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Research Article

Positive Solutions of a Singular Nonlocal Fractional Order Differential System via Schauder's Fixed Point Theorem

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We establish the existence of positive solutions to a class of singular nonlocal fractional order differential system depending on two parameters. Our methods are based on Schauder's fixed point theorem.

1. Introduction

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Particularly, fractional derivatives provide an excellent tool for the description of memory and hereditary properties of many materials and processes. With this advantage, fractional order models are more realistic and practical than the classical integer-order models in physics, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, fitting of experimental data, and so forth [1–16]. Recently, Rehman and Khan [17] studied the fractional order multipoint boundary value problem:

$$\begin{aligned} &D_t^{\alpha} y(t) = f\left(t, y(t), D_t^{\beta} y(t)\right), \quad t \in (0,1), \\ &y(0) = 0, \qquad D_t^{\beta} y(1) - \sum_{i=1}^{m-2} \zeta_i D_t^{\beta} y(\xi_i) = y_0, \end{aligned} \tag{1}$$

where $1 < \alpha \le 2$, $0 < \beta < 1$, $\zeta_i \in [0, +\infty)$, $0 < \xi_i < 1$, with $\sum_{i=1}^{m-2} \zeta_i \xi_i < 1$. By using the contraction mapping principle, the existence and uniqueness of positive solutions were established. In [18], Zhang et al. discussed the existence and uniqueness of positive solutions for the following fractional differential equation with derivatives:

$$\begin{aligned} &-\mathscr{D}_{\mathbf{t}}{}^{\alpha} x(t) = f\left(t, x(t), -\mathscr{D}_{\mathbf{t}}{}^{\beta} x(t)\right), \quad t \in (0,1), \\ &\mathscr{D}_{\mathbf{t}}{}^{\beta} x(0) = 0, \qquad \mathscr{D}_{\mathbf{t}}{}^{\gamma} x(1) = \sum_{j=1}^{p-2} a_j \mathscr{D}_{\mathbf{t}}{}^{\gamma} x\left(\xi_j\right), \end{aligned} \tag{2}$$

where $1 < \alpha \le 2$, $\alpha - \beta > 1$, $0 < \beta \le \gamma < 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{p-2} < 1$, $a_j \in [0, +\infty)$ with $c = \sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\gamma-1} < 1$, and $\mathscr{D}_{\mathbf{t}}$ is the standard Riemann-Liouville derivative. $f : (0,1) \times [0, +\infty) \times (-\infty, +\infty) \to [0, +\infty)$ is continuous, and $f(t, u, v)$ may be singular at $t = 0, 1$. By means of monotone iterative technique, the existence and uniqueness of the positive solution for a fractional differential equation with derivatives are established, and the iterative sequence of the solution, an error estimation, and the convergence rate of the positive solution are also given.

However, the research on the systems of fractional differential equations has not received much attention. So motivated by the results mentioned above, in this paper, we study the existence of positive solutions for the following singular

nonlocal fractional order differential system depending on two parameters:

$$\begin{aligned} -D_t^{\alpha} u(t) &= \lambda f\left(t, D_t^{\beta_1} v(t)\right), \\ -D_t^{\beta} v(t) &= \mu g\left(t, D_t^{\alpha_1} u(t)\right), \\ & 0 < t < 1, \\ D_t^{\alpha_1} u(0) = 0, \qquad & D_t^{\gamma_1} u(1) = \sum_{j=1}^{m-2} b_j D_t^{\gamma_1} u\left(\zeta_j\right), \\ D_t^{\beta_1} v(0) = 0, \qquad & D_t^{\delta_1} v(1) = \sum_{j=1}^{m-2} a_j D_t^{\delta_1} v\left(\xi_j\right), \end{aligned} \tag{3}$$

where $1 < \alpha, \beta \le 2$, $\alpha - \gamma_1 \ge 1$, $\beta - \delta_1 \ge 1$, $0 \le \beta_1 \le \delta_1$, $0 \le \alpha_1 \le \gamma_1$, $a_j, b_j \in [0, +\infty)$, $\xi_i, \zeta_i \in (0,1)$ with $0 < \sum_{j=1}^{m-2} a_j \xi_j, \sum_{j=1}^{m-2} b_j \zeta_j < 1$, D_t^{α}, D_t^{β} are the standard Riemann-Liouville derivatives, and λ and μ are positive parameters. $f : (0,1) \times (0, +\infty) \to [0, +\infty)$ and $g : [0,1] \times [0, +\infty) \to (0, +\infty)$ are continuous and $f(t, x)$ may be singular at $t = 0, 1$ and $x = 0$. The system (3) is an abstract model arising from biological dynamic system, which was introduced by Perelson [5] to describe the primary infection with HIV in integer-order version, and was extended to a fractional order version of HIV-1 infection of $CD4^+$ T-cells by Arafa et al. [11].

The present paper has several interesting features. Firstly, the system depends on two parameters and the nonlinear terms f and g are allowed to have different nonlinear character; that is, f is decreasing on y and g is increasing on x; secondly, f may be singular at $y = 0$ and $t = 0, 1$; so far fewer work was done when f can be singular at $y = 0$; thirdly, the boundary conditions of the system are nonlocal and involve fractional derivatives of the unknown functions.

2. Preliminaries and Lemmas

In this section, we firstly define an appropriate invariant set and then make a change of variables for the system (3) so that Schauder's fixed point theorem can be applied. Our work is based on fractional framework; for further background knowledge of fractional calculus, we refer readers to the monographs [1–4] or the papers [6, 8, 17, 18] and the references therein.

