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Research Article

On an Iterative Method for Finding a Zero to the Sum of Two Maximal Monotone Operators

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In this paper we consider a problem that consists of finding a zero to the sum of two monotone operators. One method for solving such a problem is the forward-backward splitting method. We present some new conditions that guarantee the weak convergence of the forward-backward method. Applications of these results, including variational inequalities and gradient projection algorithms, are also considered.

1. Introduction

It is well known that monotone inclusions problems play an important role in the theory of nonlinear analysis. This problem consists of finding a zero of maximal monotone operators. However, in some examples such as convex programming and variational inequality problems, the operator is needed to be decomposed of the sum of two monotone operators (see, e.g., [1–6]). In this way, one needs to find $x \in \mathscr{H}$ so that

$$0 \in Ax + Bx, \tag{1}$$

where A and B are two monotone operators on a Hilbert space $\mathscr{H}$. To solve such problem, the splitting method, such as Peaceman-Rachford algorithm [7] and Douglas-Rachford algorithm [8], is usually considered. We consider a special case whenever $B : \mathscr{H} \to 2^{\mathscr{H}}$ is multivalued and $A : \mathscr{H} \to \mathscr{H}$ is single-valued. A classical way to solve problem (1) under our assumption is the forward-backward splitting (FBS) (see [2, 9]). Starting with an arbitrary initial $x_0 \in \mathscr{H}$, the FBS generates a sequence (x_n) satisfying

$$x_{n+1} = (I + rB)^{-1} (I - rA)\, x_n, \tag{2}$$

where r is some properly chosen real number. Then the FBS converges weakly to a solution of problem (1) whenever such point exists.

On the other hand, we observe that problem (1) is equivalent to the fixed point equation:

$$(I + rB)^{-1} (I - rA)\, x = x, \tag{3}$$

for the single-valued operator $(I + rB)^{-1}(I - rA)$. Moreover, if r is properly chosen, the operator $(I + rB)^{-1}(I - rA)$ should be nonexpansive. Motivated by this assumption, by using the techniques of the fixed point theory for nonexpansive operators, we try to investigate and study various monotone inclusion problems.

The rest of this paper is organized as follows. In Section 2, some useful lemmas are introduced. In Section 3, we consider the modified forward-backward splitting method and prove its weak convergence under some new conditions. In Section 4, some applications of our results in finding a solution of the variational inequality problem are included.

2. Preliminary and Notation

Throughout the paper, I denotes the identity operator, $\mathrm{Fix}(S)$ the set of the fixed points of an operator S, and ∇f the gradient of the functional $f : \mathscr{H} \to \mathbb{R}$. The notation "$\to$" denotes strong convergence and "$\rightharpoonup$" weak convergence. Denote by $\omega_w(x_n)$ the set of the cluster points of (x_n) in the weak topology (i.e., the set $\{x : \exists x_{n_j} \rightharpoonup x\}$, where (x_{n_j}) means a subsequence of (x_n)).

Let C be a nonempty closed convex subset of $\mathscr{H}$. Denote by P_C the projection from $\mathscr{H}$ onto C; namely, for $x \in \mathscr{H}$, $P_C x$ is the unique point in C with the property

$$\left\| x - P_C x \right\| = \min_{y \in C} \left\| x - y \right\|. \tag{4}$$

It is well-known that $P_C x$ is characterized by the inequality

$$\left\langle x - P_C x, z - P_C x \right\rangle \leq 0, \quad z \in C. \tag{5}$$

A single-valued operator $S : \mathscr{H} \to \mathscr{H}$ is called nonexpansive if

$$\left\| Sx - Sy \right\| \leq \left\| x - y \right\| \quad (\forall x, y \in \mathscr{H}); \tag{6}$$

firmly nonexpansive if

$$\left\langle Sx - Sy, x - y \right\rangle \geq \left\| Sx - Sy \right\|^2 \quad (\forall x, y \in \mathscr{H}); \tag{7}$$

and κ-averaged if there exists a constant $\kappa \in (0,1)$ and a nonexpansive operator R such that $S = (1-\kappa)I + \kappa R$. Firmly nonexpansive operators are $(1/2)$-averaged.

