Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2014, Article ID 468694, 8 pages
http://dx.doi.org/10.1155/2014/468694

Research Article

Convergence Theorem for a Family of New Modified Halley's Method in Banach Space

Rongfei Lin,[1] Yueqing Zhao,[1] Qingbiao Wu,[2] and Jueliang Hu[3]

[1] *Department of Mathematics, Taizhou University, Linhai, Zhejiang 317000, China*
[2] *Department of Mathematics, Zhejiang University, Hangzhou, Zhejiang 310027, China*
[3] *Department of Mathematics, Zhejiang Sci-Tech University, Hangzhou, Zhejiang 310018, China*

Correspondence should be addressed to Qingbiao Wu; qbwu@zju.edu.cn

Received 18 February 2014; Revised 8 May 2014; Accepted 22 May 2014; Published 17 June 2014

Academic Editor: Alicia Cordero

We establish convergence theorems of Newton-Kantorovich type for a family of new modified Halley's method in Banach space to solve nonlinear operator equations. We present the corresponding error estimate. To show the application of our theorems, two numerical examples are given.

1. Introduction

Solving the nonlinear operator equation is an important issue in the engineering and technology field. Finding solutions of the nonlinear operator equation in Banach spaces also is a very general subject which is widely used in both theoretical and applied areas of mathematics for many years. Many problems may be formulated in terms of finding zeros. These roots cannot be expressed in closed form generally. Thus, in order to solve nonlinear equations, we have to use approximate methods (see [1]). In this study, we consider establishing the Newton-Kantorovich convergence theorems for a family of new modified Halley's method with third-order Banach space by using majorizing function which is used to solve the nonlinear operator equation. Consider

$$F(x) = 0, \tag{1}$$

where F is defined as an open convex Ω of a Banach space X with values in a Banach space Y.

There are kinds of methods to find a solution of (1). Iterative methods are often used to solve this problem (see [1]). If we use the famous Newton's method, we can do as follows:

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n), \quad (n \geq 0) \ (x_0 \in \Omega). \tag{2}$$

Under the reasonable hypothesis, Newton's method is second-order convergence.

Since Kantorovich presented the famous convergence result (see [2]), many Newton-Kantorovich type convergence theorems were gotten (see [3–11]). To improve the convergence order, many modified methods have been presented (see [12–15]). The famous Halley's method is third-order convergence which was widely discussed (see [16–23]). The famous Halley's method is defined as follows:

$$\begin{aligned} x_{n+1} = x_n - &\left[I + \frac{1}{2} L_F(x_n) \left(I - \frac{1}{2} L_F(x_n) \right)^{-1} \right] \\ &\times F'(x_n)^{-1} F(x_n), \quad n = 0, 1, \ldots, \end{aligned} \tag{3}$$

where

$$L_F(x) = F'(x)^{-1} F''(x) F'(x)^{-1} F(x), \quad x \in \Omega. \tag{4}$$

In the year of 2001, Gutiérrez and Hernández [24] proposed the Super-Halley method, which is an acceleration of Newton's method with third order. Consider

$$x_{n+1} = x_n - \left[I + \frac{1}{2} L_F(x_n) (I - L_F(x_n))^{-1} \right] F'(x_n)^{-1} F(x_n). \tag{5}$$

Using the majorant principle, the authors also established a semilocal convergence theorem for the Super-Halley method under weaker conditions, which is defined as follows:

$$\left\| F'(x_0)^{-1} \left(F''(x) - F''(y) \right) \right\| \le k \left\| x - y \right\|. \tag{6}$$

Extending the family of scalar iterative processes considered by Hernandez and Salanova in [25], Gutiérrez and Hernández [26] presented a one-parameter family of iterative processes

$$\begin{aligned} x_{\alpha,n+1} = x_{\alpha,n} - &\left[I + \frac{1}{2} L_F(x_{\alpha,n}) \left(I - \alpha L_F(x_{\alpha,n}) \right)^{-1} \right] \\ &\times F'(x_{\alpha,n})^{-1} F x_{\alpha,n}. \end{aligned} \tag{7}$$

This family includes, as particular cases, Chebyshev's method ($\alpha = 0$), Halley's method ($\alpha = 1/2$), and convex acceleration of Newton's method ($\alpha = 1$). Under Kantorovich-type assumptions, the authors obtained results on existence and uniqueness of solution of (1).

The calculation of the second derivative of the function F is needed in the above iterative methods. For avoiding the calculation $F''(x_n)$ and keeping higher order of convergence at the same time, some authors have studied the convergence of the iteration (3) by using difference quotient of the first derivative to replace the second derivative.

Based on Halley's method in which the second derivative is replaced with a finite difference between first derivatives, that is,

$$F''(x_n) \simeq \frac{F'(y_n) - F'(x_n)}{y_n - x_n}, \tag{8}$$

a class of iterative methods with free second derivative is obtained [27]. Consider

$$x_{n+1} = x_n - \frac{2\theta F(x_n)}{(2\theta - 1) F'(x_n) + F'(y_n)}, \tag{9}$$

where $\theta \in (0, 1]$ and $y_n = x_n - \theta F(x_n)/F'(x_n)$.

For $\theta = 1$, a third-order method is obtained [28]:

$$x_{n+1} = x_n - \frac{2F(x_n)}{F'(x_n) + F'(x_n - F(x_n)/F'(x_n))}. \tag{10}$$

For $\theta = 1/2$, another third-order method is obtained [29, 30]:

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n - F(x_n)/(2F'(x_n)))}. \tag{11}$$

These methods do not require the second derivative although they can converge cubically. Thus, the approach to remove the second derivative is important and interesting for deriving more new high-order iterative methods free from second derivative from third-order iterative methods with second derivative.