Throughout this paper, we mean by $C[0,1]$ the Banach space of all continuous functions on $[0,1]$ with the usual norm $\|x\| = \max_{0 \le t \le 1} |x(t)|$. Let

$$P = \{x \in C[0,1] : x(t) \ge 0, t \in [0,1]\}; \tag{4}$$

then P is a normal cone in the Banach space E. Thus the space $C[0,1]$ can be equipped with a partial order given by

$$x, y \in C[0,1], x - y \in P \Longleftrightarrow x(t) \ge y(t), \quad \text{for } t \in [0,1]. \tag{5}$$

Now define a subcone of P as follows:

$$\begin{aligned} B = \{ & x(t) \in P : \text{there exist two} \\ & \text{positive numbers } L_x > 1 > l_x \\ & \text{such that } l_x t^{\alpha-\alpha_1-1} \le x(t) \le L_x t^{\alpha-\alpha_1-1},\ t \in [0,1] \}. \end{aligned} \tag{6}$$

Obviously, B is nonempty since $t^{\alpha-\alpha_1-1} \in B$.

Lemma 1. *Let $u(t) = I^{\alpha_1} x(t)$, $v(t) = I^{\beta_1} y(t)$, $x(t)$, and $y(t) \in C[0,1]$. Then system (3) is turned into the equivalent one:*

$$\begin{aligned} -D_t^{\alpha-\alpha_1} x(t) &= \lambda f\left(t, y(t)\right), \\ -D_t^{\beta-\beta_1} y(t) &= \mu g\left(t, x(t)\right), \\ & 0 < t < 1, \\ x(0) = 0, \qquad & D_t^{\gamma_1-\alpha_1} x(1) = \sum_{j=1}^{m-2} b_j D_t^{\gamma_1-\alpha_1} x\left(\zeta_j\right), \\ y(0) = 0, \qquad & D_t^{\delta_1-\beta_1} y(1) = \sum_{j=1}^{m-2} a_j D_t^{\delta_1-\beta_1} y\left(\xi_j\right), \end{aligned} \tag{7}$$

and if $(x, y) \in C[0,1] \times C[0,1]$ is a solution of the problem (7), then $(I^{\alpha_1} x(t), I^{\beta_1} y(t))$ is a solution of the system (3).

Proof. By using semigroup property of the fractional integration operator (see [1] page 73, Lemma 2.3 or [4] Sections 2.3 and 2.5), one has

$$\begin{aligned} D_t^{\alpha} u(t) &= \frac{d^n}{dt^n}\left(I^{n-\alpha} I^{\alpha_1} x(t)\right) \\ &= \frac{d^n}{dt^n} I^{n-\alpha+\alpha_1} x(t) = D_t^{\alpha-\alpha_1} x(t), \\ D_t^{\gamma_1} u(t) &= \frac{d^n}{dt^n}\left(I^{n-\gamma_1} I^{\alpha_1} x(t)\right) \\ &= \frac{d^n}{dt^n} I^{n-\gamma_1+\alpha_1} x(t) = D_t^{\gamma_1-\alpha_1} x(t), \\ D_t^{\alpha_1} u(t) &= D_t^{\alpha_1} I^{\alpha_1} x(t) = x(t). \end{aligned} \tag{8}$$

And then, it follows from (8) that

$$\begin{aligned} D_t^{\alpha_1} u(0) &= x(0) = 0, \\ D_t^{\gamma_1-\alpha_1} x(1) &= D_t^{\gamma_1} u(1) \\ &= \sum_{j=1}^{m-2} b_j D_t^{\gamma_1} u\left(\zeta_j\right) = \sum_{j=1}^{m-2} b_j D_t^{\gamma_1-\alpha_1} x\left(\zeta_j\right). \end{aligned} \tag{9}$$

In the same way, we also have

$$y(0) = 0, \qquad D_t^{\delta_1-\beta_1} y(1) = \sum_{j=1}^{m-2} a_j D_t^{\delta_1-\beta_1} y\left(\xi_j\right). \tag{10}$$

It follows from $u(t) = I^{\alpha_1}x(t)$ and $v(t) = I^{\beta_1}y(t)$ that $x(t) = D_t^{\alpha_1}u(t)$ and $y(t) = D_t^{\beta_1}v(t)$. So substituting the above formulas into (3), we obtain (7).

On the other hand, if (x, y) is a solution of (7), (8) yields

$$\begin{aligned}
-D_t^{\alpha}u(t) &= -\frac{d^n}{dt^n}I^{n-\alpha}I^{\alpha_1}x(t) = -\frac{d^n}{dt^n}I^{n-\alpha-\alpha_1}x(t)\\
&= -D_t^{\alpha-\alpha_1}x(t) = \lambda f\left(t, y(t)\right) = \lambda f\left(t, D_t^{\beta_1}v(t)\right),\\
-D_t^{\beta}v(t) &= -D_t^{\beta-\beta_1}y(t) = \mu g\left(t, x(t)\right) = \mu g\left(t, D_t^{\alpha_1}u(t)\right),\\
D_t^{\alpha_1}u(t) &= D_t^{\alpha_1}I^{\alpha_1}x(t) = x(t), \qquad D_t^{\gamma_1}u(t) = D_t^{\gamma_1-\alpha_1}x(t),\\
D_t^{\beta_1}v(t) &= D_t^{\beta_1}I^{\beta_1}y(t) = y(t), \qquad D_t^{\delta_1}v(t) = D_t^{\delta_1-\beta_1}y(t).
\end{aligned} \tag{11}$$

So from (11), we have

$$\begin{aligned}
&D_t^{\alpha_1}u(0) = 0, \qquad D_t^{\gamma_1}u(1) = \sum_{j=1}^{m-2}b_jD_t^{\gamma_1}u\left(\zeta_j\right),\\
&D_t^{\beta_1}v(0) = 0, \qquad D_t^{\delta_1}v(1)\sum_{j=1}^{m-2}a_jD_t^{\delta_1-\beta_1}v\left(\xi_j\right).
\end{aligned} \tag{12}$$

Moreover

$$I^{\alpha_1}x(t), \qquad I^{\beta_1}y(t) \in C[0,1]. \tag{13}$$

Consequently, $(I^{\alpha_1}x(t), I^{\beta_1}y(t))$ is a positive solution of (3). □

Now we recall some useful lemmas by [18], which are important to the proof of our main results.

