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Research Article

Banach-Saks Type and Gurariǐ Modulus of Convexity of Some Banach Sequence Spaces

Henryk Hudzik,[1] Vatan Karakaya,[2] Mohammad Mursaleen,[3] and Necip Simsek[4]

[1] *Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland*
[2] *Department of Mathematical Engineering, Yıldız Technical University, Davutpasa Campus, Esenler, 34750 Istanbul, Turkey*
[3] *Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India*
[4] *Department of Mathematics, Istanbul Ticaret University, Üsküdar, Istanbul, Turkey*

Correspondence should be addressed to Henryk Hudzik; hudzik@amu.edu.pl

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Banach-Saks type is calculated for two types of Banach sequence spaces and Gurariǐ modulus of convexity is estimated from above for the spaces of one type among them.

1. Introduction

Recently, there has been a lot of interest in investigating geometric properties of sequence spaces besides topological and some other usual properties. In the literature, there are many papers concerning geometric properties of various Banach sequence spaces. For example, geometry of Orlicz spaces and of Musielak-Orlicz spaces has been studied in [1–25]. Several authors including Cui and Hudzik [26–29], Cui and Meng [30], Suantai [31], and Lee [32] investigated the geometric properties of Cesàro sequence space ces(p). Also Cesàro-Orlicz sequence spaces equipped with Luxemburg norm have been studied in [5, 33–36]. Additionally, geometry of Orlicz-Lorentz sequence spaces and of generalized Orlicz-Lorentz sequence space were studied in [37, 38]. Furthermore, Mursaleen et al. [39] studied some geometric properties of normed Euler sequence space. Additionally, Hudzik and Narloch [40] have studied relationships between monotonicity and complex rotundity properties with some consequences. Besides, some geometrical properties of Calderon-Lozanovskii sequence spaces have been investigated in [41–43].

Quite recently, Karakaya [20] defined a new sequence space involving lacunary sequence space equipped with the Luxemburg norm and studied Kadec-Klee (H) and rotundity (R) properties of these spaces. Further information on topological and geometric properties of sequence spaces can be found in [39, 44–62].

Let X be a real Banach space and $S(X)$ and $B(X)$ be the unit sphere and the unit ball of X, respectively. Let ℓ^0, c_0, c, ℓ_∞, and ℓ_1 be the spaces of all real sequences, null, convergent, and bounded sequences and absolutely convergent series, respectively, and let c_{00} be the space of those real sequences which have only a finite number of nonzero coordinates and $\ell_p = \{x = x(i) : \sum_{i=1}^{\infty} |x(i)|^p < \infty\}$.

Note that c_0, c, and ℓ_∞ are Banach spaces with the sup-norm $\|x\|_\infty = \sup_i |x(i)|$ and ℓ_p $(1 \le p < \infty)$ are Banach spaces with the norm $\|x\|_p = (\sum |x(i)|^p)^{1/p}$, while c_{00} is not a Banach space with respect to any norm.

Let us recall that a sequence $\{v(i)\}_{i=1}^{\infty}$ in a Banach space X is called *Schauder basis* of X (or *basis* for short) if for each $x \in X$ there exists a unique sequence $\{\lambda(i)\}_{i=1}^{\infty}$ of scalars such that $x = \sum_{i=1}^{\infty} \lambda(i)v(i)$, that is, $\lim_{n\to\infty} \sum_{i=1}^{n} \lambda(i)v(i) = x$.

A sequence space X with a linear topology is called a *K-space* if each of the projection maps $P_j : X \to \mathbb{C}$ defined by $P_j(x) = x(i)$ for $x = (x(i))_{i=1}^{\infty} \in X$ is continuous for each natural j. A *Fréchet space* is a complete metric linear space and the metric is generated by an F-norm and a Fréchet space which is a K-space is called an *FK-space*; that is, a K-space X

is called an FK-space if X is a complete linear metric space. In other words, X is an FK-space if X is a Fréchet space with continuous coordinate projections. All the sequence spaces mentioned above are FK spaces except the space c_{00}.

An FK-space X which contains the space c_{00} is said to have the *property AK* if, for every sequence $\{x(i)\} \in X$, $x = \sum_{i=1}^{\infty} x(i)e(i)$, where $e(i) = (0, 0, \dots, 1(i\text{th-place}), 0, 0, \dots)$. The spaces c_0 equipped with the sup-norm, ℓ_p $(1 \leq p < \infty)$ equipped with the norm $\|x\| = (\sum_{i=1}^{\infty} |x(i)|^p)^{1/p}$ and ℓ^0 endowed with the metric $d(x, y) = \sum_{i=1}^{\infty} 2^{-i}(|x(i) - y(i)|/(1 + |x(i) - y(i)|))$ have property AK, while c and ℓ_∞ do not have property AK. Banach spaces with continuous coordinate projections are called BK-spaces.