By directly replacing $L_F(x_n)$ with a new approximation

$$L_F(x_n) = \frac{1}{2} \frac{F''(x_n) F(x_n)}{F'(x_n)^2} \simeq \frac{F(y_n) + (\theta - 1) F(x_n)}{\theta^2 F(x_n)}, \tag{12}$$

a class of modifications of Halley's method free from second derivative [31] is obtained; that is,

$$x_{n+1} = x_n - \frac{\theta^2 F(x_n)}{(\theta^2 - \theta + 1) F(x_n) - F(y_n)} \frac{F(x_n)}{F'(x_n)}, \tag{13}$$

where $\theta \in R$, $\theta \ne 0$, and $y_n = x_n - \theta F(x_n)/F'(x_n)$. This modified Halley's method is cubically convergent for any nonzero real number θ.

Now, we consider a new finite difference approximation of $F''(x)$:

$$pF''(x_n)(y_n - x_n) \simeq F'(x_n + p(y_n - x_n)) - F'(x_n), \tag{14}$$

where

$$y_n = x_n - F'(x_n)^{-1} F(x_n), \quad \lambda > 0, \tag{15}$$

and from Halley's method (3), we obtain a family of new modified Halley's method with parameters p and α:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1} F(x_n), \\ H(x_n, y_n) &= \frac{1}{p} F'(x_n)^{-1} \left[F'(x_n + p(y_n - x_n)) - F'(x_n) \right], \\ &\qquad \alpha \in [0, 1], \quad p \in (0, 1], \\ x_{n+1} &= y_n - \frac{1}{2} H(x_n, y_n) \left[I - \alpha H(x_n, y_n) \right] (y_n - x_n). \end{aligned} \tag{16}$$

This includes the modified Chebyshev iteration (see [8, 9]) for $p = 1/2$, $\alpha = 0$ and the Jarratt iteration (see [32, 33]) for $p = 2/3$, $\alpha = 1$. In this paper, we discuss the convergence of modified Halley's methods for solving nonlinear operator equations in Banach spaces and establish convergence theorems of Newton-Kantorovich's type. The corresponding error estimate is also given. Finally, two examples are provided to show the application of our theorem.

2. Convergence Theorem

In the section, we establish a Newton-Kantorovich type convergence theorem and present the error estimate. Denote $g(t) = ((1/2)Kt^2) - (t/\beta) + (\eta/\beta)$, where K, β, η are positive real numbers. Write $h = K\beta\eta$, $t^* = ((1 - \sqrt{1 - 2h})/h)\eta$, and $t^{**} = ((1 + \sqrt{1 - 2h})/h)\eta$, where t^*, t^{**} are the roots of the equation $g(t) = 0$. Let

$$\begin{aligned} s_n &= t_n - g'(t_n)^{-1} g(t_n), \quad t_0 = 0, \\ H_g(t_n, s_n) &= \frac{1}{p} g'(t_n)^{-1} \left[g'(t_n + p(s_n - t_n)) - g'(t_n) \right], \\ &\qquad \alpha \in [0, 1], \quad p \in (0, 1], \\ t_{n+1} &= s_n - \frac{1}{2} H_g(t_n, s_n) \left[1 - \alpha H_g(t_n, s_n) \right] (s_n - t_n). \end{aligned} \tag{17}$$

Firstly, we get some lemmas.

Lemma 1. *Assume $\{t_n\}$ and $\{s_n\}$ are the sequences generated by (17). If $h \leq 1/2$, then the sequences $\{t_n\}$ and $\{s_n\}$ are monotonically increasing and converge to t^*. Moreover, one has*

$$0 \leq t_n < s_n < t_{n+1} < s_{n+1} < \cdots < t^*,$$

$$\begin{aligned} t^* - t_{n+1} &= (t^* - t_n)^3 \left[(t^* - t_n)^3 + 4(t^* - t_n)^2 (t^{**} - t_n) \right. \\ &\quad + 5 (t^* - t_n)(t^{**} - t_n)^2 \\ &\quad \left. + (2 - 2\alpha)(t^{**} - t_n)^3 \right] \\ &\quad \times \left([(t^* - t_n) + (t^{**} - t_n)]^5 \right)^{-1}, \\ t^{**} - t_{n+1} &= (t^{**} - t_n)^3 \left[(t^{**} - t_n)^3 + 4(t^{**} - t_n)^2 (t^* - t_n) \right. \\ &\quad + 5 (t^{**} - t_n)(t^* - t_n)^2 \\ &\quad \left. + (2 - 2\alpha)(t^* - t_n)^3 \right] \\ &\quad \times \left([(t^* - t_n) + (t^{**} - t_n)]^5 \right)^{-1}. \end{aligned} \tag{18}$$

Proof. By (17) we can get

$$\begin{aligned} g(t) &= \frac{K}{2}(t^* - t)(t^{**} - t), \\ g'(t) &= -\frac{K}{2}[(t^* - t) + (t^{**} - t)], \\ s_n - t_n &= -g'(t_n)^{-1} g(t_n) = \frac{(t^* - t_n)(t^{**} - t_n)}{(t^* - t_n) + (t^{**} - t_n)}, \\ H_g(t_n, s_n) &= g'(t_n)^{-1} K (s_n - t_n) = \frac{-2(t^* - t_n)(t^{**} - t_n)}{[(t^* - t_n) + (t^{**} - t_n)]^2}, \\ t_{n+1} - s_n &= -\frac{1}{2} H_g(t_n, s_n)\left[1 - \alpha H_g(t_n, s_n)\right](s_n - t_n). \end{aligned} \tag{19}$$

By a simple calculation and mathematical induction, it is easy to prove Lemma 1. □