Lemma 2 (see [18]). *Let $h \in L^1[0,1]$, if $1 < \alpha - \alpha_1 \le 2$; then the unique solution of the linear problem*

$$\begin{aligned}
&-D_t^{\alpha-\alpha_1}x(t) = h(t), \quad t \in (0,1),\\
&x(0) = 0, \qquad D_t^{\gamma_1-\alpha_1}x(1) = \sum_{j=1}^{m-2}b_jD_t^{\gamma_1-\alpha_1}x\left(\zeta_j\right),
\end{aligned} \tag{14}$$

is given by

$$x(t) = \int_0^1 K_{\alpha}(t,s)\,h(s)\,ds, \tag{15}$$

where

$$\begin{aligned}
&K_{\alpha}(t,s)\\
&= K_{\alpha1}(t,s) + \frac{t^{\alpha-\alpha_1-1}}{1-\sum_{j=1}^{m-2}b_j\zeta_j^{\alpha-\gamma_1-1}}\sum_{j=1}^{m-2}b_jK_{\alpha2}\left(\zeta_j,s\right),\\
&K_{\alpha1}(t,s)\\
&= \frac{1}{\Gamma(\alpha-\alpha_1)}\begin{cases}t^{\alpha-\alpha_1-1}(1-s)^{\alpha-\gamma_1-1} - (t-s)^{\alpha-\alpha_1-1}, & 0 \le s \le t \le 1,\\ t^{\alpha-\alpha_1-1}(1-s)^{\alpha-\gamma_1-1}, & 0 \le t \le s \le 1,\end{cases}\\
&K_{\alpha2}(t,s)\\
&= \frac{1}{\Gamma(\alpha-\alpha_1)}\begin{cases}(t(1-s))^{\alpha-\gamma_1-1} - (t-s)^{\alpha-\gamma_1-1}, & 0 \le s \le t \le 1,\\ (t(1-s))^{\alpha-\gamma_1-1}, & 0 \le t \le s \le 1.\end{cases}
\end{aligned} \tag{16}$$

Moreover, for any $t, s \in [0,1]$,

$$K_{\alpha i}(t,s) \le \frac{(1-s)^{\alpha-\gamma_1-1}}{\Gamma(\alpha-\alpha_1)}, \quad i = 1,2. \tag{17}$$

By Lemma 2, similar results are valid for the problem

$$\begin{aligned}
&-D_t^{\beta-\beta_1}y(t) = h(t), \quad t \in [0,1],\\
&y(0) = 0, \qquad D_t^{\delta_1-\beta_1}y(1) = \sum_{j=i}^{m-2}a_jD_t^{\delta_1-\beta_1}y\left(\xi_j\right).
\end{aligned} \tag{18}$$

For convenience, we adopt the following corresponding notations:

$$\begin{aligned}
&K_{\beta}(t,s)\\
&= K_{\beta1}(t,s) + \frac{t^{\beta-\beta_1-1}}{1-\sum_{j=1}^{m-2}a_j\xi_j^{\beta-\delta_1-1}}\sum_{j=1}^{m-2}a_jK_{\beta2}\left(\xi_j,s\right),\\
&K_{\beta1}(t,s)\\
&= \frac{1}{\Gamma(\beta-\beta_1)}\begin{cases}t^{\beta-\beta_1-1}(1-s)^{\beta-\delta_1-1} - (t-s)^{\beta-\beta_1-1}, & 0 \le s \le t \le 1,\\ t^{\beta-\beta_1-1}(1-s)^{\beta-\delta_1-1}, & 0 \le t \le s \le 1,\end{cases}\\
&K_{\beta2}(t,s)\\
&= \frac{1}{\Gamma(\beta-\beta_1)}\begin{cases}(t(1-s))^{\beta-\delta_1-1} - (t-s)^{\beta-\delta_1-1}, & 0 \le s \le t \le 1,\\ (t(1-s))^{\beta-\delta_1-1}, & 0 \le t \le s \le 1.\end{cases}
\end{aligned} \tag{19}$$

Lemma 3 (see [18]). *The Green functions $K_{\alpha}(t,s)$ and $K_{\beta}(t,s)$ have the following properties:*

(1) *$K_{\alpha}(t,s) > 0$ and $K_{\beta}(t,s) > 0$ for $t, s \in (0,1)$,*

(2) *there exist functions* $m_\alpha(s)$, $m_\beta(s)$, $M_\alpha(s)$, *and* $M_\beta(s)$ *such that*

$$m_\alpha(s)\,t^{\alpha-\alpha_1-1} \le K_\alpha(t,s) \le M_\alpha(s)\,t^{\alpha-\alpha_1-1}, \quad t,s\in(0,1),$$
$$m_\beta(s)\,t^{\beta-\beta_1-1} \le K_\beta(t,s) \le M_\beta(s)\,t^{\beta-\beta_1-1}, \quad t,s\in(0,1), \tag{20}$$