A Banach space X is said to be a *Köthe sequence space* (see [28, 63]) if X is a subspace of ℓ^0 such that

(i) if $x \in \ell^0$, $y \in X$, and $|x(i)| \leq |y(i)|$, for all $i \in \mathbb{N}$, then $x \in X$ and $\|x\| \leq \|y\|$;

(ii) there exists an element $x \in X$ such that $x(i) > 0$ for all $i \in \mathbb{N}$.

We say that $x \in X$ is order continuous if, for any sequence (x_n) in X_+ (the positive cone in X) such that $x_n(i) \leq |x(i)|$ and $x_n(i) \to 0$ $(n \to \infty)$ for each $i \in \mathbb{N}(n \to \infty)$, we have $\|x_n\| \to 0$ $(n \to \infty)$ holds.

A Köthe sequence space X is said to be order continuous that if all sequences in X are order continuous. It is easy to see that $x \in X$ is order continuous if and only if $\|(0, 0, \dots, 0, x(n+1), x(n+2), \dots)\| \to 0$ as $n \to \infty$.

A Köthe sequence space X is said to have the *Fatou property* if, for any real sequence $x \in \ell^0$ and any $\{x_n\}$ in X such that $x_n \uparrow x$ coordinatewisely and $\sup_n \|x_n\| < \infty$, we have the fact that $x \in X$ and $\|x_n\| \to \|x\|$.

A Banach space X is said to have the *Banach-Saks property* if every bounded sequence $\{x_n\}$ in X admits a subsequence $\{z_n\}$ such that the sequence $\{t_k(z)\}$ is convergent in X with respect to the norm, where

$$t_k(z) = \frac{1}{k}(z_1 + z_2 + \cdots + z_k), \quad \forall k \in \mathbb{N}. \tag{1}$$

A Banach space X is said to have the *weak Banach-Saks property* whenever given any weakly null sequence $\{x_n\}$ in X there exists its subsequence $\{z_n\}$ such that the sequence $\{t_k(z)\}$ converges to zero strongly.

Given any $p \in (1, \infty)$, we say that a Banach space $(X, \|\cdot\|)$ has the *Banach-Saks property of type p* if there exists a constant $c > 0$ such that every weakly null sequence $\{x_k\}$ has a subsequence $\{x_{k_\ell}\}$ such that (see [22])

$$\left\| \sum_{\ell=1}^{n} x_{k_\ell} \right\| \leq c n^{1/p} \quad (\forall n \in \mathbb{N}). \tag{2}$$

The Banach-Saks property of type $p \in (1, \infty)$ and the weak Banach-Saks property for Cesàro sequence spaces have been considered in [28]. These properties and stronger property (S_p) for Musielak-Orlicz and Nakano sequence spaces have been studied in [17].

We say that a Banach space X has the *weak fixed point property* if every nonexpansive self-mapping defined on a nonempty weakly compact convex subset A of X has a fixed point in A.

In [64], Garcia-Falset introduced the following coefficient for a Banach space $(X, \|\cdot\|)$:

$$R(X) = \sup\left\{\liminf_{n\to\infty} \|x_n - x\| : x \in B(X), \{x_n\} \subset B(X),\ x_n \longrightarrow 0 \text{ weakly}\right\} \tag{3}$$

and he proved (see [64, 65]) that a Banach space X with $R(X) < 2$ has the weak fixed point property.

Clarkson *modulus of convexity* of a normed space $(X, \|\cdot\|)$ is defined (see Clarkson [66] and Day [67]) by the formula

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x + y\|}{2}; x, y \in S(X), \|x - y\| = \varepsilon\right\} \tag{4}$$

for any $\varepsilon \in [0, 2]$. The inequality $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$ characterizes the uniform convexity of X and the equality $\delta_X(2) = 1$ characterizes strict convexity (=rotundity) of X.

The Gurariĭ modulus of convexity of a normed space X is defined (see [68, 69]) by

$$\beta_X(\varepsilon) = \inf\left\{1 - \inf_{\alpha\in[0,1]} \|\alpha x + (1 - \alpha) y\|; x, y \in S(X), \|x - y\| = \varepsilon\right\} \tag{5}$$

for any $\varepsilon \in [0, 2]$. It is obvious that $\delta_X(\varepsilon) \leq \beta_X(\varepsilon)$ for any Banach space X and any $\varepsilon \in [0, 2]$. It is also known that $\beta_X(\varepsilon) \leq 2\delta_X(\varepsilon)$ for any $\varepsilon \in [0, 2]$ and that X is rotund if and only if $\beta_X(\varepsilon) = 2$ and as well as that X is uniformly convex if and only if $\beta_X(\varepsilon) > 0$ for any $\varepsilon \in [0, 2]$. Gurariĭ [68] proved that if $X = c_0$ is renormed by the norm

$$\|x\| = \|x\|_\infty + \left(\sum_{n=0}\left(\frac{Px_n}{2^n}\right)^2\right)^{1/2}, \quad \forall \{x_n\} \in c_0, \tag{6}$$

then $\beta_X(\varepsilon) = 0$ for any $\varepsilon \in [0, 2)$ and $\beta_X(2) = 1$.