Lemma 1 (see [10]). *The following assertions hold.*

(i) *S is κ-averaged for $\kappa \in (0,1)$ if and only if*

$$\left\| Sx - Sy \right\|^2 \leq \left\| x - y \right\|^2 - \frac{1-\kappa}{\kappa} \left\| (I-S)x - (I-S)y \right\|^2 \tag{8}$$

for all $x, y \in \operatorname{dom} S$.

(ii) *Assume that S_i is κ_i-averaged for $\kappa_i \in (0,1)$, $i = 1, 2$. Then $S_1 S_2$ is κ-averaged with $\kappa = \kappa_1 + \kappa_2 - \kappa_1 \kappa_2$.*

The following lemma is known as the demiclosedness principle for nonexpansive mappings.

Lemma 2. *Let C be a nonempty closed convex subset of $\mathscr{H}$ and S a nonexpansive operator with $\operatorname{Fix}(S) \neq \emptyset$. If (x_n) is a sequence in C such that $x_n \rightharpoonup x$ and $(I-S)x_n \to y$, then $(I-S)x = y$. In particular, if $y = 0$, then $x \in \operatorname{Fix}(S)$.*

A multivalued operator $B : \mathscr{H} \to 2^{\mathscr{H}}$ is called monotone if

$$\left\langle u - v, x - y \right\rangle \geq 0, \quad (\forall u \in Bx, v \in By); \tag{9}$$

κ-inverse strongly monotone (κ-ism), if there exists a constant $\kappa > 0$ so that

$$\left\langle u - v, x - y \right\rangle \geq \kappa \| u - v \|^2, \quad (\forall u \in Bx, v \in By); \tag{10}$$

and maximal monotone if it is monotone and its graph $G(B) = \{(x,y) : y \in Bx\}$ is not properly contained in the graph of any other monotone operator.

In what follows, we shall assume that

(i) $A : \mathscr{H} \to \mathscr{H}$ is single-valued and κ-ism;

(ii) $B : \mathscr{H} \to 2^{\mathscr{H}}$ is multivalued and maximal monotone.

Hereafter, if no confusion occurs, we denote by

$$J_r := (I + rB)^{-1} \tag{11}$$

the resolvent of B for any given $r > 0$. It is known that J_r is single-valued and firmly nonexpansive; moreover $\operatorname{dom}(I + rB) = \mathscr{H}$ (see [11]).

Lemma 3 (see [12]). *For $r > 0$, let $T_r = J_r(I - rA)$. Then*

(i) $\operatorname{Fix}(T_r) = (A+B)^{-1}(0)$;

(ii) *T_r is $(2\kappa + r)/4\kappa$-averaged;*

(iii) *For $0 < r \leq s \leq 2\kappa$, $x \in \mathscr{H}$, it follows that*

$$\begin{gathered} \left\| x - T_r x \right\| \leq 2 \left\| x - T_s x \right\|, \\ \left\| T_s x - T_r x \right\| \leq \left| 1 - \frac{r}{s} \right| \left\| x - T_s x \right\|. \end{gathered} \tag{12}$$

Definition 4. Assume that (x_n) is a sequence in $\mathscr{H}$ and that (ϵ_n) is a real sequence with $\sum_n \epsilon_n < \infty$. Then (x_n) is called *quasi Fejér monotone* w.r.t. C, if

$$\left\| x_{n+1} - z \right\| \leq \left\| x_n - z \right\| + \epsilon_n \quad (\forall z \in C). \tag{13}$$

Lemma 5 (see [13]). *Let C be a nonempty closed convex subset of $\mathscr{H}$. If the sequence (x_n) is quasi-Fejér-monotone w.r.t. C, then the following hold:*

(i) *$x_n \rightharpoonup x^* \in C$ if and only if $\omega_w(x_n) \subseteq C$;*

(ii) *the sequence $(P_C x_n)$ converges strongly;*

(iii) *if $x_n \rightharpoonup x^* \in C$, then $x^* = \lim_{n\to\infty} P_C x_n$.*

3. Weak Convergence Theorem

In [10] Combettes considered a modified FBS: for any initial guess $x_0 \in \mathscr{H}$, set

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n J_{r_n} (x_n - r_n A x_n) + e_n, \tag{14}$$

where $(\alpha_n) \subseteq [0,1]$, $(r_n) \subseteq [0, 2\kappa]$ and $(e_n) \subseteq \mathscr{H}$ is computation error. He proved the weak convergence of algorithm (14) provided that

(1) $\underline{\lim}_n \alpha_n > 0$,

(2) $0 < \underline{\lim}_n r_n \leq \overline{\lim}_n r_n < 2\kappa$,

(3) $\sum_n \| e_n \| < \infty$.