where

$$\begin{aligned} m_\alpha(s) &= \frac{\sum_{j=1}^{m-2} b_j K_{\alpha 2}\left(\zeta_j,s\right)}{1-\sum_{j=1}^{m-2} b_j \zeta_j^{\alpha-\gamma_1-1}},\\ M_\alpha(s) &= \frac{1}{\Gamma(\alpha-\alpha_1)} + \frac{\sum_{j=1}^{m-2} b_j K_{\alpha 2}\left(\zeta_j,s\right)}{1-\sum_{j=1}^{m-2} b_j \zeta_j^{\alpha-\gamma_1-1}},\\ m_\beta(s) &= \frac{\sum_{j=1}^{m-2} a_j K_{\beta 2}\left(\xi_j,s\right)}{1-\sum_{j=1}^{m-2} a_j \xi_j^{\beta-\delta_1-1}},\\ M_\beta(s) &= \frac{1}{\Gamma(\beta-\beta_1)} + \frac{\sum_{j=1}^{m-2} a_j K_{\beta 2}\left(\xi_j,s\right)}{1-\sum_{j=1}^{m-2} a_j \xi_j^{\beta-\delta_1-1}}. \end{aligned} \tag{21}$$

Clearly, the following maximum principle is direct conclusion of Lemma 2.

Lemma 4. *If* $1<\alpha-\alpha_1\le 2$ *and* $x\in C([0,1],R)$ *satisfies*

$$x(0)=0, \qquad D_t^{\gamma_1-\alpha_1}x(1)=\sum_{j=1}^{m-2} b_j D_t^{\gamma_1-\alpha_1}x\left(\zeta_j\right), \tag{22}$$

and $-D_t^{\alpha-\alpha_1}x(t)\ge 0$ *for any* $t\in(0,1)$, *then* $x(t)\ge 0$ *for* $t\in(0,1)$.

It is well known that $(x,y)\in C(0,1)\times C[0,1]$ is a solution of the system (7) if and only if (x,y) is a solution of the nonlinear integral system of equations

$$\begin{aligned} x(t) &= \lambda\int_0^1 K_\alpha(t,s)\,f\left(s,y(s)\right)ds,\\ y(t) &= \mu\int_0^1 K_\beta(t,s)\,g\left(s,x(s)\right)ds, \end{aligned} \tag{23}$$

and the system (23) is equivalent to the following nonlinear integral equation

$$x(t)=\lambda\int_0^1 K_\alpha(t,s)\,f\left(s,\mu\int_0^1 K_\beta(s,\tau)\,g\left(\tau,x(\tau)\right)d\tau\right)ds, \quad t\in[0,1]. \tag{24}$$

Let us define a nonlinear operator $(Fx)(t)$ by

$$(Fx)(t) = \lambda\int_0^1 K_\alpha(t,s)\,f\left(s,\mu\int_0^1 K_\beta(s,\tau)\,g\left(\tau,x(\tau)\right)d\tau\right)ds, \quad t\in[0,1]. \tag{25}$$

Then the existence of solutions to the system of (7) is equivalent to the existence of fixed point of the nonlinear operator F; that is, if $x^*(t)$ is a fixed point of F in $C[0,1]$, then system (7) has at least one solution $(x^*(t), y^*(t))$ which can be written by

$$\begin{aligned} x(t) &= x^*(t),\\ y^*(t) &= \mu\int_0^1 K_\beta(t,s)\,g\left(s,x^*(s)\right)ds, \end{aligned} \tag{26}$$

and then system (3) has at least one solution:

$$\left(I^{\alpha_1}x^*(t), I^{\beta_1}y^*(t)\right). \tag{27}$$

In order to find the fixed point of F, we need the definitions of the upper solution and lower solution for the following integrodifferential equation:

$$\begin{aligned} -D_t^{\alpha-\alpha_1}x(t) &= \lambda f\left(t,\mu\int_0^1 K_\beta(t,s)\,g\left(s,x(s)\right)ds\right),\\ x(0)=0, &\qquad D_t^{\gamma_1-\alpha_1}x(1)=\sum_{j=1}^{m-2} b_j D_t^{\gamma_1-\alpha_1}x\left(\zeta_j\right). \end{aligned} \tag{28}$$

Definition 5. A continuous function $\phi(t)$ is called a lower solution of the problem (28), if it satisfies

$$\begin{aligned} -D_t^{\alpha-\alpha_1}\phi(t) &\le \lambda f\left(t,\mu\int_0^1 K_\beta(t,s)\,g\left(s,\phi(s)\right)ds\right),\\ \phi(0)\ge 0, &\qquad D_t^{\gamma_1-\alpha_1}\phi(1)\ge\sum_{j=1}^{m-2} b_j D_t^{\gamma_1-\alpha_1}\phi\left(\zeta_j\right). \end{aligned} \tag{29}$$

Definition 6. A continuous function $\varphi(t)$ is called an upper solution of the problem (28), if it satisfies

$$\begin{aligned} -D_t^{\alpha-\alpha_1}\varphi(t) &\ge \lambda f\left(t,\mu\int_0^1 K_\beta(t,s)\,g\left(s,\varphi(s)\right)ds\right),\\ \varphi(0)\le 0, &\qquad D_t^{\gamma_1-\alpha_1}\varphi(1)\le\sum_{j=1}^{m-2} b_j D_t^{\gamma_1-\alpha_1}\varphi\left(\zeta_j\right). \end{aligned} \tag{30}$$

3. Main Result

For the convenience in presentation, we now present some assumptions to be used in the rest of the paper.

(A1) $f:(0,1)\times(0,+\infty)\to[0,+\infty)$ is continuous and decreasing on y in $(0,+\infty)$ and $g:[0,1]\times[0,+\infty)\to(0,+\infty)$ is continuous and increasing on x in $[0,+\infty)$.

