s + \sum_{t=s}^{\infty}P_tF_t\right)\right]; \quad (66)$$

$$-\frac{1}{bT^2}\sum_{u=T}^{\infty}\sum_{s=u}^{\infty}\left(W_s + \sum_{t=s}^{\infty}\left(Q_t + |c_t|\right)\right) < \min\left\{\left(1+\frac{1}{b}\right)M - L, L - \frac{1}{b}M - N\right\}; \quad (67)$$

$$b_n \geq b > 1, \quad \forall n \geq T. \quad (68)$$

Define a mapping $S_L : A(N,M) \to l_\beta^\infty$ by

$$S_Lx_n = \begin{cases} -n^2L - \dfrac{x_{n+\tau}}{b_{n+\tau}} - \dfrac{1}{b_{n+\tau}} \times \displaystyle\sum_{u=n+\tau}^{\infty}\sum_{s=u}^{\infty}\Big\{h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right) - \displaystyle\sum_{t=s}^{\infty}\left[f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right) - c_t\right]\Big\}, & n \geq T, \\ S_Lx_T, & \beta \leq n < T, \end{cases} \quad (69)$$

for each $x = \{x_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$. Making use of (15), (66), (68), and (69), we conclude that

$$\begin{aligned} &\left|\frac{S_Lx_n}{n^2} - \frac{S_Ly_n}{n^2}\right| \\ &\leq -\frac{1}{b_{n+\tau}}\left|\frac{x_{n+\tau} - y_{n+\tau}}{n^2}\right| - \frac{1}{b_{n+\tau}n^2}\sum_{u=n+\tau}^{\infty}\sum_{s=u}^{\infty}\Big[\left|h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right) - h\left(s, y_{h_{1s}}, y_{h_{2s}}, \ldots, y_{h_{ks}}\right)\right| \\ &\qquad + \sum_{t=s}^{\infty}\left|f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right) - f\left(t, y_{f_{1t}}, y_{f_{2t}}, \ldots, y_{f_{kt}}\right)\right|\Big] \\ &\leq -\frac{1}{b_{n+\tau}}\left|\frac{x_{n+\tau} - y_{n+\tau}}{(n+\tau)^2}\right|\frac{(n+\tau)^2}{n^2} - \frac{1}{b_{n+\tau}n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\Big[R_s\max\left\{\left|x_{h_{ls}} - y_{h_{ls}}\right| : 1 \leq l \leq k\right\} \\ &\qquad + \sum_{t=s}^{\infty}P_t\max\left\{\left|x_{f_{lt}} - y_{f_{lt}}\right| : 1 \leq l \leq k\right\}\Big] \\ &\leq -\frac{1}{b}\left[\left(1+\frac{\tau}{T}\right)^2 + \frac{1}{T^2}\sum_{u=T}^{\infty}\sum_{s=u}^{\infty}\left(R_sH_s + \sum_{t=s}^{\infty}P_tF_t\right)\right]\|x-y\| \\ &= \theta\|x-y\|, \end{aligned}$$

$$\begin{aligned} &\frac{S_Lx_n}{n^2} \\ &\leq -L - \frac{M}{b} - \frac{1}{bn^2} \times \sum_{u=n+\tau}^{\infty}\sum_{s=u}^{\infty}\Big\{\left|h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right)\right| + \sum_{t=s}^{\infty}\left[\left|f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right)\right| + |c_t|\right]\Big\} \end{aligned}$$

$$\leq -L - \frac{M}{b} - \frac{1}{bT^2}\sum_{u=T}^{\infty}\sum_{s=u}^{\infty}\left(W_s + \sum_{t=s}^{\infty}(Q_t + |c_t|)\right)$$
$$< -L - \frac{M}{b} + \min\left\{\left(1+\frac{1}{b}\right)M + L, -L-N\right\} \leq M,$$
$$\frac{S_L x_n}{n^2}$$
$$\geq -L$$
$$+ \frac{1}{bn^2}\sum_{u=n+\tau}^{\infty}\sum_{s=u}^{\infty}\left\{\left|h\left(s, x_{h_{1s}}, x_{h_{2s}}, \ldots, x_{h_{ks}}\right)\right|\right.$$
$$\left. + \sum_{t=s}^{\infty}\left[\left|f\left(t, x_{f_{1t}}, x_{f_{2t}}, \ldots, x_{f_{kt}}\right)\right| + |c_t|\right]\right\}$$
$$\geq -L + \frac{1}{bT^2}\sum_{u=T}^{\infty}\sum_{s=u}^{\infty}\left[W_s + \sum_{t=s}^{\infty}(Q_t + |c_t|)\right]$$
$$> -L - \min\left\{\left(1+\frac{1}{b}\right)M + L, -L-N\right\} \geq N, \tag{70}$$