Lemma 2. *Let $F(x)$ be a nonlinear operator defined on a convex domain Ω of a Banach space X with values in a Banach space Y. Assume that F has second-order continuous Frechet derivatives on Ω. If $\{x_n\}$, $\{y_n\}$ are the sequences generated by (16), then*

$$\begin{aligned} &F(x_{n+1}) \\ &= \int_0^1 F''(y_n + t(x_{n+1} - y_n))(1 - t)\,dt (x_{n+1} - y_n)^2 \\ &\quad + \int_0^1 \left[F''(x_n + t(y_n - x_n))(1 - t) \right. \\ &\qquad \left. - \frac{1}{2} F''(x_n + pt(y_n - x_n)) \right] dt (y_n - x_n)^2 \\ &\quad - \frac{1 - \alpha}{2} \int_0^1 F''(x_n + pt(y_n - x_n))\,dt \\ &\quad \times (y_n - x_n) H(x_n, y_n)(y_n - x_n) \\ &\quad - \frac{\alpha}{2} \int_0^1 \left[F''(x_n + t(y_n - x_n)) \right. \\ &\qquad \left. - F''(x_n + pt(y_n - x_n)) \right] dt \\ &\quad \times (y_n - x_n) H(x_n, y_n)(y_n - x_n) \\ &\quad + \frac{\alpha}{2} \int_0^1 F''(x_n + t(y_n - x_n))\,dt \\ &\quad \times (y_n - x_n) H(x_n, y_n) H(x_n, y_n)(y_n - x_n). \end{aligned} \tag{20}$$

Proof. Consider

$$\begin{aligned} &F(x_{n+1}) \\ &= F(x_{n+1}) - F(y_n) - F'(y_n)(x_{n+1} - y_n) \\ &\quad + F(y_n) + F'(y_n)(x_{n+1} - y_n) \\ &= \int_0^1 F''(y_n + t(x_{n+1} - y_n))(1 - t)\,dt (x_{n+1} - y_n)^2 \\ &\quad + F(y_n) + F'(y_n)(x_{n+1} - y_n), \\ &F'(x_n) H(x_n, y_n) \\ &= \frac{1}{p}\left[F'(x_n + p(y_n - x_n)) - F'(x_n)\right] \\ &= \int_0^1 F''(x_n + pt(y_n - x_n))\,dt (y_n - x_n), \\ &F(y_n) + F'(y_n)(x_{n+1} - y_n) \\ &= F(y_n) - F(x_n) - F'(x_n)(y_n - x_n) \\ &\quad - \frac{1}{2} F'(y_n) H(x_n, y_n)[I - \alpha H(x_n, y_n)](y_n - x_n) \\ &= \int_0^1 F''(x_n + t(y_n - x_n))(1 - t)\,dt (y_n - x_n)^2 \\ &\quad - \frac{1}{2}\left[F'(y_n) - F'(x_n)\right] H(x_n, y_n)[I - \alpha H(x_n, y_n)] \\ &\quad \times (y_n - x_n) \\ &\quad - \frac{1}{2} F'(x_n) H(x_n, y_n)[I - \alpha H(x_n, y_n)](y_n - x_n) \\ &= \int_0^1 F''(x_n + t(y_n - x_n))(1 - t)\,dt (y_n - x_n)^2 \\ &\quad - \frac{1}{2}\int_0^1 F''(x_n + pt(y_n - x_n))\,dt (y_n - x_n)^2 \\ &\quad + \frac{\alpha}{2}\int_0^1 F''(x_n + pt(y_n - x_n))\,dt (y_n - x_n) \end{aligned}$$

$$
\begin{aligned}
&\times H(x_n, y_n)(y_n - x_n) \\
&- \frac{1}{2}\int_0^1 F''(x_n + t(y_n - x_n))\, dt\, (y_n - x_n) \\
&\times H(x_n, y_n)(y_n - x_n) \\
&+ \frac{\alpha}{2}\int_0^1 F''(x_n + t(y_n - x_n))\, dt\, (y_n - x_n) H(x_n, y_n) \\
&\times H(x_n, y_n)(y_n - x_n).
\end{aligned} \tag{21}
$$

Hence,

$$
\begin{aligned}
&F(x_{n+1}) \\
&\quad = \int_0^1 F''(y_n + t(x_{n+1} - y_n))(1-t)\, dt (x_{n+1} - y_n)^2 \\
&\quad + \int_0^1 \Big[F''(x_n + t(y_n - x_n))(1-t) \\
&\qquad - \frac{1}{2}F''(x_n + pt(y_n - x_n))\Big] dt (y_n - x_n)^2 \\
&\quad - \frac{1-\alpha}{2}\int_0^1 F''(x_n + t(y_n - x_n))\, dt \\
&\quad \times (y_n - x_n) H(x_n, y_n)(y_n - x_n) \\
&\quad - \frac{\alpha}{2}\int_0^1 \big[F''(x_n + t(y_n - x_n)) \\
&\qquad - F''(x_n + pt(y_n - x_n))\big]\, dt \\
&\quad \times (y_n - x_n) H(x_n, y_n)(y_n - x_n) \\
&\quad + \frac{\alpha}{2}\int_0^1 F''(x_n + t(y_n - x_n))\, dt \\
&\quad \times (y_n - x_n) H(x_n, y_n) H(x_n, y_n)(y_n - x_n).
\end{aligned} \tag{22}
$$

This completes the proof. □

Theorem 3. *Let X and Y be Banach space and let $\Omega \subset X$ be an open convex domain. Assume $F : \Omega \subset X \to Y$ has second-order continuous Frechet derivatives. For an initial value $x_0 \in \Omega$ and fixed parameters $\alpha \in [0, 1)$, $p \in (0, 1]$, if $F'(x_0)^{-1}$ exists and the conditions*

$$
\begin{gathered}
\left\|F'(x_0)^{-1}\right\| \le \beta, \qquad \left\|y_0 - x_0\right\| = \left\|F'(x_0)^{-1} F(x_0)\right\| \le \eta, \\
\left\|F''(x)\right\| \le M, \qquad \left\|F''(x) - F''(y)\right\| \le N\left\|x - y\right\|, \\
x, y \in \Omega, \\
\max\left\{M\sqrt{1 + \frac{(2+3p)N}{6\beta(1-\alpha)M^2}}, M\sqrt[3]{1 + \frac{(1-p)N}{2\beta M^2}}\right\} \le K, \\
\overline{S(x_0, t^*)} \subset \Omega,
\end{gathered} \tag{23}
$$

are satisfied and $h = K\beta\eta \le 1/2$, then the sequence $\{x_n\}_{n\ge0}$ generated by (16) is well defined and converges to a unique solution x^* of (1) in $S(x_0, t^{**})$.