(A2) For any real numbers l, $\mu>0$,

$$\int_0^1 M_\alpha(s)\,f\left(s,\mu\int_0^1 K_\beta(s,\tau)\,g\left(\tau,l\tau^{\alpha-\alpha_1-1}\right)d\tau\right)ds<+\infty. \tag{31}$$

Theorem 7. *Suppose (A1) and (A2) hold; then for any* $(\lambda,\mu)\in(0,+\infty)\times(0,+\infty)$, *the system* (3) *has at least one positive*

solution (u^*, v^*)*, and there exist positive constants* r_1^*, r_2^*, d_1^*, *and* d_2^* *such that*

$$\begin{aligned} r_1^* t^{\alpha-1} \le u^*(t) \le r_2^* t^{\alpha-1}, \\ d_1^* t^{\beta-1} \le v^*(t) \le d_2^* t^{\beta-1}, \\ t \in [0,1]. \end{aligned} \tag{32}$$

Proof. We start by showing that (3) has at least one positive solution (u^*, v^*). For this purpose, we firstly prove that the operator F is well defined and $F(B) \subset B$.

For any $x \in B$, there exist two positive numbers $l_x < 1 < L_x$ such that $l_x t^{\alpha-\alpha_1-1} \le x(t) \le L_x t^{\alpha-\alpha_1-1}$, so it follows from (25), (20), and (31) that

$$\begin{aligned} &(Fx)(t) \\ &= \lambda \int_0^1 K_\alpha(t,s) \\ &\qquad \times f\left(s, \mu \int_0^1 K_\beta(s,\tau)\, g(\tau, x(\tau))\, d\tau\right) ds \\ &\le \lambda t^{\alpha-\alpha_1-1} \\ &\quad \times \int_0^1 M_\alpha(s) \\ &\qquad \times f\left(s, \mu \int_0^1 K_\beta(s,\tau)\, g\left(\tau, l_x \tau^{\alpha-\alpha_1-1}\right) d\tau\right) ds \\ &< +\infty, \\ &(Fx)(t) \\ &\ge \lambda t^{\alpha-\alpha_1-1} \int_0^1 m_\alpha(s) f\left(s, \mu \int_0^1 K_\beta(s,\tau)\, g(\tau, x(\tau))\, d\tau\right) ds \\ &\ge \lambda t^{\alpha-\alpha_1-1} \\ &\quad \times \int_0^1 m_\alpha(s) f\left(s, \mu \int_0^1 K_\beta(s,\tau)\, g\left(\tau, L_x \tau^{\alpha-\alpha_1-1}\right) d\tau\right) ds. \end{aligned} \tag{33}$$

Take

$$\begin{aligned} &l_x' \\ &= \min\left\{1, \lambda \right. \\ &\qquad \times \int_0^1 m_\alpha(s) \\ &\qquad \left. \times f\left(s, \mu \int_0^1 K_\beta(s,\tau)\, g\left(\tau, L_x \tau^{\alpha-\alpha_1-1}\right) d\tau\right) ds\right\}, \end{aligned}$$

$$\begin{aligned} &L_x' \\ &= \max\left\{1, \lambda \right. \\ &\qquad \times \int_0^1 M_\alpha(s) \\ &\qquad \left. \times f\left(s, \mu \int_0^1 K_\beta(s,\tau)\, g\left(\tau, l_x \tau^{\alpha-\alpha_1-1}\right) d\tau\right) ds\right\}; \end{aligned} \tag{34}$$

then

$$l_x' t^{\alpha-\alpha_1-1} \le (Fx)(t) \le L_x' t^{\alpha-\alpha_1-1}. \tag{35}$$

So $(Fx)(t)$ is well defined and $F(B) \subseteq B$. It follows from (A1) that the operator $(Fx)(t)$ is decreasing on x. Moreover, by Lemma 2, we have

$$\begin{aligned} &-D_t^{\alpha-\alpha_1}(Fx)(t) = \lambda f\left(t, \mu \int_0^1 K_\beta(t,s)\, g(s, Fx(s))\, ds\right), \\ &(Fx)(0) = 0, \qquad D_t^{\gamma_1-\alpha_1}(Fx)(1) = \sum_{j=1}^{m-2} b_j D_t^{\gamma_1-\alpha_1} Fx\left(\zeta_j\right). \end{aligned} \tag{36}$$

Let

$$\begin{aligned} \underline{e}(t) &= \min\left\{t^{\alpha-\alpha_1-1}, F\left(t^{\alpha-\alpha_1-1}\right)\right\}, \\ \overline{e}(t) &= \max\left\{t^{\alpha-\alpha_1-1}, F\left(t^{\alpha-\alpha_1-1}\right)\right\}; \end{aligned} \tag{37}$$

if $t^{\alpha-\alpha_1-1} = F(t^{\alpha-\alpha_1-1})$, then

$$x^*(t) = t^{\alpha-\alpha_1-1}, \qquad y^*(t) = \mu \int_0^1 K_\beta(t,s)\, g\left(s, s^{\alpha-\alpha_1-1}\right) ds, \tag{38}$$

is positive solution of (7). Thus the system (3) has at least one solution

$$\left(I^{\alpha_1} x^*(t), I^{\beta_1} y^*(t)\right) \tag{39}$$

which satisfies Theorem 7. If $t^{\alpha-\alpha_1-1} \ne F(t^{\alpha-\alpha_1-1})$, we have

$$\underline{e}(t), \overline{e}(t) \in B, \quad \underline{e}(t) \le t^{\alpha-\alpha_1-1} \le \overline{e}(t). \tag{40}$$