which yield (27). The rest of the proof is similar to that of Theorem 2 and is omitted. This completes the proof. □

3. Examples

In this section, we suggest six examples to explain the results presented in Section 2.

Example 1. Consider the third order nonlinear neutral delay difference equation

$$\Delta^3\left(x_n - x_{n-\tau}\right) + \Delta\left(\frac{\sin^2 x_{n-3}}{n^7}\right) + \frac{1}{\left(n^9 + 2n^5 + 1\right)\left(1 + x_{n^2}^2\right)} = \frac{n^2 - 2n}{n^8 + n^3 + 1}, \quad \forall n \geq 4, \tag{71}$$

where $\tau \in \mathbb{N}$ is fixed. Let $n_0 = 4, k = 1$, and $\beta = \min\{4-\tau, 1\}$, and let M and N be two positive constants with $M > N$ and

$$b_n = -1, \qquad c_n = \frac{n^2 - 2n}{n^8 + n^3 + 1},$$
$$f(n, u) = \frac{1}{\left(n^9 + 2n^5 + 1\right)\left(1 + u^2\right)},$$
$$h(n, u) = \frac{\sin^2 u}{n^7}, \qquad f_{1n} = n^2, \qquad F_n = n^4,$$
$$h_{1n} = n - 3, \qquad H_n = (n-3)^2, \qquad P_n = \frac{2M}{n^9},$$
$$Q_n = \frac{1}{n^9}, \qquad R_n = \frac{2}{n^7}, \qquad W_n = \frac{1}{n^7},$$
$$\forall (n, u) \in \mathbb{N}_{n_0} \times \mathbb{R}. \tag{72}$$

It is easy to see that (14), (15), and (18) are satisfied. Note that

$$\frac{1}{n^2}\sum_{t=n+\tau}^{\infty} t^2 \max\{R_t H_t, W_t\}$$
$$= \frac{1}{n^2}\sum_{t=n+\tau}^{\infty} t^2 \max\left\{\frac{2(t-3)^2}{t^7}, \frac{1}{t^7}\right\}$$
$$= \sum_{t=n+\tau}^{\infty}\frac{2(t-3)^2 + 1}{t^5}$$
$$\leq \frac{2}{n^2}\sum_{t=n+\tau}^{\infty}\frac{1}{t^3} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{73}$$
$$\frac{1}{n^2}\sum_{t=n+\tau}^{\infty} t^3 \max\{P_t F_t, Q_t, |c_t|\}$$
$$= \frac{1}{n^2}\sum_{t=n+\tau}^{\infty} t^3 \max\left\{\frac{2M}{t^5}, \frac{1}{t^9}, \frac{\left|t^2 - 2t\right|}{t^8 + t^3 + 1}\right\}$$
$$\leq \frac{\max\{1, 2M\}}{n^2}\sum_{t=n+\tau}^{\infty}\frac{1}{t^2} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,$$

which together with Lemma 1 yield that (16) and (17) hold. It follows from Theorem 2 that (71) possesses uncountably many positive solutions in $A(N, M)$. On the other hand, for any $L \in (N, M)$, there exist $\theta \in (0, 1)$ and $T \geq n_0 + \tau + \beta$ such that, for each $x_0 = \{x_{0n}\}_{n\in\mathbb{N}_\beta} \in A(N, M)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0} = \{\{x_{mn}\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0}$ generated by (19) converges to a positive solution $z = \{z_n\}_{n\in\mathbb{N}_\beta} \in A(N, M)$ of (71) with $\lim_{n\to\infty} z_n = +\infty$ and has the error estimate (20), where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $[0, 1]$ satisfying (21).