Gurariĭ and Sozonov [70] proved that a normed linear space $(X, \|\cdot\|)$ is an inner product space if and only if, for every $x, y \in S(X)$

$$\inf_{\alpha\in[0,1]} \|\alpha x + (1 - \alpha) y\| = \frac{\|x + y\|}{2}. \tag{7}$$

Zanco and Zucchi [71] showed an example of a normed space X with $\delta_X(2) \neq \beta_X(2)$.

Now, we will define Köthe sequence spaces $m(\phi, p)$ and $\ell_p(u, v)$ that will be considered in this paper.

Let $\mathscr{C}$ denote the set whose elements are finite sets of distinct positive integers. Given any element σ of $\mathscr{C}$, we denote by $c(\sigma)$ the sequence $\{c_n(\sigma)\}$ such that $c_n(\sigma) = 1$ for $n \in \sigma$, and $c_n(\sigma) = 0$ otherwise. Further, we define

$$\mathscr{C}_\tau = \left\{\sigma \in \mathscr{C} : \sum_{n=1}^{\infty} c_n(\sigma) \leq \tau\right\}; \tag{8}$$

that is, $\mathscr{C}_\tau$ is the set of those σ whose support has cardinality at most τ. Let us define

$$\Phi = \left\{\phi = \{\phi_n\}_{n=1}^{\infty} \in \ell^0 : \phi_1 > 0, \right. \quad \left. \Delta\phi_k \geq 0, \Delta\left(\frac{\phi_k}{k}\right) \leq 0, \forall k \in \mathbb{N}\right\}, \tag{9}$$

where $\Delta\phi_n = \phi_n - \phi_{n-1}$.

Given any $\phi \in \Phi$, we define the following sequence space, introduced in [55]:

$$m(\phi) = \left\{x = \{x_n\}_{n=1}^{\infty} \in \ell^0 : \sup_{\tau\geq 1}\left(\sup_{\sigma\in\mathscr{C}_\tau}\left(\frac{1}{\phi_\tau}\sum_{n\in\sigma}|x_n|\right)\right) < \infty\right\}. \tag{10}$$

Sargent [55] established the relationship of this space to the space ℓ_p ($1 \leq p \leq \infty$) and characterized some matrix transformations. In [49], matrix classes $(X, m(\phi))$ have been characterized, where X is assumed to be any FK-space.

Recently in [52], some of the geometric properties of $m(\phi)$ have been investigated. In [61], Tripathy and Sen extended the space $m(\phi)$ to $m(\phi, p)$ as follows:

$$m(\phi, p) := \left\{x = \{x_n\}_{n=1}^{\infty} \in \ell^0 : \sup_{\tau\geq 1}\left(\sup_{\sigma\in\mathscr{C}_\tau}\left(\frac{1}{\phi_\tau}\sum_{n\in\sigma}|x_n|^p\right)\right) < \infty\right\}, \tag{11}$$

for $\phi \in \Phi$ and $p > 0$.

It has been proved in [61] that, for $1 \leq p < \infty$, $m(\phi, p)$ is a Banach space if it is endowed with the norm

$$\|x\|_{m(\phi,p)} = \sup_{\tau\geq 1}\left(\sup_{\sigma\in\mathscr{C}_\tau}\left(\frac{1}{\phi_\tau}\sum_{n\in\sigma}|x_n|^p\right)^{1/p}\right), \tag{12}$$

and that one has the following.

(i) If $\phi_n = 1$, for all $n \in \mathbb{N}$, then $m(\phi, p) = \ell_p$. Moreover, $\ell_p \subseteq m(\phi, p) \subseteq \ell_\infty$.

(ii) If $p = 1$, then $m(\phi, p) = m(\phi)$. Also, for any $p \geq 1, m(\phi) \subseteq m(\phi, p)$.

(iii) $m(\phi, p) \subseteq m(\psi, p)$ if and only if $\sup_{\tau\geq 1}(\phi_\tau/\psi_\tau) < \infty$.

It is easy to see that $m(\phi, p)$ is a Köthe sequence space, indeed a BK-space with respect to its natural norm (see [55]). Note that throughout the present paper we will study the space $m(\phi, p)$ except the case $\phi_n = n$, for which it is reduced to the space ℓ^∞.