We observe that (14) is in fact a Mann-type iteration. In the following we shall prove the convergence of (14) under some sightly weak conditions.

Theorem 6. *Suppose the following conditions are satisfied:*

(C1) $\overline{\lim}_n \alpha_n > 0$;

(C2) $0 \leq r_n \leq 2\kappa$;

(C3) $0 \leq \alpha_n \leq 4\kappa/(2\kappa + r_n)$;

(C4) $\overline{\lim}_n (4\kappa/(2\kappa + r_n) - \alpha_n) > 0$;

(C5) $\sum_n \| e_n \| < \infty$.

If in addition $\Omega := (A+B)^{-1}(0) \neq \emptyset$, then the sequence (x_n) generated by (14) *converges weakly to $x^* := P_\Omega x_n$.*

Proof. We first show that (x_n) is quasi-Fejér-monotone. Let $T_n = J_{r_n}(I - r_n A)$. Then It follows from Lemma 3 that T_n is β_n-averaged with $\beta_n = (2\kappa + r_n)/4\kappa$ and $\Omega = \mathrm{Fix}(T_n)$. Letting $z \in \Omega$, we have that

$$\begin{aligned}\|x_{n+1} - z\| &= \|(1-\alpha_n)(x_n - z) + \alpha_n (T_n x_n - z) + e_n\| \\ &\leq (1-\alpha_n)\|x_n - z\| + \alpha_n \|x_n - z\| + \|e_n\| \\ &= \|x_n - z\| + \|e_n\|.\end{aligned} \tag{15}$$

By condition (C5), we conclude that (x_n) is quasi-Fejér-monotone w.r.t Ω.

Next let us show $\omega_w(x_n) \subseteq \Omega$. To see this, choose $M > 0$ so that

$$2 \sup_{n \geq 0} (\|x_n - z\|) \leq M. \tag{16}$$

Let $R_n = (1-\alpha_n)I + \alpha_n T_n$. Obviously R_n is $\alpha_n\beta_n$-averaged. According to Lemma 1, we deduce that

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|((1-\alpha_n)I + \alpha_n T_n)x_n - z + e_n\|^2 \\ &\leq \|R_n x_n - z\|^2 + 2\langle x_{n+1} - z, e_n\rangle \\ &\leq \|R_n x_n - z\|^2 + M\|e_n\| \\ &\leq \|x_n - z\|^2 - \frac{1-\alpha_n\beta_n}{\alpha_n\beta_n}\|R_n x_n - x_n\|^2 + M\|e_n\| \\ &\leq \|x_n - z\|^2 - \alpha_n\left(\frac{1}{\beta_n} - \alpha_n\right)\|T_n x_n - x_n\|^2 \\ &\quad + M\|e_n\|,\end{aligned} \tag{17}$$

which in turn implies that

$$\sum_{i=0}^{n} \alpha_i \left(\frac{1}{\beta_i} - \alpha_i\right)\|T_i x_i - x_i\|^2 \leq \|x_0 - z\|^2 + M\sum_{i=0}^{\infty}\|e_i\| \tag{18}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ yields

$$\sum_{n=0}^{\infty} \alpha_n \left(\frac{1}{\beta_n} - \alpha_n\right)\|T_n x_n - x_n\|^2 < \infty. \tag{19}$$