Example 2. Consider the third order nonlinear neutral delay difference equation

$$\Delta^3\left(x_n + x_{n-\tau}\right) + \Delta\left(\frac{\sin^2 x_{3n^3+1}}{n^3\left(n^6 + 2\right)\left(1 + x_{2n^2-3}^4\right)}\right) + \frac{(-1)^n n^3\left(x_{n^2-n-1} + x_{(n+1)(n+2)}\right)}{\left(n^{13} + n^5 + 1\right)\left(1 + x_{n^2-n-1}^2 + x_{(n+1)(n+2)}^2\right)} = \frac{n^2 - \ln n}{n^6 + n^5 + 1}, \quad \forall n \geq 5, \tag{74}$$

where $\tau \in \mathbb{N}$ is fixed. Let $n_0 = 5, k = 2$, and $\beta = 5 - \tau$, and let M and N be two positive constants with $M > N$ and

$$b_n = 1, \qquad c_n = \frac{n^2 - \ln n}{n^6 + n^5 + 1},$$
$$f(n, u, v) = \frac{(-1)^n n^3 (u + v)}{\left(n^{13} + n^5 + 1\right)\left(1 + u^2 + v^2\right)},$$

$$
\begin{gathered}
h(n,u,v)=\frac{\sin^2 v}{n^3\left(n^6+2\right)\left(1+u^4\right)},\qquad f_{1n}=n^2-n-1,\\
f_{2n}=(n+1)(n+2),\\
F_n=(n+1)^2(n+2)^2,\qquad h_{1n}=2n^2-3,\\
h_{2n}=3n^3+1,\\
H_n=\left(3n^3+1\right)^2,\qquad P_n=Q_n=\frac{4}{n^{10}},\qquad R_n=W_n=\frac{10}{n^9},\\
\forall (n,u,v)\in\mathbb{N}_{n_0}\times\mathbb{R}^2.
\end{gathered}\tag{75}
$$

It is clear that (14), (15), and (40) are fulfilled. Note that

$$
\begin{aligned}
&\frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\max\left\{R_sH_s,W_s\right\}\\
&\quad=\frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\max\left\{\frac{10\left(3s^3+1\right)^2}{s^9},\frac{10}{s^9}\right\}\\
&\quad\le\frac{160}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\frac{1}{s^3}\le\frac{160}{n^2}\sum_{s=n}^{\infty}\frac{1}{s^2}\longrightarrow 0\quad\text{as } n\longrightarrow\infty,
\end{aligned}\tag{76}
$$

which means that

$$
\lim_{n\to\infty}\frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\max\left\{R_sH_s,W_s\right\}=0.\tag{77}
$$

Observe that

$$
\begin{aligned}
&\frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\max\left\{P_tF_t,Q_t,\left|c_t\right|\right\}\\
&=\frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\max\left\{\frac{4(t+1)^2(t+2)^2}{t^{10}},\frac{4}{t^{10}},\frac{t^2-\ln t}{t^6+t^5+1}\right\}\\
&\le\frac{196}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{t^4}=\frac{196}{n^2}\sum_{u=n}^{\infty}\sum_{t=u}^{\infty}\frac{t-u+1}{t^4}\\
&\le\frac{196}{n^2}\sum_{u=n}^{\infty}\sum_{t=u}^{\infty}\frac{1}{t^3}\le\frac{196}{n^2}\sum_{t=n}^{\infty}\frac{1}{t^2}\longrightarrow 0\quad\text{as } n\longrightarrow\infty,
\end{aligned}\tag{78}
$$

which yields that

$$
\lim_{n\to\infty}\frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\max\left\{P_tF_t,Q_t,\left|c_t\right|\right\}=0.\tag{79}
$$

Thus Theorem 3 guarantees that (74) possesses uncountably positive solutions in $A(N,M)$. On the other hand, for any $L\in(N,M)$, there exist $\theta\in(0,1)$ and $T\ge\tau+n_0+\beta$ such that the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0}=\{\{x_{mn}\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0}$ generated by (41) converges to a positive solution $z=\{z_n\}_{n\in\mathbb{N}_\beta}\in A(N,M)$ of (74) with $\lim_{n\to\infty}z_n=+\infty$ and has the error estimate (20), where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $[0,1]$ satisfying (21).