Theorem 4. *Assume F satisfies conditions in Theorem 3. Denoting $\theta = (t^*/t^{**}) = (1 - \sqrt{1-2h})/(1 + \sqrt{1-2h})$, one has the following:*

(i) *when $h = K\beta\eta < 6\sqrt{2} - 8 = 0.4852\cdots$,*

$$
\| x_n - x^* \| \le t^* - t_n \le \frac{(1-\theta^2)\eta}{(1 - (1/\sqrt{2}))[\sqrt{2}\theta]^{3^n}}[\sqrt{2}\theta]^{3^n - 1}, \tag{24}
$$

(ii) *when $\alpha \ge 3/8, h \le 4\,(9\sqrt{5} - 20) = 0.4984\cdots$,*

$$
\begin{aligned}
&\left\|x_n - x^*\right\| \le t^* - t_n \\
&\qquad \le \frac{(1-\theta^2)\eta}{(1 - (2/\sqrt{5}))[(\sqrt{5}/2)\theta]^{3^n}}\left[\frac{\sqrt{5}}{2}\theta\right]^{3^n - 1}.
\end{aligned} \tag{25}
$$

3. Proof of Theorem

Proof of Theorem 3. To prove Theorem 3, we first prove that following items are true for all $n \ge 0$:

(I_n) $x_n \in \overline{S(x_0, t_n)}$;

(II_n) $\|F'(x_n)^{-1}\| \le -g'(t_n)^{-1}$;

(III_n) $\|y_n - x_n\| \le s_n - t_n$;

(IV_n) $y_n \in \overline{S(x_0, s_n)}$;

(V_n) $\|x_{n+1} - y_n\| \le t_{n+1} - s_n$.

It is easy to check for the case $n = 0$ by the initial conditions. By using mathematical induction, assume that the above statements are true for some fixed $n \ge 0$. Then we have

(I_{n+1}):

$$
\begin{aligned}
\left\|x_{n+1} - x_0\right\| &\le \left\|x_{n+1} - y_n\right\| + \left\|y_n - x_0\right\| \\
&\le t_{n+1} - s_n + s_n = t_{n+1},
\end{aligned} \tag{26}
$$

(II_{n+1}):

$$
\begin{aligned}
\left\|F'(x_{n+1}) - F'(x_0)\right\| &\le M\left\|x_{n+1} - x_0\right\| \\
&\le Mt_{n+1} < Kt^* = K\eta\frac{1 - \sqrt{1-2h}}{h} \\
&\le \frac{1}{\beta} = \frac{1}{\left\|f'(x_0)^{-1}\right\|}.
\end{aligned} \tag{27}
$$

By Banach Lemma, we get that $F'(x_{n+1})^{-1}$ exists and

$$\begin{aligned}
&\left\|F'(x_{n+1})^{-1}\right\| \\
&\leq \frac{\left\|F'(x_0)^{-1}\right\|}{1-\left\|F'(x_0)^{-1}\right\|\cdot\left\|F'(x_{n+1})-F'(x_0)\right\|} \\
&\leq \frac{\beta}{1-\beta M\left\|x_{n+1}-x_0\right\|} \\
&\leq \frac{1}{(1/\beta)-K\left\|x_{n+1}-x_0\right\|} \leq \frac{1}{(1/\beta)-Kt_{n+1}} \\
&= -g'(t_{n+1})^{-1};
\end{aligned} \tag{28}$$

(III_{n+1}):

$$\begin{aligned}
&\left\|\int_0^1\left[F''(x_n+t(y_n-x_n))(1-t)\right.\right. \\
&\qquad\left.\left.-\frac{1}{2}F''(x_n+pt(y_n-x_n))\right]dt\right\| \\
&\leq \left\|\int_0^1\left[F''(x_n+t(y_n-x_n))-F''(x_n)\right](1-t)\,dt\right\| \\
&\quad+\frac{1}{2}\left\|\int_0^1\left[F''\left[(x_n+pt(y_n-x_n))-F''(x_n)\right]dt\right\| \\
&\leq \frac{(2+3p)N}{12}\left\|y_n-x_n\right\|, \\
&\left\|H(x_{n+1},y_{n+1})\right\| \\
&= \frac{1}{p}\left\|F'(x_{n+1})^{-1}\int_0^1F''(x_{n+1}+pt(y_{n+1}-x_{n+1}))\right. \\
&\qquad\left.\times p(y_{n+1}-x_{n+1})\,dt\right\| \\
&\leq -\frac{M}{g'(t_{n+1})}\left\|y_{n+1}-x_{n+1}\right\|.
\end{aligned} \tag{29}$$