Now we will introduce the space $\ell_p(u, v)$. Let $u = \{u_n\}_{n=0}^{\infty}$ and let $v = \{v_n\}_{n=0}^{\infty}$ be arbitrary real sequences with all coordinates u_k and v_k different from zero and let, for any $p \in [1, \infty)$,

$$\ell_p(u, v) = \left\{x = \{x_k\}_{k=1}^{\infty} : \sum_{n=1}^{\infty}\left|\sum_{k=1}^{n} u_n v_k x(k)\right|^p < \infty\right\}. \tag{13}$$

It is obvious that this is a linear space. It is known (see [48]) that the functional

$$\|x\|_{\ell_p(u,v)} = \left(\sum_n\left|\sum_{k=1}^{n} u_n v_k x(k)\right|^p\right)^{1/p} \tag{14}$$

is a norm in $\ell_p(u, v)$ and that the couple $(\ell_p(u, v), \|\cdot\|_{\ell_p(u,v)})$ is a Banach space. The space $\ell_p(u, v)$ is a generalization of three spaces. Namely, one has the following.

(i) If $(v_k) = (1, 1, 1, \ldots)$ and $(u_n) = (1/n)$, then $\ell_p(u, v)$ is the Cesáro sequence space X_p of nonabsolute type (see [32]) and $\|x\|_{\ell_p(u,v)} = \|x\|_{X_p}$.

(ii) Let $\{p_n\}$ be a real sequence with $p_1 > 0$ and $p_n \neq 0$ for all $n \in \mathbb{N}$. If $(v_k) = \{p_k\}$ and $(u_n) = (1/Q_n)$, where $Q_n = p_1 + p_2 + \cdots + p_n$ for any $n \in \mathbb{N}$. Then we obtain that $\ell_p(u, v)$ is the Riesz sequence space of nonabsolute type denoted by $\gamma^p(v)$ and that $\|x\|_{\ell_p(u,v)} = \|x\|_{\gamma^p(v)}$ (see [44]).

(iii) Let $\{p_n\}$ be a real sequence with $p_1 > 0$ and $p_n \neq 0$, for all $n \in \mathbb{N}$, $n \geq 2$. If $(v_k) = (p_{n-k+1})$ and $(u_n) = (1/Q_n)$, where $Q_n = p_n + p_{n-1} + \cdots + p_1$ for any $n \in \mathbb{N}$. Then $\ell_p(u, v)$ reduces to the Nörlund sequence space of nonabsolute type denoted by N_p and $\|x\|_{\ell_p(u,v)} = \|x\|_{N_p}$ (see [60]).

2. Banach-Saks Type of Sequence Space $m(\phi,p)$

In this section, we investigate some properties of the space $m(\phi, p)$ such as the Fatou property, the Banach-Saks property of type p, and the weak fixed point property. Let us start with the following lemma.

Lemma 1. *If a BK-space X containing c_{00} has the property AK, then it is order continuous, that is, $\|(0, 0, \ldots, x(n), x(n+1), \ldots)\| \to 0$ as $n \to \infty$ for any $x \in X$.*

Proof. From the definition of property AK, we have that every $x = \{x(i)\} \in X$ has the unique representation $x = \sum_{i=1}^{\infty} x(i)e(i)$, that is, $x^{[n]} = \sum_{i=1}^{n} x(i)e(i) \to x$ as $n \to \infty$. Hence $\|x - x^{[n]}\|_X \to 0$ as $n \to \infty$, that is, $\|(0, 0, \ldots, x(n), x(n+1), \ldots)\| \to 0$ as $n \to \infty$, which means that X is order continuous. □

Corollary 2. *The space $m(\phi, p)$ is order continuous.*

Proof. It is easy to see that $m(\phi, p)$ contains c_{00} and that every $x = \{x(i)\} \in m(\phi, p)$ has the unique representation $x = \sum_{i=1}^{\infty} x(i)e(i)$, that is, $x^{[n]} = \sum_{i=1}^{n} x(i)e(i) \to x$ as $n \to \infty$, which means that $m(\phi, p)$ has the property AK. Hence, from the above lemma, $m(\phi, p)$ is order continuous. □

Theorem 3. *The space $m(\phi, p)$ has the Fatou property.*

Proof. Let x be any real sequence from $(l^0)_+$ and $\{x_n\}$ be any non-decreasing sequence of non-negative elements from $m(\Phi, p)$ such that $x_n(i) \to x(i)$ as $n \to \infty$ coordinatewisely and $\sup_n\|x_n\|_{m(\Phi,p)} < \infty$.