Example 3. Consider the third order nonlinear neutral delay difference equation

$$
\begin{aligned}
&\Delta^3\left(x_n+\frac{1+3\ln n}{2+4\ln n}x_{n-\tau}\right)\\
&\quad+\Delta\left(\frac{(-1)^n\sin\left(e^{-n^2\left|x_{5n^2-3}\right|}\right)}{n^{15}-\sqrt{n}+3}+\frac{n^2+(-1)^{n(n+1)/2}}{\left(n^{12}+6n^{10}+7\right)e^{\left|x_{2n^3+1}\right|}}\right)\\
&\quad+\frac{(-1)^n}{n^6\left(1+x_{n-3}^2\right)}-\frac{1}{\left(n^7+2n^4-1\right)\left(1+x_{n+4}^2\right)}\\
&\quad=\frac{3(-1)^n n^2}{9n^{10}\ln^3 n},\quad\forall n\ge 7,
\end{aligned}\tag{80}
$$

where $\tau\in\mathbb{N}$ is fixed. Let $n_0=7$, $k=2$, $b=3/4$, and $\beta=\min\{7-\tau,4\}$, and let M and N be two positive constants with $M>4N$ and

$$
\begin{gathered}
b_n=\frac{1+3\ln n}{2+4\ln n},\qquad c_n=\frac{3(-1)^n n^2}{9n^{10}\ln^3 n},\\
f(n,u,v)=\frac{(-1)^n}{n^6\left(1+u^2\right)}-\frac{1}{\left(n^7+2n^4-1\right)\left(1+v^2\right)},\\
h(n,u,v)=\frac{(-1)^n\sin\left(e^{-n^2|u|}\right)}{n^{15}-\sqrt{n}+3}+\frac{n^2+(-1)^{n(n+1)/2}}{\left(n^{12}+6n^{10}+7\right)e^{|v|}},\\
f_{1n}=n-3,\qquad f_{2n}=n+4,\\
F_n=(n+4)^2,\qquad h_{1n}=5n^2-3,\\
h_{2n}=2n^3+1,\qquad H_n=\left(2n^3+1\right)^2,\\
P_n=Q_n=\frac{3}{n^6},\qquad R_n=W_n=\frac{2}{n^{10}},\\
\forall (n,u,v)\in\mathbb{N}_{n_0}\times\mathbb{R}^2.
\end{gathered}\tag{81}
$$

It is not difficult to verify that (14), (15), and (47) are fulfilled. Note that

$$
\begin{aligned}
&\frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\max\left\{R_sH_s,W_s\right\}\\
&\quad=\frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\max\left\{\frac{2\left(2s^3+1\right)^2}{s^{10}},\frac{2}{s^{10}}\right\}\\
&\quad\le\frac{18}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\frac{1}{s^4}\le\frac{18}{n^2}\sum_{s=n}^{\infty}\frac{1}{s^3}\longrightarrow 0\quad\text{as } n\longrightarrow\infty,
\end{aligned}\tag{82}
$$

which implies that

$$
\lim_{n\to\infty}\frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\max\left\{R_sH_s,W_s\right\}=0.\tag{83}
$$

Observe that

$$\begin{aligned}
&\frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\max\left\{P_tF_t, Q_t, |c_t|\right\} \\
&\quad = \frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\max\left\{\frac{3(t+4)^2}{t^6}, \frac{3}{t^6}, \left|\frac{3(-1)^t t^2}{9t^{10}\ln^3 t}\right|\right\} \\
&\quad \le \frac{12}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{t^4} \\
&\quad \le \frac{12}{n^2}\sum_{u=n}^{\infty}\sum_{t=u}^{\infty}\frac{1}{t^3} \le \frac{12}{n^2}\sum_{t=n}^{\infty}\frac{1}{t^2} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,
\end{aligned} \tag{84}$$

which means that

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\max\left\{P_tF_t, Q_t, |c_t|\right\} = 0. \tag{85}$$

That is, (38) and (39) hold. Consequently Theorem 4 implies that (80) possesses uncountably many positive solutions in $A(N, M)$. On the other hand, for any $L \in ((3/4)M + N, M)$, there exist $\theta \in (0,1)$ and $T \ge n_0 + \tau + \beta$ such that the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0} = \{\{x_{mn}\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0}$ generated by (41) converges to a positive solution $z = \{z_n\}_{n\in\mathbb{N}_\beta} \in A(N, M)$ of (80) with $\lim_{n\to\infty} z_n = +\infty$ and has the error estimate (20), where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $[0,1]$ satisfying (21).