By Lemma 2 and $0<-g'(t_n)\leq 1/\beta$,

$$\begin{aligned}
&\left\|F(x_{n+1})\right\| \\
&\leq \frac{M}{2}\left\|x_{n+1}-y_n\right\|^2+\frac{(2+3p)N}{12} \\
&\quad\times\left\|y_n-x_n\right\|^3+\frac{(1-\alpha)M^2}{-2g'(t_n)}\left\|y_n-x_n\right\|^3 \\
&\quad+\frac{\alpha M^3}{2\left[g'(t_n)\right]^2}\left\|y_n-x_n\right\|^4+\frac{\alpha MN(1-p)}{-4g'(t_n)}\left\|y_n-x_n\right\|^4 \\
&\leq \frac{K}{2}\left\|x_{n+1}-y_n\right\|^2+\left[1+\frac{N(1-p)}{2\beta M^2}\right]\frac{\alpha M^3\left\|y_n-x_n\right\|^4}{2\left[g'(t_n)\right]^2} \\
&\quad+\left[1+\frac{(2+3p)N}{6\beta(1-\alpha)M^2}\right]\frac{(1-\alpha)M^2}{-2g'(t_n)}\left\|y_n-x_n\right\|^3 \\
&\leq \frac{K}{2}(t_{n+1}-s_n)^2+\frac{\alpha K^3}{2}\frac{(s_n-t_n)^4}{\left[g'(t_n)\right]^2}-\frac{(1-\alpha)K^2}{2g'(t_n)}(s_n-t_n)^3 \\
&= g(t_{n+1}).
\end{aligned} \tag{30}$$

Hence, we deduce that

$$\begin{aligned}
\left\|y_{n+1}-x_{n+1}\right\| &\leq \left\|-F'(x_{n+1})^{-1}\right\|\ \left\|F(x_{n+1})\right\| \\
&\leq -g'(t_{n+1})^{-1}g(t_{n+1}) = s_{n+1}-t_{n+1}.
\end{aligned} \tag{31}$$

Moreover, we have (IV_{n+1}):

$$\begin{aligned}
\left\|y_{n+1}-x_0\right\| &\leq \left\|y_{n+1}-x_{n+1}\right\|+\left\|x_{n+1}-x_0\right\| \\
&\leq (s_{n+1}-t_{n+1})+t_{n+1} = s_{n+1},
\end{aligned} \tag{32}$$

(V_{n+1}):

$$\begin{aligned}
&\left\|x_{n+2}-y_{n+1}\right\| \\
&= \left\|-\frac{1}{2}H(x_{n+1},y_{n+1})\left[I-\alpha H(x_{n+1},y_{n+1})\right](y_{n+1}-x_{n+1})\right\| \\
&\leq \frac{1}{2}\frac{M\left\|y_{n+1}-x_{n+1}\right\|}{-g'(t_{n+1})}\left[1+\alpha\frac{M\left\|y_{n+1}-x_{n+1}\right\|}{-g'(t_{n+1})}\right] \\
&\quad\times\left\|y_{n+1}-x_{n+1}\right\| \\
&\leq -\frac{1}{2}\frac{K(s_{n+1}-t_{n+1})}{g'(t_{n+1})}\left[1-\alpha\frac{K(s_{n+1}-t_{n+1})}{g'(t_{n+1})}\right](s_{n+1}-t_{n+1}) \\
&= t_{n+2}-s_{n+1}.
\end{aligned} \tag{33}$$

By Lemma 1, if $h\leq 1/2$, then the sequence $\{x_n,\}_{n\geq 0}$ generated by (16) is well defined, remains in $\overline{S(x_0,t^*)}$ for all $n\geq 0$, and converges to a solution x^* of (1).

To show uniqueness, let us assume that there exists a second solution y^* of (1) in $S(x_0,t^{**})$. Then

$$\begin{aligned}
&\left\|F'(x_0)^{-1}\int_0^1F'(x^*+t(y^*-x^*))\,dt-I\right\| \\
&\leq \left\|F'(x_0)^{-1}\right\|\left\|\int_0^1F'\left[x^*+t(y^*-x^*)\right]-F'(x_0)\,dt\right\| \\
&\leq \beta M\int_0^1\left\|x^*+t(y^*-x^*)-x_0\right\|dt \\
&\leq \beta M\int_0^1\left[(1-t)\left\|x^*-x_0\right\|+t\left\|y^*-x_0\right\|\right]dt \\
&< \frac{\beta M}{2}(t^*+t^{**})\leq 1.
\end{aligned} \tag{34}$$

By Banach Lemma, we can obtain that the inverse of the linear operator $\int_0^1 F'[x^* + t(y^* - x^*)]dt$ exists and

$$F(y^*) - F(x^*) = \int_0^1 F'[x^* + t(y^* - x^*)]\,dt\,(y^* - x^*). \tag{35}$$

We conclude that $x^* = y^*$. The proof of Theorem 3 is completed. □

Proof of Theorem 4. By Lemma 1, we have

$$\begin{aligned} t^* - t_{n+1} &= (t^* - t_n)^3 \left[(t^* - t_n)^3 + 4(t^* - t_n)^2 (t^{**} - t_n) + 5(t^* - t_n)(t^{**} - t_n)^2 + (2 - 2\alpha) \times (t^{**} - t_n)^3\right] \\ &\quad \times \left(\left[(t^* - t) + (t^{**} - t)\right]^5\right)^{-1} \\ &< (t^* - t_n)^3 \left[5(t^* - t_n)^2 (t^{**} - t_n) + 5(t^* - t_n)(t^{**} - t_n)^2 + 2(t^{**} - t_n)^3\right] \\ &\quad \times \left(\left[(t^* - t) + (t^{**} - t)\right]^5\right)^{-1}. \end{aligned} \tag{36}$$

Similarly,

$$\begin{aligned} t^{**} - t_{n+1} &= (t^{**} - t_n)^3 \left[(t^{**} - t_n)^3 + 4(t^{**} - t_n)^2 (t^* - t_n) + 5(t^{**} - t_n)(t^* - t_n)^2 + (2 - 2\alpha)(t^* - t_n)^3\right] \\ &\quad \times \left(\left[(t^* - t) + (t^{**} - t)\right]^5\right)^{-1} \\ &> (t^{**} - t_n)^3 \left[(t^{**} - t_n)^3 + 4(t^{**} - t_n)^2 (t^* - t_n) + 5(t^{**} - t_n)(t^* - t_n)^2\right] \\ &\quad \times \left(\left[(t^* - t) + (t^{**} - t)\right]^5\right)^{-1}. \end{aligned} \tag{37}$$