Let

$$\varphi(t) = \left(F\underline{e}\right)(t), \qquad \phi(t) = \left(F\overline{e}\right)(t). \tag{41}$$

From (A1), we know that F is nonincreasing on x; thus by (37) and (40), one gets

$$\begin{aligned} &\phi(t) = \left(F\overline{e}\right)(t) \le F\underline{e}(t) = \varphi(t), \\ &\left(F\overline{e}\right)(t) \le F\left(t^{\alpha-\alpha_1-1}\right) \le \overline{e}(t), \\ &\left(F\underline{e}\right)(t) \ge F\left(t^{\alpha-\alpha_1-1}\right) \ge \underline{e}(t), \end{aligned} \tag{42}$$

and $\phi(t)$ and $\varphi(t) \in B$. Thus from (36)–(42), we have

$$\begin{aligned}
&D_t^{\alpha-\alpha_1}\varphi(t) + \lambda f\left(t, \mu\int_0^1 K_\beta(t,s)\, g\left(s, \varphi(s)\right) ds\right)\\
&\quad = D_t^{\alpha-\alpha_1}\left(F\underline{e}\right)(t) + \lambda f\left(t, \mu\int_0^1 K_\beta(t,s)\, g\left(s, \left(F\underline{e}\right)(s)\right) ds\right)\\
&\quad \le D_t^{\alpha-\alpha_1}\left(F\underline{e}\right)(t) + \lambda f\left(t, \mu\int_0^1 K_\beta(t,s)\, g\left(s, \underline{e}(s)\right) ds\right) = 0,\\
&\qquad \varphi(0) = 0, \qquad D_t^{\gamma_1-\alpha_1}\varphi(1) = \sum_{j=1}^{m-2} b_j D_t^{\gamma_1-\alpha_1}\varphi\left(\zeta_j\right),\\
&D_t^{\alpha-\alpha_1}\phi(t) + \lambda f\left(t, \mu\int_0^1 K_\beta(t,s)\, g\left(s, \phi(s)\right) ds\right)\\
&\quad = D_t^{\alpha-\alpha_1}\left(F\overline{e}\right)(t) + \lambda f\left(t, \mu\int_0^1 K_\beta(t,s)\, g\left(s, \left(F\overline{e}\right)(s)\right) ds\right)\\
&\quad \ge D_t^{\alpha-\alpha_1}\left(F\overline{e}\right)(t) + \lambda f\left(t, \mu\int_0^1 K_\beta(t,s)\, g\left(s, \overline{e}(s)\right) ds\right) = 0,\\
&\qquad \phi(0) = 0, \qquad D_t^{\gamma_1-\alpha_1}\phi(1) = \sum_{j=1}^{m-2} b_j D_t^{\gamma_1-\alpha_1}\phi\left(\zeta_j\right).
\end{aligned} \tag{43}$$

So it follows from (42)-(43) that $\varphi(t)$ and $\phi(t)$ are, respectively, upper solution and lower solution of the problem (28) and $\varphi(t), \phi(t) \in B$.

Define a function $W : (0,1) \times E \to E$ as follows:

$$W(t, x(t)) = \begin{cases} f\left(t, \mu\int_0^1 K_\beta(t,s)\, g\left(s, \phi(s)\right) ds\right), & x(t) < \phi(t),\\ f\left(t, \mu\int_0^1 K_\beta(t,s)\, g\left(s, x(s)\right) ds\right), & \phi(t) \le x(t) \le \varphi(t),\\ f\left(t, \mu\int_0^1 K_\beta(t,s)\, g\left(s, \varphi(s)\right) ds\right), & x(t) > \varphi(t). \end{cases} \tag{44}$$

Obviously $W \in C((0,1) \times [0, +\infty))$.

Next we define an operator T in $C[0,1]$ by

$$(Tx)(t) = \lambda\int_0^1 K_\alpha(t,s)\, W(s, x(s))\, ds \tag{45}$$

and consider the following boundary value problem:

$$\begin{aligned}
&-D_t^{\alpha-\alpha_1}x(t) = \lambda W(t, x(t)),\\
&x(0) = 0, \qquad D_t^{\gamma_1-\alpha_1}x(1) = \sum_{j=1}^{m-2} b_j D_t^{\gamma_1-\alpha_1}x\left(\zeta_j\right).
\end{aligned} \tag{46}$$

It is easy to see that the fixed point of T is the solution of (46).

Note that

$$\begin{aligned}
(Tx)(t) &\le \lambda t^{\alpha-\alpha_1-1}\int_0^1 M_\alpha(s)\, W(s, x(s))\, ds\\
&\le \lambda t^{\alpha-\alpha_1-1}\int_0^1 M_\alpha(s)\\
&\qquad \times f\left(s, \mu\int_0^1 K_\beta(s,\tau)\, g\left(\tau, \phi(\tau)\right) d\tau\right) ds\\
&\le \lambda t^{\alpha-\alpha_1-1}\int_0^1 M_\alpha(s)\\
&\qquad \times f\left(s, \mu\int_0^1 K_\beta(s,\tau)\, g\left(\tau, l_\phi \tau^{\alpha-\alpha_1-1}\right) d\tau\right) ds\\
&< +\infty;
\end{aligned} \tag{47}$$

then T is uniformly bounded. From the uniform continuity of $K_\alpha(t,s)$ and Lebesgue dominated convergence theorem, we get T is equicontinuous. So T is completely continuous. It follows from Schauder's fixed point theorem that T has at least one fixed point $x^*(t)$ such that $x^*(t) = (Tx^*)(t)$.