Example 4. Consider the third order nonlinear neutral delay difference equation

$$\begin{aligned}
&\Delta^3\left(x_n + \frac{1-5n^3}{2+6n^3}x_{n-\tau}\right) + \Delta\left(\frac{2n^2+n-1}{(n^8+3n^6+2)(1+x_{3n-7}^2)}\right) \\
&\quad + \frac{\sin\left(n^2 x_{3n^2-2}\right)}{\left(\sqrt{n}+14\right)^{22}} \\
&\quad = \frac{(-1)^n n^3 + 5n^2 + 4n - 2}{n^9 + n^8 + 2n^5 + n^3 + 7}, \quad \forall n \ge 9,
\end{aligned} \tag{86}$$

where $\tau \in \mathbb{N}$ is fixed. Let $n_0 = 9$, $k = 1$, $b = -5/6$, and $\beta = 9 - \tau$, and let M and N be two positive constants with $M > 6N$ and

$$\begin{gathered}
b_n = \frac{1-5n^3}{2+6n^3}, \qquad c_n = \frac{(-1)^n n^3 + 5n^2 + 4n - 2}{n^9 + n^8 + 2n^5 + n^3 + 7}, \\
f(n,u) = \frac{\sin\left(n^2 u\right)}{\left(\sqrt{n}+14\right)^{22}}, \\
h(n,u) = \frac{2n^2+n-1}{(n^8+3n^6+2)(1+u^2)}, \\
f_{1n} = 3n^2 - 2, \qquad F_n = \left(3n^2-2\right)^2, \qquad h_{1n} = 3n - 7, \\
H_n = (3n-7)^2, \qquad P_n = Q_n = \frac{3}{n^9}, \quad R_n = W_n = \frac{1}{n^5}, \\
\forall (n,u) \in \mathbb{N}_{n_0} \times \mathbb{R}.
\end{gathered} \tag{87}$$

Obviously, (14), (15), and (52) are satisfied. Note that

$$\begin{aligned}
&\frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\max\left\{R_sH_s, W_s\right\} \\
&\quad = \frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\max\left\{\frac{(3s-7)^2}{s^5}, \frac{1}{s^5}\right\} \\
&\quad \le \frac{9}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\frac{1}{s^3} \le \frac{9}{n^2}\sum_{s=n}^{\infty}\frac{1}{s^2} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,
\end{aligned} \tag{88}$$

which implies that

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\max\left\{R_sH_s, W_s\right\} = 0. \tag{89}$$

Notice that

$$\begin{aligned}
&\frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\max\left\{P_tF_t, Q_t, |c_t|\right\} \\
&\quad = \frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\max\left\{\frac{3\left(3t^2-2\right)^2}{t^9}, \frac{3}{t^9},\right. \\
&\qquad\qquad \left.\left|\frac{(-1)^t t^3 + 5t^2 + 4t - 2}{t^9 + t^8 + 2t^5 + t^3 + 7}\right|\right\} \\
&\quad \le \frac{27}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{t^5} \le \frac{27}{n^2}\sum_{u=n}^{\infty}\sum_{t=u}^{\infty}\frac{1}{t^4} \\
&\quad \le \frac{27}{n^2}\sum_{t=n}^{\infty}\frac{1}{t^3} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,
\end{aligned} \tag{90}$$

which gives that

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\max\left\{P_tF_t, Q_t, |c_t|\right\} = 0. \tag{91}$$

That is, (38) and (39) hold. Thus Theorem 5 shows that (86) possesses uncountably many positive solutions in $A(N, M)$. On the other hand, for any $L \in (N, (1/6)M)$, there exist $\theta \in (0,1)$ and $T \ge n_0 + \tau + \beta$ such that the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0} = \{\{x_{mn}\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0}$ generated by (48) converges to a positive solution $z = \{z_n\}_{n\in\mathbb{N}_\beta} \in A(N, M)$ of (86) with $\lim_{n\to\infty} z_n = +\infty$ and has the error estimate (20), where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $[0,1]$ satisfying (21).