By conditions (C1) and (C4), we check that

$$\liminf_{n\to\infty} \alpha_n \left(\frac{1}{\beta_n} - \alpha_n\right) > 0, \tag{20}$$

which yields that $\|T_n x_n - x_n\| \to 0$. By condition (C2), we find $n_0 \in \mathbb{N}$ and $r \in (0, 2\kappa]$ so that $r_n \geq r$ for all $n \geq n_0$. Let $T_r = J_r(I - rB)$. It then follows from Lemma 3 that

$$\|T_r x_n - x_n\| \leq 2\|T_n x_n - x_n\| \tag{21}$$

for all $n \geq n_0$. Letting $n \to \infty$, we have $\|T_r x_n - x_n\| \to 0$ as $n \to \infty$. Take $x' \in \omega_w(x_n)$ and a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \rightharpoonup x'$. Since T_r is nonexpansive, applying Lemmas 2 and 3 yields $x' \in \mathrm{Fix}(T_r) = \Omega$ and thus $\omega_w(x_n) \subseteq \Omega$. By Lemma 5 the proof is complete. □

We can also present another condition for the weak convergence of (14).

Theorem 7. *Suppose that the following conditions are satisfied:*

(C1) $\sum_n |r_{n+1} - r_n| < \infty$;
(C2) $0 < \lim_n r_n \leq 2\kappa$;
(C3) $0 \leq \alpha_n \leq 4\kappa/(2\kappa + r_n)$;
(C4) $\sum_n \alpha_n (4\kappa/(2\kappa + r_n) - \alpha_n) = \infty$;
(C5) $\sum_n \|e_n\| < \infty$.

If in addition $\Omega := (A+B)^{-1}(0) \neq \emptyset$, then the sequence (x_n) generated by (14) *converges weakly to $x^* := P_\Omega x_n$.*

Proof. Compared with the proof of Theorem 6, it suffices to show that $\|T_n x_n - x_n\| \to 0$ as $n \to \infty$. Observe that the estimate

$$\sum_{n=0}^{\infty} \alpha_n \left(\frac{4\kappa}{2\kappa + r_n} - \alpha_n\right)\|T_n x_n - x_n\|^2 < \infty \tag{22}$$

still holds. Let $a_n := \|x_n - T_n x_n\|$. Then by condition (C4) $\underline{\lim}_n a_n = 0$. According to Lemma 3, we have

$$\begin{aligned} a_{n+1} &= \|(1-\alpha_n)(x_n - T_n x_n) + (T_n x_n - T_{n+1}x_{n+1}) + e_n\| \\ &\leq (1-\alpha_n)\|x_n - T_n x_n\| + \|T_n x_n - T_{n+1}x_{n+1}\| \\ &\quad + \|e_n\| \\ &\leq (1-\alpha_n)\|x_n - T_n x_n\| + \|T_{n+1}x_n - T_{n+1}x_{n+1}\| \\ &\quad + \|T_n x_n - T_{n+1}x_n\| + \|e_n\| \\ &\leq (1-\alpha_n)\|x_n - T_n x_n\| + \|x_n - x_{n+1}\| \\ &\quad + \|T_n x_n - T_{n+1}x_n\| + \|e_n\| \\ &= (1-\alpha_n)\|x_n - T_n x_n\| + \|\alpha_n(x_n - T_n x_n) - e_n\| \\ &\quad + \left|1 - \frac{r_{n+1}}{r_n}\right|\|x_n - T_n x_n\| + \|e_n\| \\ &\leq (1-\alpha_n)\|x_n - T_n x_n\| + \alpha_n\|x_n - T_n x_n\| \\ &\quad + \left|1 - \frac{r_{n+1}}{r_n}\right|\|x_n - T_n x_n\| + 2\|e_n\| \\ &\leq a_n + M(|r_n - r_{n+1}| + \|e_n\|),\end{aligned} \tag{23}$$

where $M > 0$ is properly chosen real number. Then, for any given $p \in \mathbb{N}$, we arrive at

$$a_{n+p} \leq a_n + M\sum_{i=n}^{\infty}(|r_i - r_{i+1}| + \|e_i\|). \tag{24}$$

Conditions (C1) and (C5) imply that $\lim_{n\to\infty} a_n$ exists and therefore $\|x_n - T_n x_n\| \to 0$ as $n \to \infty$. Hence the proof is complete. □

Applying Theorem 7, one can easily get the following.

Corollary 8. *Suppose that the following conditions are satisfied:*

(1) $0 < r < 2\kappa$,

(2) $0 \le \alpha_n \le 4\kappa/(2\kappa + r)$;

(3) $\sum_n \alpha_n(4\kappa/(2\kappa + r) - \alpha_n) = \infty$,

(4) $\Omega := (A + B)^{-1}(0) \ne \emptyset$.