In what follows, we prove

$$\phi(t) \le x^*(t) \le \varphi(t), \quad t \in [0,1]. \tag{48}$$

In fact, as $x^*(t)$ is a fixed point of T, we have

$$x^*(0) = 0, \qquad D_t^{\gamma_1-\alpha_1}x^*(t) = \sum_{j=1}^{m-2} b_j D_t^{\gamma_1-\alpha_1}x^*\left(\zeta_j\right). \tag{49}$$

We firstly prove $x^*(t) \le \varphi(t)$. Otherwise, suppose $x^*(t) > \varphi(t)$. According to the definition of W, we have

$$\begin{aligned}
-D_t^{\alpha-\alpha_1}x^*(t) &= \lambda W\left(t, x^*(t)\right)\\
&= \lambda f\left(t, \mu\int_0^1 K_\beta(t,s)\, g\left(s, \varphi(s)\right) ds\right).
\end{aligned} \tag{50}$$

On the other hand, as φ is an upper solution of (28), we have

$$-D_t^{\alpha-\alpha_1}\varphi(t) \ge \lambda f\left(t, \mu\int_0^1 K_\beta(t,s)\, g\left(s, \varphi(s)\right) ds\right). \tag{51}$$

Letting $z(t) = \varphi(t) - x^*(t)$, (50)-(51) imply that

$$\begin{aligned}
&D_t^{\alpha-\alpha_1}z(t) = D_t^{\alpha-\alpha_1}\varphi(t) - D_t^{\alpha-\alpha_1}x^*(t) \le 0,\\
&z(0) = 0, \qquad D_t^{\gamma_1-\alpha_1}z(t) = \sum_{j=1}^{m-2} b_j D_t^{\gamma_1-\alpha_1}z\left(\zeta_j\right).
\end{aligned} \tag{52}$$

It follows from Lemma 4 that

$$z(t) \ge 0; \tag{53}$$

that is, $x^*(t) \le \varphi(t)$ on $[0,1]$, which contradicts $x^*(t) > \varphi(t)$. Thus $\varphi(t) \ge x^*(t)$, $t \in (0,1)$. In the same way, $x^*(t) \ge \phi(t)$. Consequently,

$$\phi(t) \le x^*(t) \le \varphi(t), \tag{54}$$

and then $x^*(t)$ is a positive solution of the problem (28).

It follows from $\phi(t), \varphi(t) \in B$ that there exist four numbers $0 < l_\phi, l_\varphi < 1, L_\phi, L_\varphi > 1$ such that

$$\begin{aligned} l_\phi t^{\alpha-\alpha_1-1} &\le \phi(t) \le L_\phi t^{\alpha-\alpha_1-1}, \\ l_\varphi t^{\alpha-\alpha_1-1} &\le \varphi(t) \le L_\varphi t^{\alpha-\alpha_1-1}, \end{aligned} \tag{55}$$

and (54)-(55) yield

$$r_1 t^{\alpha-\alpha_1-1} = l_\phi t^{\alpha-\alpha_1-1} \le x^*(t) \le L_\varphi t^{\alpha-\alpha_1-1} = r_2 t^{\alpha-\alpha_1-1}. \tag{56}$$

Furthermore,

$$\begin{aligned} &d_1 t^{\beta-\beta_1-1} \\ &\quad \le \mu t^{\beta-\beta_1-1} \int_0^1 m_\beta(s)\, g\left(s, r_1 s^{\alpha-\alpha_1-1}(s)\right) ds \le y^*(t) \\ &\quad \le \mu t^{\beta-\beta_1-1} \int_0^1 M_\beta(s)\, g\left(s, r_2 s^{\alpha-\alpha_1-1}(s)\right) ds \le d_2 t^{\beta-\beta_1-1}, \end{aligned} \tag{57}$$

where

$$\begin{aligned} d_1 &= \min\left\{\frac{1}{2}, \mu \int_0^1 m_\beta(s)\, g\left(s, r_1 s^{\alpha-\alpha_1-1}(s)\right) ds\right\}, \\ d_2 &= \max\left\{2, \mu \int_0^1 M_\beta(s)\, g\left(s, r_2 s^{\alpha-\alpha_1-1}(s)\right) ds\right\}. \end{aligned} \tag{58}$$

By Lemma 1, (56), and (57), we obtain that the problem (3) has a positive solution $(u^*(t), v^*(t)) = (I^{\alpha_1} x^*(t), I^{\beta_1} y^*(t))$, which satisfies

$$\begin{aligned} r_1^* t^{\alpha-1} &= \frac{\Gamma(\alpha-\alpha_1) r_1}{\Gamma(\alpha)} t^{\alpha-1} \le u^*(t) \\ &= \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} x^*(s)\, ds \\ &\le \frac{\Gamma(\alpha-\alpha_1) r_2}{\Gamma(\alpha)} t^{\alpha-1} = r_2^* t^{\alpha-1}, \\ d_1^* t^{\beta-1} &= \frac{\Gamma(\beta-\beta_1) d_1}{\Gamma(\beta)} t^{\beta-1} \le v^*(t) \\ &= \frac{1}{\Gamma(\beta_1)} \int_0^t (t-s)^{\beta_1-1} y^*(s)\, ds \\ &\le \frac{\Gamma(\beta-\beta_1) d_2}{\Gamma(\beta)} t^{\beta-1} = d_2^* t^{\beta-1}. \end{aligned} \tag{59}$$