Example 5. Consider the third order nonlinear neutral delay difference equation

$$\begin{aligned}\Delta^3 & \left(x_n + \left(\frac{\pi}{2} + n\sin\frac{1}{n}\right)x_{n-\tau}\right) \\ & + \Delta\left(\frac{(-1)^{n(n+1)/2}}{(n+4)^8(n+5)^3\left(1+\cos\left(n^2 x_{2n+1}\right)\right)}\right) \\ & + \frac{n\sin\left(n x_{n-2}\right)}{2+(n+5)^{16}} \\ & = \frac{(-1)^{n-1}\cos^3\left(n^2+1\right)}{n^{16}+\ln n}, \quad \forall n \geq 3,\end{aligned} \tag{92}$$

where $\tau \in \mathbb{N}$ is fixed. Let $n_0 = 3$, $k = 1$, $b = \pi/2$, and $\beta = \min\{3-\tau, 1\}$, and let M and N be two positive constants with $(1-2/\pi)M > N$ and

$$\begin{gathered} b_n = \frac{\pi}{2} + n\sin\frac{1}{n}, \qquad c_n = \frac{(-1)^{n-1}\cos^3\left(n^2+1\right)}{n^{16}+\ln n}, \\ f(n,u) = \frac{n\sin(nu)}{2+(n+5)^{16}}, \\ h(n,u) = \frac{(-1)^{n(n+1)/2}}{(n+4)^8(n+5)^3\left(1+\cos\left(n^2 u\right)\right)}, \\ f_{1n} = n-2, \qquad F_n = (n-2)^2, \\ h_{1n} = 2n+1, \qquad H_n = (2n+1)^2, \\ P_n = Q_n = \frac{1}{n^{14}}, \quad R_n = W_n = \frac{2}{n^9}, \quad \forall (n,u) \in \mathbb{N}_{n_0} \times \mathbb{R}. \end{gathered} \tag{93}$$

Clearly, (14), (15), and (61) are satisfied. Note that

$$\begin{aligned} & \frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\max\left\{R_s H_s, W_s\right\} \\ & \quad = \frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\max\left\{\frac{2(2s+1)^2}{s^9}, \frac{2}{s^9}\right\} \\ & \quad \leq \frac{18}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\frac{1}{s^7} \leq \frac{18}{n^2}\sum_{s=n}^{\infty}\frac{1}{s^6} \longrightarrow 0 \quad \text{as}\, n \longrightarrow \infty, \end{aligned} \tag{94}$$

which means that

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\max\left\{R_s H_s, W_s\right\} = 0,$$

$$\begin{aligned} & \frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\max\left\{P_t F_t, Q_t, \left|c_t\right|\right\} \\ & \quad = \frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\max\left\{\frac{(t-2)^2}{t^{14}}, \frac{1}{t^{14}}, \right. \\ & \qquad \left.\left|\frac{(-1)^{t-1}\cos^3\left(t^2+1\right)}{t^{16}+\ln t}\right|\right\} \\ & \quad \leq \frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{t^{14}} \leq \frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{t=u}^{\infty}\frac{1}{t^{13}} \\ & \quad \leq \frac{1}{n^2}\sum_{t=n}^{\infty}\frac{1}{t^{12}} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \end{aligned} \tag{95}$$

which implies that

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\max\left\{P_t F_t, Q_t, \left|c_t\right|\right\} = 0. \tag{96}$$

That is, (38) and (39) hold. Consequently Theorem 6 implies that (92) possesses uncountably many positive solutions in $A(N,M)$. On the other hand, for any $L \in ((2/\pi)M + N, M)$, there exist $\theta \in (0,1)$ and $T \geq n_0 + \tau + \beta$ such that the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0} = \{\{x_{mn}\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0}$ generated by (58) converges to a positive solution $z = \{z_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$ of (92) with $\lim_{n\to\infty} z_n = +\infty$ and has the error estimate (20), where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $[0,1]$ satisfying (21).