Let us denote $s = \sup_n \|x_n\|_{m(\Phi,p)}$. Then, since the supremum is homogeneous, we have

$$\begin{aligned}
&\frac{1}{s}\sup_{\tau\geq 1}\left(\sup_{\sigma\in\mathscr{C}_\tau}\left(\frac{1}{\Phi_\tau}\sum_{i\in\sigma}|x_n(i)|^p\right)^{1/p}\right) \\
&= \sup_{\tau\geq 1}\left(\sup_{\sigma\in\mathscr{C}_\tau}\left(\frac{1}{\Phi_\tau}\sum_{i\in\sigma}\left|\frac{x_n(i)}{s}\right|^p\right)^{1/p}\right) \\
&\leq \sup_{\tau\geq 1}\left(\sup_{\sigma\in\mathscr{C}_\tau}\left(\frac{1}{\Phi_\tau}\sum_{i\in\sigma}\left|\frac{x_n(i)}{\|x_n\|_{m(\Phi,p)}}\right|^p\right)^{1/p}\right) \\
&= \frac{1}{\|x_n\|_{m(\Phi,p)}}\|x_n\|_{m(\Phi,p)} = 1.
\end{aligned} \tag{15}$$

Moreover, by the assumptions that $\{x_n\}$ is non-decreasing and convergent to x coordinatewisely and by the Beppo-Levi theorem, we have

$$\begin{aligned}
&\frac{1}{s}\lim_{n\to\infty}\left[\sup_{\tau\geq 1}\left(\sup_{\sigma\in\mathscr{C}_\tau}\left(\frac{1}{\Phi_\tau}\sum_{i\in\sigma}|x_n(i)|^p\right)^{1/p}\right)\right] \\
&= \sup_{\tau\geq 1}\left(\sup_{\sigma\in\mathscr{C}_\tau}\left(\frac{1}{\Phi_\tau}\sum_{i\in\sigma}\left|\frac{x(i)}{s}\right|^p\right)^{1/p}\right) \\
&= \left\|\frac{x}{s}\right\|_{m(\Phi,p)} \leq 1,
\end{aligned} \tag{16}$$

whence

$$\|x\|_{m(\Phi,p)} \leq s = \sup_n \|x_n\|_{m(\Phi,p)} = \lim_{n\to\infty}\|x_n\|_{m(\Phi,p)} < \infty. \tag{17}$$

Therefore, $x \in m(\Phi,p)$. On the other hand, since $0 \leq x_n \leq x$ for any natural number n and the sequence $\{x_n\}$ is non-decreasing, we obtain that the sequence $\{\|x_n\|_{m(\Phi,p)}\}$ is bounded from above by $\|x\|_{m(\Phi,p)}$. In consequence $\lim_{n\to\infty}\|x_n\|_{m(\Phi,p)} \leq \|x\|_{m(\Phi,p)}$, which together with the opposite inequality proved already, yields that $\|x\|_{m(\Phi,p)} = \lim_n \|x_n\|_{m(\Phi,p)}$. □

Theorem 4. *The space $m(\phi,p)$ has the Banach-Saks property of the type p.*

Proof. Let $\{\epsilon_n\}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty}\epsilon_n \leq 1/2$. Let $\{x_n\}$ be a weakly null sequence in $B(m(\phi,p))$. Let us set $x_0 = 0$ and $z_1 = x_{n_1} = x_1$. Then, there exists $\tau_1 \in \mathbb{N}$ such that

$$\left\|\sum_{i\in s_1} z_1(i)\,e(i)\right\|_{m(\phi,p)} < \epsilon_1, \tag{18}$$

where s_1 consist of the elements of σ which exceed τ_1. Since $x_n \xrightarrow{\omega} 0$ implies that $x_n \to 0$ coordinatewise, there is $n_2 \in \mathbb{N}$ such that

$$\left\|\sum_{i=1}^{\tau_1} x_n(i)\,e(i)\right\|_{m(\phi,p)} < \epsilon_1, \tag{19}$$

for all $n \geq n_2$. Set $z_2 = x_{n_2}$. Then there exists $\tau_2 > \tau_1$ such that

$$\left\|\sum_{i\in s_2} z_2(i)\,e(i)\right\|_{m(\phi,p)} < \epsilon_2, \tag{20}$$

where s_2 consist of all elements of σ which exceed τ_2. Using again the fact that $x_n \to 0$ coordinatewise, there exists $n_3 > n_2$ such that

$$\left\|\sum_{i=1}^{\tau_2} x_n(i)\,e(i)\right\|_{m(\phi,p)} < \epsilon_2, \tag{21}$$

for all $n \geq n_3$.

Continuing this process, we can find two increasing sequences $\{\tau_j\}$ and $\{n_j\}$ such that

$$\begin{aligned}
&\left\|\sum_{i=1}^{\tau_j} x_n(i)\,e(i)\right\|_{m(\phi,p)} < \epsilon_j, \quad \text{for each } n \geq n_{j+1}, \\
&\left\|\sum_{i\in s_j} z_j(i)\,e(i)\right\|_{m(\phi,p)} < \epsilon_{j,}
\end{aligned} \tag{22}$$

where $z_j = x_{n_j}$ and s_j consist of the elements of σ which exceed τ_j. Since $\epsilon_{j-1} + \epsilon_j < 1$, we have

$$\left(\frac{1}{\phi_\tau}\sum_{n\in\sigma}\left|z_j(n)\right|\right) \leq \epsilon_{j-1} + \epsilon_j < 1, \tag{23}$$