Hence,

$$\begin{aligned} \frac{t^* - t_{n+1}}{t^{**} - t_{n+1}} &< 2\left[\frac{t^* - t_n}{t^{**} - t_n}\right]^3 < 2 \cdot 2^3 \left[\frac{t^* - t_{n-1}}{t^{**} - t_{n-1}}\right]^{3^2} \\ &< \cdots < 2 \cdot 2^3 \cdots 2^{3^n} \left[\frac{t^* - t_0}{t^{**} - t_0}\right]^{3^{n+1}} = \sqrt{2}^{3^{n+1}-1} \theta^{3^{n+1}}. \end{aligned} \tag{38}$$

Because $t^{**} - t_n = t^* - t_n + t^{**} - t^* = t^* - t_n + ((1 - \theta^2)\eta)/\theta$, we obtain

$$t^* - t_n \le \frac{(1 - \theta^2)\eta}{(1 - (1/\sqrt{2}))\left[\sqrt{2}\theta\right]^{3^n}} \left[\sqrt{2}\theta\right]^{3^n - 1}. \tag{39}$$

When $\alpha \ge 3/8$,

$$\begin{aligned} t^* - t_{n+1} &= (t^* - t_n)^3 \left[(t^* - t_n)^3 + 4(t^* - t_n)^2 (t^{**} - t_n) + 5(t^* - t_n)(t^{**} - t_n)^2 + (2 - 2\alpha)(t^{**} - t_n)^3\right] \\ &\quad \times \left(\left[(t^* - t) + (t^{**} - t)\right]^5\right)^{-1} \\ &< (t^* - t_n)^3 \left[5(t^* - t_n)^2 (t^{**} - t_n) + 5(t^* - t_n)(t^{**} - t_n)^2 + \left(\frac{5}{4}\right)(t^{**} - t_n)^3\right] \\ &\quad \times \left(\left[(t^* - t) + (t^{**} - t)\right]^5\right)^{-1} \\ &= \frac{5}{4}(t^* - t_n)^3 \left[4(t^* - t_n)^2 (t^{**} - t_n) + 4(t^* - t_n)(t^{**} - t_n)^2 + (t^{**} - t_n)^3\right] \\ &\quad \times \left(\left[(t^* - t) + (t^{**} - t)\right]^5\right)^{-1}. \end{aligned} \tag{40}$$

Hence, we get

$$\begin{aligned} \frac{t^* - t_{n+1}}{t^{**} - t_{n+1}} &< \frac{5}{4}\left[\frac{t^* - t_n}{t^{**} - t_n}\right]^3 < \left(\frac{\sqrt{5}}{2}\right)^{3^{n+1}-1} \theta^{3^{n+1}}, \\ \|x_n - x^*\| \le t^* - t_n &\le \frac{(1 - \theta^2)\eta}{1 - (2/\sqrt{5})\left[(\sqrt{5}/2)\theta\right]^{3^n}} \left[\frac{\sqrt{5}}{2}\theta\right]^{3^n - 1}. \end{aligned} \tag{41}$$

The proof of Theorem 4 is completed. □

4. Applications

Example 1. Consider the case as follows:

$$x(s) = 1 + \frac{1}{4}x(s)\int_0^1 \frac{s}{s+t}x(t)\,dt, \tag{42}$$

where the space is $X = C[0,1]$ with norm

$$\|x\| = \max_{0 \le s \le 1} |x(s)|. \tag{43}$$

This equation arises in the theory of the radiative transfer, neutron transport, and kinetic theory of gasses. Let us define the operator F on X by

$$F(x) = \frac{1}{4}x(s)\int_0^1 \frac{s}{s+t}x(t)\,dt - x(s) + 1. \tag{44}$$

Table 1: Error computing results of (42).

n	Newton method	Method (16) with $p = 2/3$, $\alpha = 1/2$
1	5.215×10^{-1}	2.2371×10^{-3}
2	6.610×10^{-2}	1.3667×10^{-9}
3	14.017×10^{-4}	3.1229×10^{-28}

Table 2: Error computing results ($|x_n - x^*|$).

Step	Newton's method	Halley's method (3)	Method (16) with $p = 2/3, \alpha = 1$	Method (16) with $p = 1/2, \alpha = 0$
$n = 1$	2.56574×10^{-2}	2.13749×10^{-3}	1.58011×10^{-3}	5.74362×10^{-3}
$n = 2$	4.8337×10^{-4}	3.04697×10^{-9}	1.19327×10^{-11}	1.98875×10^{-7}
$n = 3$	1.75164×10^{-7}	8.84002×10^{-27}	3.90354×10^{-44}	8.35734×10^{-21}
$n = 4$	2.30119×10^{-14}	2.15878×10^{-79}	4.47038×10^{-174}	6.20201×10^{-61}

Then, for $x_0 = 1$, we can obtain

$$\begin{gathered} M = 2 \cdot \frac{1}{4} \max_{0 \le s \le 1} \left| \int_0^1 \frac{s}{s+t} dt \right| = \frac{\ln 2}{2} = 0.3465, \\ N = 0, \quad K = M = 0.3465, \quad \beta = \left\| F'(x_0)^{-1} \right\| = 1.5304, \\ \eta = \left\| F'(x_0)^{-1} F(x_0) \right\| = 0.2652, \\ h = K\beta\eta = 0.1406 < 0.4852, \quad t^* = 0.2870, \\ t^{**} = 3.485, \quad \theta = 0.08240. \end{gathered} \tag{45}$$