Example 6. Consider the third order nonlinear neutral delay difference equation

$$\begin{aligned} \Delta^3 & \left(x_n - \frac{2n^5+9n^2-1}{n^5+3n^2+2}x_{n-\tau}\right) + \Delta\left(\frac{\cos\left((-1)^n e^n\right)}{(n+7)^6\sqrt{1+\left|x_{n-2}\right|}}\right) \\ & + \frac{\sin\left(n^2 x_{n-1}\right)}{n^9+3n^5+2n^2+1} \\ & = \frac{(-1)^{n-1}n^4+4n^2+n-1}{n^{11}+6n^3+7n+2}, \quad \forall n \geq 6, \end{aligned} \tag{97}$$

where $\tau \in \mathbb{N}$ is fixed. Let $n_0 = 6$, $k = 1$, $b = -2$, and $\beta = \min\{6-\tau, 3\}$, and let M and N be two positive constants with $(1/2)M > N$ and

$$\begin{gathered} b_n = -\frac{2n^5+9n^2-1}{n^5+3n^2+2}, \\ c_n = \frac{(-1)^{n-1}n^4+4n^2+n-1}{n^{11}+6n^3+7n+2}, \\ f(n,u) = \frac{\sin\left(n^2 u\right)}{n^9+3n^5+2n^2+1}, \end{gathered}$$

$$h(n,u) = \frac{\cos\left((-1)^n e^n\right)}{(n+7)^6 \sqrt{1+|u|}},$$
$$f_{1n} = n-1, \qquad F_n = (n-1)^2,$$
$$h_{1n} = n-2, \qquad H_n = (n-2)^2,$$
$$P_n = Q_n = \frac{1}{n^7}, \quad R_n = W_n = \frac{1}{n^6},$$
$$\forall (n,u) \in \mathbb{N}_{n_0} \times \mathbb{R}. \tag{98}$$

Obviously, (14), (15), and (64) are satisfied. Note that

$$\begin{aligned} &\frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\max\{R_sH_s, W_s\} \\ &= \frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\max\left\{\frac{(s-2)^2}{s^6}, \frac{1}{s^6}\right\} \\ &\le \frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\frac{1}{s^4} \le \frac{1}{n^2}\sum_{s=n}^{\infty}\frac{1}{s^3} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \end{aligned} \tag{99}$$

which means that

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\max\{R_sH_s, W_s\} = 0. \tag{100}$$

It is clear that

$$\begin{aligned} &\frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\max\{P_tF_t, Q_t, |c_t|\} \\ &= \frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\max\left\{\frac{(t-1)^2}{t^7}, \frac{1}{t^7}, \left|\frac{(-1)^{t-1}t^4+4t^2+t-1}{t^{11}+6t^3+7t+2}\right|\right\} \\ &\le \frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\frac{1}{t^5} \le \frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{t=u}^{\infty}\frac{1}{t^4} \\ &\le \frac{1}{n^2}\sum_{t=n}^{\infty}\frac{1}{t^3} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \end{aligned} \tag{101}$$

which implies that

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{u=n}^{\infty}\sum_{s=u}^{\infty}\sum_{t=s}^{\infty}\max\{P_tF_t, Q_t, |c_t|\} = 0. \tag{102}$$

That is, (38) and (39) hold. Consequently Theorem 7 implies that (97) possesses uncountably many positive solutions in $A(N,M)$. On the other hand, for any $L \in (-M/2, -N)$, there exist $\theta \in (0,1)$ and $T \ge n_0 + \tau + \beta$ such that the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0} = \{\{x_{mn}\}_{n\in\mathbb{N}_\beta}\}_{m\in\mathbb{N}_0}$ generated by (65) converges to a positive solution $z = \{z_n\}_{n\in\mathbb{N}_\beta} \in A(N,M)$ of (97) with $\lim_{n\to\infty} z_n = +\infty$ and has the error estimate (20), where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in$[0,1]$ satisfying (21).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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