for all $j \in \mathbb{N}$ and $\tau \geq 1$. Hence

$$\begin{aligned}
&\left\|\sum_{j=1}^{n} z_j\right\|_{m(\phi,p)} \\
&= \left\|\sum_{j=1}^{n}\left(\sum_{i=1}^{\tau_{j-1}} z_j(i)\,e(i) + \sum_{i=\tau_{j-1}+1}^{\tau_j} z_j(i)\,e(i) + \sum_{i\in s_j} z_j(i)\,e(i)\right)\right\|_{m(\phi,p)} \\
&\leq \left\|\sum_{j=1}^{n}\left(\sum_{i=1}^{\tau_{j-1}} z_j(i)\,e(i)\right)\right\|_{m(\phi,p)} + \left\|\sum_{j=1}^{n}\left(\sum_{i=\tau_{j-1}+1}^{\tau_j} z_j(i)\,e(i)\right)\right\|_{m(\phi,p)} \\
&\leq \sum_{j=1}^{n}\left\|\left(\sum_{i=\tau_{j-1}+1}^{\tau_j} z_j(i)\,e(i)\right)\right\|_{m(\phi,p)} + 2\sum_{j=1}^{n}\epsilon_j.
\end{aligned} \tag{24}$$

By using the norm of the space $m(\phi, p)$, we have

$$\begin{aligned}\sum_{j=1}^{n}\left\|\left(\sum_{i=\tau_{j-1}+1}^{\tau_j} z_j(i)\, e(i)\right)\right\|_{m(\phi,p)} &= \sum_{j=1}^{n}\sup_{\tau\geq 1}\sup_{s_{j-1}\in\mathscr{C}_\tau}\left(\frac{1}{\phi_\tau}\sum_{n\in s_{j-1}}\left|z_j(n)\right|^p\right) \\ &\leq \sum_{j=1}^{n}\sup_{\tau\geq 1}\sup_{\sigma\in\mathscr{C}_\tau}\left(\frac{1}{\phi_\tau}\sum_{n\in\sigma}\left|z_j(n)\right|^p\right)\leq n.\end{aligned} \tag{25}$$

Therefore,

$$\left\|\sum_{j=1}^{n} z_j\right\|_{m(\phi,p)} \leq n^{1/p}+1\leq 2n^{1/p}. \tag{26}$$

This completes the proof of the theorem. □

Theorem 5. *For $1 < p < \infty$, the space $m(\phi, p)$ has the weak fixed point property, if $K > 2^{1-p}$, where $K = \sup_{\tau\geq 1}\phi_\tau < \infty$.*

Proof. If $\psi_\tau = 1$, for all $\tau \in \mathbb{N}$, it follows that

$$\begin{aligned} m(\phi,p)\subseteq \ell_p \quad &\text{iff } \sup_{\tau\geq 1}(\phi_\tau)^{1/p}<\infty, \\ \|x\|_{m(\phi,p)} &= \sup_{\tau\geq 1}\left(\frac{1}{\phi_\tau}\right)^{1/p}\|x\|_{\ell_p}.\end{aligned} \tag{27}$$

Hence

$$\begin{aligned} R(m(\phi,p)) &= \sup_{\tau\geq 1}\left(\frac{1}{\phi_\tau}\right)^{1/p} R\left(\ell_p\right) \\ &= \left(\frac{2}{K}\right)^{1/p} \\ &< 2, \quad \text{since } R\left(\ell_p\right)=2^{1/p},\end{aligned} \tag{28}$$

where $R(X)$ stands for the Garcia-Falset coefficient of X. Therefore, $m(\phi, p)$ has in this case the weak fixed point property. □

3. Banach-Saks Type and Gurariĭ Modulus of Sequence Spaces $\ell_p(u,v)$

Theorem 6. *The space $\ell_p(u, v)$ has the Banach-Saks property of the type p.*

Proof. Let (ε_n) be a sequence of positive numbers for which $\sum_{n=1}^{\infty}\varepsilon_n \leq (1/2)$. Let $\{x_n\}$ be a weakly null sequence in $B(\ell_p(u, v))$. Set $t_0 = x_0 = 0$ and $t_1 = x_{n_1} = x_1$. Then there exists $r_1 \in \mathbb{N}$ such that

$$\left\|\sum_{i=r_1+1}^{\infty} t_1(i)\, e(i)\right\|_{\ell_p(u,v)} < \varepsilon_{1}. \tag{29}$$

Since the fact that $\{x_n\}$ is a weakly null sequence implies that $x_n \to 0$, coordinatewise, there is an $n_2 \in \mathbb{N}$ such that

$$\left\|\sum_{i=1}^{r_1} x_n(i)\, e(i)\right\|_{\ell_p(u,v)} < \varepsilon_1, \tag{30}$$

for all $n \geq n_2$. Set $t_2 = x_{n_2}$. Then there exists an $r_2 > r_1$ such that

$$\left\|\sum_{i=r_2+1}^{\infty} t_2(i)\, e(i)\right\|_{\ell_p(u,v)} < \varepsilon_{2}. \tag{31}$$