That means the hypotheses of the theorem are satisfied and for $\alpha = 1/2 > 3/8$ the error bound becomes

$$\begin{aligned} &\left\| x_n - x^* \right\| \\ &\le t^* - t_n \le \frac{\left(1 - \theta^2\right)\eta}{\left(1 - \left(2/\sqrt{5}\right)\right)\left[\left(\sqrt{5}/2\right)\theta\right]^{3^n}} \left[\frac{\sqrt{5}}{2}\theta\right]^{3^n - 1} \\ &= \frac{0.2634}{1 - 0.8944 \cdot (0.07370)^{3^n}} (0.07370)^{3^n - 1}. \end{aligned} \tag{46}$$

For $n = 1, 2, 3$, we get

$$\begin{gathered} \left\| x_1 - x^* \right\| \le t^* - t_1 \le 2.2371 \times 10^{-3}, \\ \left\| x_2 - x^* \right\| \le t^* - t_2 \le 1.3667 \times 10^{-9}, \\ \left\| x_3 - x^* \right\| \le t^* - t_3 \le 3.1229 \times 10^{-28}. \end{gathered} \tag{47}$$

In practical computation, we use a discretization process. By Gauss-Legendre quadrature formula with 8 nodes:

$$\int_0^1 \varphi(t)\, dt \simeq \sum_{j=1}^{8} \omega_j \varphi\left(t_j\right), \tag{48}$$

we approximate the integral equation (42), where the nodes t_j and the weights ω_j are known. Denote $x(t_i)$ by x_i, $i = 1, 2, \ldots 8$, so we can transform (42) into the following system of nonlinear equations:

$$x_i = 1 + \frac{1}{4} x_i \sum_{j=1}^{8} a_{ij} x_j, \tag{49}$$

where $a_{ij} = \omega_j (t_i/(t_i + t_j))$. Then, we rewrite the above system in the matrix form. Consider

$$F(X) = X - V - \frac{1}{4} AW, \tag{50}$$

where $X = (x_1, x_2, \ldots, x_8)^T$, $V = (1, 1, \ldots, 1)^T$, $A = (a_{ij})_{i,j=1}^{8}$, and $W = (x_1^2, x_2^2, \ldots, x_8^2)^T$. We also get

$$F'(X) = I - \frac{1}{2} A \operatorname{diag}\{x_1, x_2, \ldots, x_8\}. \tag{51}$$

After 3 iterative steps, we can obtain the numerical solution of (42) by method (16) with $p = 2/3$, $\alpha = 1/2$, and the solution $X_* = (1.2751, 1.4418, 1.5595, 1.6486, 1.7189, 1.7761, 1.8236, 1.8638)^T$, and we present the error computing result of (42) in Table 1.

Example 2. Now we employ iterative methods (16) to solve the equation and compare these methods with Newton's method, Halley's method, and modified Halley's methods (16). We define as follows:

$$f(x) = x^n - R, \quad n > 2, \; R \in (0, +\infty). \tag{52}$$

Denote $x^* = \sqrt[n]{R}$, by (16) $x_{n+1} = v(x_n)$, where $v(x) = x - [1 - (1/2)H(x) + (\alpha/2)H^2(x)](f(x))/(f'(x))$, $H(x) = (f'(x - pf(x)/f'(x)) - f'(x))/(pf'(x))$. We have $v'(x^*) = v''(x^*) = v'''(x^*) = 0$ if

$$p = \frac{2}{3}, \quad \alpha = 1. \tag{53}$$

So, we get the convergence of the sequence $\{x_n\}$ generated by modified Halley's method (16) with four orders when $p = 2/3$, $\alpha = 1$.

Now, we compare some of these methods for the calculus of $x^* = \sqrt[4]{16} = 2$. We analyze the errors $x_n - x^*$ for these methods with Newton's method, Halley's method, and the family of new modified Halley's methods (16). In these cases, we have taken $n = 4$, $x_0 = 2.2$ (see Table 2).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is supported by the National Basic Research 973 Program of China (no. 2011JB105001), National Natural Science Foundation of China (Grant no. 11371320), Zhejiang Natural Science Foundation (Grant no. LZ14A010002), Foundation of Science and Technology Department (Grant no. 2013C31084) of Zhejiang Province, and Scientific Research Fund of Zhejiang Provincial Education Department (nos. Y201431077 and Y201329420).

References

[1] J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, NY, USA, 1970.

[2] L. V. Kantorovič, "On Newton's method," *Akademiya Nauk SSSR. Trudy Matematicheskogo Instituta Imeni V. A. Steklova*, vol. 28, pp. 104–144, 1949 (Russian).

[3] I. K. Argyros and S. Hilout, "Extending the Newton-Kantorovich hypothesis for solving equations," *Journal of Computational and Applied Mathematics*, vol. 234, no. 10, pp. 2993–3006, 2010.

[4] I. K. Argyros, "On the Newton-Kantorovich hypothesis for solving equations," *Journal of Computational and Applied Mathematics*, vol. 169, no. 2, pp. 315–332, 2004.

[5] I. K. Argyros, "On the comparison of a weak variant of the Newton-Kantorovich and Miranda theorems," *Journal of Computational and Applied Mathematics*, vol. 166, no. 2, pp. 585–589, 2004.

[6] J. M. Gutiérrez and M. A. Hernández, "New recurrence relations for Chebyshev method," *Applied Mathematics Letters*, vol. 10, no. 2, pp. 63–65, 1997.

[7] J. A. Ezquerro, D. González, and M. A. Hernández, "A modification of the classic conditions of Newton-Kantorovich for Newton's method," *Mathematical and Computer Modelling*, vol. 57, no. 3-4, pp. 584–594, 2013.