The proof is completed. □

Example 8. Consider the following singular fractional order differential system

$$\begin{aligned} -D_t^{7/4} u(t) &= \lambda\left[\sin t + \left(D_t^{1/3} v(t)\right)^{-1/3}\right], \\ -D_t^{5/3} v(t) &= \mu\left[t^{-1/2} + \left(D_t^{1/2} u(t)\right)^2\right], \\ D_t^{1/2} u(0) = 0, &\qquad D_t^{2/3} u(1) = \sum_{j=1}^{10} b_j D_t^{2/3} u\left(\zeta_j\right), \\ D_t^{1/3} v(0) = 0, &\qquad D_t^{3/5} v(1) = \sum_{j=1}^{10} a_j D_t^{3/5} v\left(\xi_j\right), \end{aligned} \tag{60}$$

where $\lambda > 0$ and $\mu > 0$ are parameters, $\xi_j = j/12$, $a_j = 1/15 j^{1/15}$, $\zeta_j = 1/(j+1)$, and $b_j = (1/2)^j$, $j = 1, 2, \ldots, 10$. Obviously,

$$\begin{aligned} 0 < \sum_{j=1}^{10} a_j \xi_j^{1/15} &= \sum_{j=1}^{10} \frac{1}{15 j^{1/15}} \left(\frac{j}{12}\right)^{1/15} = \frac{1}{15} \sum_{j=1}^{10} \frac{1}{12^{1/15}} < 1, \\ 0 < \sum_{j=1}^{10} b_j \zeta_j^{1/12} &= \sum_{j=1}^{10} \left(\frac{1}{2}\right)^j \left(\frac{1}{j+1}\right)^{1/12} \le \left(\frac{1}{2}\right)^{1/13} < 1. \end{aligned} \tag{61}$$

Letting $u(t) = I^{1/2} x(t)$, $v(t) = I^{1/3} y(t)$, then (60) is turned into

$$\begin{aligned} -D_t^{5/4} x(t) &= \lambda\left(\sin t + y^{-1/3}(t)\right), \\ -D_t^{4/3} y(t) &= \mu\left(t^{-1/2} + x^2(t)\right), \\ x(0) = 0, &\qquad D_t^{1/3} x(1) = \sum_{j=1}^{10} b_j D_t^{1/3} x\left(\zeta_j\right), \\ y(0) = 0, &\qquad D_t^{1/10} y(1) = \sum_{j=1}^{10} a_j D_t^{1/10} y\left(\xi_j\right), \end{aligned} \tag{62}$$

where $f(t, y) = \sin t + y^{-1/3}$ and $g(t, x) = t^{-1/2} + x^2$; it is easy to see that $f(t, y)$ is decreasing with respect to y and $g(t, x)$ is increasing with respect to x, and then (A1) is satisfied.

It follows from (20) and (21) that there exists a positive number $m_0 < +\infty$ such that

$$K_\beta(t, s) \ge m_\beta(s)\, t^{1/3} \ge m_0 t^{1/3}, \tag{63}$$

and for any constants $\mu, l > 0$, we have

$$\begin{aligned} &f\left(t, \mu \int_0^1 K_\beta(t, s)\, g\left(s, l s^{\alpha-\alpha_1-1}\right) ds\right) \\ &\quad = \sin t + \left(\mu \int_0^1 K_\beta(t, s)\, g\left(s, l s^{\alpha-\alpha_1-1}\right) ds\right)^{-1/3} \\ &\quad = \sin t + \left(\mu \int_0^1 K_\beta(t, s)\left(s^{-1/2} + l^2 s^2\right) ds\right)^{-1/3} \end{aligned}$$

$$\begin{aligned}
&\le \sin t + (\mu m_0)^{-1/3}\left(\int_0^1 s^{-1/2} + l^2 s^2 ds\right)^{-1/3} \\
&\le \sin 1 + (\mu m_0)^{-1/3}\left(\frac{1}{2} + \frac{1}{3}l^2\right)^{-1/3} \\
&< \sin 1 + \left(\frac{1}{2}\mu m_0\right)^{-1/3}, \\
M_\alpha(s) &= \frac{1}{\Gamma(5/4)} + \frac{\sum_{j=1}^{10} b_j K_{\alpha 2}\left(\zeta_j, s\right)}{1-c_2} \le \frac{1}{\Gamma(5/4)} \\
&\quad + \frac{\left((1-s)^{1/12}/\Gamma(5/4)\right)\sum_{j=1}^{10}(j+1)(1/2)^j}{1-c_2} \\
&\le \frac{1}{\Gamma(5/4)}\left(1 + \frac{2^{10}-9}{2^{11}(1-c_2)}\right),
\end{aligned} \tag{64}$$

where $c_2 = \sum_{j=1}^{10} b_j \zeta_j^{1/12}$. So

$$\begin{aligned}
&\int_0^1 M_\alpha(s) f\left(s, \mu\int_0^1 K_\beta(s,\tau) g\left(\tau, l\tau^{\alpha-\alpha_1-1}\right) d\tau\right) ds \\
&\le \frac{1}{\Gamma(5/4)}\int_0^1 \left(\sin 1 + \left(\frac{1}{2}\mu m_0\right)^{-1/3}\right)\left(1 + \frac{2^{10}-9}{2^{11}(1-c_2)}\right) ds \\
&< +\infty,
\end{aligned} \tag{65}$$

and thus (A2) is satisfied. So from Theorem 7, for any $(\lambda, \mu) \in (0, +\infty) \times (0, +\infty)$, the system (60) has at least one positive solution (u^*, v^*), and there exist positive constants r_1^*, r_2^*, d_1^*, and d_2^* such that

$$\begin{aligned}
&r_1^* t^{3/4} \le u^*(t) \le r_2^* t^{3/4}, \qquad d_1^* t^{2/3} \le v^*(t) \le d_2^* t^{2/3}, \\
&t \in [0,1].
\end{aligned} \tag{66}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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