By using the fact that $x_n \to 0$ coordinatewise, there exists an $n_3 > n_2$ such that

$$\left\|\sum_{i=1}^{r_2} x_n(i)\, e(i)\right\|_{\ell_p(u,v)} < \varepsilon_{2}, \tag{32}$$

for all $n \geq n_3$. Continuing this process, we can find by induction two increasing subsequences (r_i) and (n_i) of natural numbers such that

$$\left\|\sum_{i=1}^{r_j} x_n(i)\, e(i)\right\|_{\ell_p(u,v)} < \varepsilon_{j}, \tag{33}$$

for all $n \geq n_{j+1}$ and

$$\left\|\sum_{i=r_j+1}^{\infty} t_j(i)\, e(i)\right\|_{\ell_p(u,v)} < \varepsilon_{j}, \tag{34}$$

where $t_j = x_{n_j}$. Hence,

$$\begin{aligned}\left\|\sum_{j=0}^{n} t_j\right\|_{\ell_p(u,v)} &= \left\|\sum_{j=0}^{n}\left(\sum_{i=0}^{r_{j-1}} t_j(i)\, e(i) + \sum_{i=r_{j-1}+1}^{r_j} t_j(i)\, e(i) + \sum_{i=r_j+1}^{\infty} t_j(i)\, e(i)\right)\right\|_{\ell_p(u,v)} \\ &\leq \left\|\sum_{j=0}^{n}\left(\sum_{i=r_{j-1}+1}^{r_j} t_j(i)\, e(i)\right)\right\|_{\ell_p(u,v)} + \left\|\sum_{j=0}^{n}\left(\sum_{i=0}^{r_{j-1}} t_j(i)\, e(i)\right)\right\|_{\ell_p(u,v)} + \left\|\sum_{j=0}^{n}\left(\sum_{i=r_j+1}^{\infty} t_j(i)\, e(i)\right)\right\|_{\ell_p(u,v)} \\ &\leq \left\|\sum_{j=0}^{n}\left(\sum_{i=r_{j-1}+1}^{r_j} t_j(i)\, e(i)\right)\right\|_{\ell_p(u,v)} + 2\sum_{j=0}^{n}\varepsilon_j.\end{aligned} \tag{35}$$

On the other hand, since $\|x_n\| = (\sum_{i=0}^{\infty} | \sum_{k=0}^{i} u_i v_k x_{n_j}(k)|^p)^{1/p}$, it can be easily seen that $\|x_n\| < 1$. Therefore, $\|x_n\|^p < 1$ and

$$\begin{aligned} \left\| \sum_{j=0}^{n} \left(\sum_{i=r_{j-1}+1}^{r_j} t_j(i) e(i) \right) \right\|_{\ell_p(u,v)}^p &= \sum_{j=0}^{n} \sum_{i=r_{j-1}+1}^{r_j} \left| \sum_{k=0}^{i} u_i v_k t_j(k) \right|^p \\ &\leq \sum_{j=0}^{n} \sum_{i=0}^{\infty} \left| \sum_{k=0}^{i} u_i v_k t_j(k) \right|^p \leq (n+1). \end{aligned} \tag{36}$$

Hence we obtain

$$\begin{aligned} \left\| \sum_{j=0}^{n} \left(\sum_{i=r_{j-1}+1}^{r_j} t_j(i) e(i) \right) \right\| \leq \left(\sum_{j=0}^{n} 1 \right)^{1/p} = (n+1)^{1/p}. \end{aligned} \tag{37}$$

By using the inequality $1 \leq (n+1)^{1/p}$ for all $n \in \mathbb{N}$ and $1 \leq p < \infty$, we have

$$\left\| \sum_{j=0}^{n} t_j \right\|_{l_p(u,v)} \leq (n+1)^{1/p} + 1 \leq 2(n+1)^{1/p}. \tag{38}$$

Therefore, the space $\ell_p(u,v)$ has the Banach-Saks type p, which completes the proof of the theorem. □

Let us define the matrix $G = G(u,v) = \{g_{nk}\}$ by

$$g_{nk} = \begin{cases} u_n v_k; & 0 \leq k \leq n, \\ 0; & k > n \end{cases} \tag{39}$$

for all $k, n \in \mathbb{N}$, where u_n depends only on n and v_k depends only on k. The matrix G is called generalized weighted mean or factorable matrix. By $H = H(v,u) = (h_{nk})$, we denote the inverse of the matrix $G(u,v)$ as follows:

$$h_{nk} = \begin{cases} \dfrac{(-1)^{n-k}}{v_n u_k}; & n-1 \leq k \leq n, \\ 0; & 0 \leq n < k \text{ or } n > k+1. \end{cases} \tag{40}$$

Theorem 7. *For $x \in \ell_p(u,v)$, by (39), one has the fact that the Gurariĭ modulus of convexity for the normed space $\ell_p(u,v)$ satisfies the inequality*

$$\beta_{\ell_p(u,v)}(\varepsilon) \leq 1 - \left(1 - \left(\frac{\varepsilon}{2} \right)^p \right)^{1/p} \tag{41}$$

for any $0 \leq \varepsilon \leq 2$.