[8] S. Amat, I. K. Argyros, S. Busquier, R. Castro, S. Hilout, and S. Plaza, "Newton-type methods on Riemannian manifolds under Kantorovich-type conditions," *Applied Mathematics and Computation*, vol. 227, pp. 762–787, 2014.

[9] M. A. Hernández and M. A. Salanova, "Modification of the Kantorovich assumptions for semilocal convergence of the Chebyshev method," *Journal of Computational and Applied Mathematics*, vol. 126, no. 1-2, pp. 131–143, 2000.

[10] Q. B. Wu and Y. Q. Zhao, "Third-order convergence theorem by using majorizing function for a modified Newton method in Banach space," *Applied Mathematics and Computation*, vol. 175, no. 2, pp. 1515–1524, 2006.

[11] R. Lin, Y. Zhao, Z. Šmarda, Y. Khan, and Q. Wu, "Newton-Kantorovich and Smale uniform type convergence theorem for a deformed Newton method in Banach spaces," *Abstract and Applied Analysis*, vol. 2013, Article ID 923898, 8 pages, 2013.

[12] I. K. Argyros and S. K. Khattri, "Weaker Kantorovich type criteria for inexact Newton methods," *Journal of Computational and Applied Mathematics*, vol. 261, pp. 103–117, 2014.

[13] S. Weerakoon and T. G. I. Fernando, "A variant of Newton's method with accelerated third-order convergence," *Applied Mathematics Letters*, vol. 13, no. 8, pp. 87–93, 2000.

[14] H. Ren and I. K. Argyros, "On the semi-local convergence of Halley's method under a center-Lipschitz condition on the second Fréchet derivative," *Applied Mathematics and Computation*, vol. 218, no. 23, pp. 11488–11495, 2012.

[15] J. S. Kou, Y. T. Li, and X. H. Wang, "On modified Newton methods with cubic convergence," *Applied Mathematics and Computation*, vol. 176, no. 1, pp. 123–127, 2006.

[16] D. Chen, I. K. Argyros, and Q. S. Qian, "A note on the Halley method in Banach spaces," *Applied Mathematics and Computation*, vol. 58, no. 2-3, pp. 215–224, 1993.

[17] I. K. Argyros, "The super-Halley method using divided differences," *Applied Mathematics Letters*, vol. 10, no. 4, pp. 91–95, 1997.

[18] J. M. Gutiérrez and M. A. Hernández, "Recurrence relations for the super-Halley method," *Computers & Mathematics with Applications*, vol. 36, no. 7, pp. 1–8, 1998.

[19] S. Amat, S. Busquier, and J. M. Gutiérrez, "Third-order iterative methods with applications to Hammerstein equations: a unified approach," *Journal of Computational and Applied Mathematics*, vol. 235, no. 9, pp. 2936–2943, 2011.

[20] J. A. Ezquerro and M. A. Hernández, "A modification of the super-Halley method under mild differentiability conditions," *Journal of Computational and Applied Mathematics*, vol. 114, no. 2, pp. 405–409, 2000.

[21] J. A. Ezquerro and M. A. Hernández, "On the R-order of the Halley method," *Journal of Mathematical Analysis and Applications*, vol. 303, no. 2, pp. 591–601, 2005.

[22] B. Neta and M. Scott, "On a family of Halley-like methods to find simple roots of nonlinear equations," *Applied Mathematics and Computation*, vol. 219, no. 15, pp. 7940–7944, 2013.

[23] R. F. Lin, Y. Q. Zhao, Z. Smarda, Q. B. Wu, and Y. Khan, "Newton-Kantorovich convergence theorem of a new modified Halley's method family in a Banach space," *Advances in Difference Equations*, vol. 2013, article 325, 2013.

[24] J. M. Gutiérrez and M. A. Hernández, "An acceleration of Newton's method: super-Halley method," *Applied Mathematics and Computation*, vol. 117, no. 2-3, pp. 223–239, 2001.

[25] M. A. Hernandez and M. A. Salanova, "A family of Chebyshev-Halley type methods," *International Journal of Computer Mathematics*, vol. 47, pp. 59–63, 1993.

[26] J. M. Gutiérrez and M. A. Hernández, "A family of Chebyshev-Halley type methods in Banach spaces," *Bulletin of the Australian Mathematical Society*, vol. 55, no. 1, pp. 113–130, 1997.

[27] J. A. Ezquerro and M. A. Hernández, "A uniparametric Halley-type iteration with free second derivative," *International Journal of Pure and Applied Mathematics*, vol. 6, no. 1, pp. 103–114, 2003.

[28] S. Weerakoon and T. G. I. Fernando, "A variant of Newton's method with accelerated third-order convergence," *Applied Mathematics Letters*, vol. 13, no. 8, pp. 87–93, 2000.

[29] M. Frontini and E. Sormani, "Some variant of Newton's method with third-order convergence," *Applied Mathematics and Computation*, vol. 140, no. 2-3, pp. 419–426, 2003.

[30] H. H. H. Homeier, "A modified Newton method for rootfinding with cubic convergence," *Journal of Computational and Applied Mathematics*, vol. 157, no. 1, pp. 227–230, 2003.

[31] J. Kou, Y. Li, and X. Wang, "Modified Halley's method free from second derivative," *Applied Mathematics and Computation*, vol. 183, no. 1, pp. 704–708, 2006.

[32] I. K. Argyros, D. Chen, and Q. Qian, "An inverse-free Jarratt type approximation in a Banach space," *Approximation Theory and Its Applications*, vol. 12, no. 1, pp. 19–30, 1996.

[33] J. A. Ezquerro and M. A. Hernández, "Relaxing convergence conditions for an inverse-free Jarratt-type approximation," *Journal of Computational and Applied Mathematics*, vol. 83, no. 1, pp. 131–135, 1997.