Proof. Let $x \in \ell_p(u,v)$. By using (39), we have

$$\begin{aligned} \|x\|_{\ell_p(u,v)} &= \|G(u,v)x\|_{\ell_p} \\ &= \left(\sum_n |(G(u,v)x)_n|^p \right)^{1/p}. \end{aligned} \tag{42}$$

Let $0 \leq \varepsilon \leq 2$. Then using (40), let us consider the following sequences:

$$\begin{aligned} x = (x_n) &= \left(H\left(\left(1 - \left(\frac{\varepsilon}{2} \right)^p \right)^{1/p} \right), H\left(\frac{\varepsilon}{2} \right), 0, 0, \ldots \right), \\ t = (t_n) &= \left(H\left(\left(1 - \left(\frac{\varepsilon}{2} \right)^p \right)^{1/p} \right), H\left(-\frac{\varepsilon}{2} \right), 0, 0, \ldots \right). \end{aligned} \tag{43}$$

Since $y_n = (Gx)_n$ and $z_n = (Gt)_n$, we have

$$\begin{aligned} y = (y_n) &= \left(\left(1 - \left(\frac{\varepsilon}{2} \right)^p \right)^{1/p}, \left(\frac{\varepsilon}{2} \right), 0, 0, \ldots \right), \\ z = (z_n) &= \left(\left(1 - \left(\frac{\varepsilon}{2} \right)^p \right)^{1/p}, \left(-\frac{\varepsilon}{2} \right), 0, 0, \ldots \right). \end{aligned} \tag{44}$$

By using the sequences given above, we obtain the following equalities:

$$\begin{aligned} \|x\|_{\ell_p(u,v)}^p &= \|G(u,v)x\|_{\ell_p}^p \\ &= \left| \left(1 - \left(\frac{\varepsilon}{2} \right)^p \right)^{1/p} \right|^p + \left| \frac{\varepsilon}{2} \right|^p \\ &= 1 - \left(\frac{\varepsilon}{2} \right)^p + \left(\frac{\varepsilon}{2} \right)^p = 1; \\ \|t\|_{\ell_p(u,v)}^p &= \|G(u,v)t\|_{\ell_p}^p \\ &= \left| \left(1 - \left(\frac{\varepsilon}{2} \right)^p \right)^{1/p} \right|^p + \left| -\frac{\varepsilon}{2} \right|^p \\ &= 1 - \left(\frac{\varepsilon}{2} \right)^p + \left(\frac{\varepsilon}{2} \right)^p = 1; \\ \|x - t\|_{\ell_p(u,v)} &= \|G(u,v)x - G(u,v)t\|_{\ell_p} \\ &= \left(\left| \left(1 - \left(\frac{\varepsilon}{2} \right)^p \right)^{1/p} - \left(1 - \left(\frac{\varepsilon}{2} \right)^p \right)^{1/p} \right|^p \right. \\ &\quad \left. + \left| \frac{\varepsilon}{2} - \left(-\frac{\varepsilon}{2} \right) \right|^p \right)^{1/p} = \varepsilon. \end{aligned} \tag{45}$$

To complete the upper estimate of the Gurariĭ modulus of convexity, it remains to calculate the infimum of $\|\alpha x + (1-\alpha)t\|_{\ell_p(u,v)}$ for $0 \le \alpha \le 1$. We have

$$\begin{aligned}
&\inf_{0\le\alpha\le1} \|\alpha x + (1-\alpha)\, t\|_{\ell_p(u,v)} \\
&\quad = \inf_{0\le\alpha\le1} \|\alpha G(u,v)\, x + (1-\alpha)\, G(u,v)\, t\|_{\ell_p} \\
&\quad = \inf_{0\le\alpha\le1} \left[\left| \alpha \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{1/p} + (1-\alpha)\left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{1/p} \right|^p + \left| \alpha\left(\frac{\varepsilon}{2}\right) + (1-\alpha)\left(-\frac{\varepsilon}{2}\right) \right|^p \right]^{1/p} \\
&\quad = \inf_{0\le\alpha\le1} \left[1 - \left(\frac{\varepsilon}{2}\right)^p + |2\alpha - 1|^p \left(\frac{\varepsilon}{2}\right)^p \right]^{1/p} \\
&\quad = \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{1/p}.
\end{aligned} \tag{46}$$

Consequently, we get for $p \ge 1$ the inequality

$$\beta_{l_p(u,v)}(\varepsilon) \le 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{1/p}, \tag{47}$$

which is the desired result. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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