Then the sequence (x_n) generated by

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n J_r (x_n - rAx_n) \tag{25}$$

converges weakly to some point in Ω.

Remark 9. Corollary 8 implies that our condition is slightly weaker than that of Combettes' whenever the sequence (r_n) approaches to some constant.

4. Application

Let C be a nonempty closed convex subset of $\mathscr{H}$. A variational inequality problem (VIP) is formulated as a problem of finding a point $x^* \in C$ with the property

$$\langle Ax^*, z - x^* \rangle \ge 0, \quad \forall z \in C, \tag{26}$$

where $A : \mathscr{H} \to \mathscr{H}$ is a nonlinear operator. We shall denote by Ω the solution set of VIP (26). One method for solving VIP is the projection algorithm which generates, starting with an arbitrary initial $x_0 \in \mathscr{H}$, a sequence (x_n) satisfying

$$x_{n+1} = P_C (x_n - rAx_n), \tag{27}$$

where r is properly chosen real number. If, in addition, A is κ-ism, then the iteration (27) with $0 < r < 2\kappa$ converges weakly to a point in Ω, whenever such point exists.

Let B be the normal cone for C, that is, $B := \{w \in \mathscr{H} : \langle x - z, w\rangle \ge 0, \forall z \in C\}$. By [14, Theorem 3], VIP (26) is equivalent to finding a zero of the maximal monotone operator $A + B$. Recalling $P_C = J_r$ for any $r > 0$, we thus can apply the previous results to get the following.

Corollary 10. *Suppose the following conditions are satisfied:*

(C1) $\overline{\lim}_n \alpha_n > 0$;

(C2) $0 \le r_n \le 2\kappa$;

(C3) $0 \le \alpha_n \le 4\kappa/(2\kappa + r_n)$;

(C4) $\overline{\lim}_n (4\kappa/(2\kappa + r_n) - \alpha_n) > 0$.

Then the sequence (x_n) generated by

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n P_C (x_n - r_n Ax_n) \tag{28}$$

converges weakly to $x^* := P_\Omega x_n$, whenever such point exists.

Corollary 11. *Suppose that the following conditions are satisfied:*

(C1) $\sum_n |r_{n+1} - r_n| < \infty$;

(C2) $0 < \lim_n r_n \le 2\kappa$;

(C3) $0 \le \alpha_n \le 4\kappa/(2\kappa + r_n)$;

(C4) $\sum_n \alpha_n(4\kappa/(2\kappa + r_n) - \alpha_n) = \infty$.

Then the sequence (x_n) generated by

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n P_C (x_n - r_n Ax_n) \tag{29}$$

converges weakly to $x^* := P_\Omega x_n$, whenever such point exists.

Consider the optimization problem of finding a point x^* with the property

$$x^* \in \arg\min_{x\in C} f(x), \tag{30}$$

where $f : \mathscr{H} \to \mathbb{R}$ is a convex and differentiable function. The gradient projection algorithm (GPA) generates a sequence (x_n) by the iterative procedure

$$x_{n+1} = P_C (x_n - r\nabla f(x_n)), \tag{31}$$

where $x_0 \in \mathscr{H}$ and r is a positive parameter. If, in addition, ∇f is $(1/\kappa)$-Lipschitz continuous, that is, for any $x, y \in \mathscr{H}$,

$$\|\nabla f(x) - \nabla f(y)\| \le \frac{1}{\kappa} \|x - y\|, \tag{32}$$

then the GPA with $0 < r < 2\kappa$ converges weakly to a minimizer of f onto C, if such minimizers exist (see, e.g., [15, Corollary 4.1]). Denote by Ω the solution set of the variational inequality

$$\langle \nabla f(x), z - x\rangle \ge 0, \quad z \in C. \tag{33}$$

According to [16, Lemma 5.13], we have $\Omega = \arg\min_{x\in C} f(x)$. Further, if ∇f is $(1/\kappa)$-Lipschitz continuous, then it is also κ-ism (see [17, Corollary 10]). Thus, we can apply the previous results by letting $A = \nabla f$.

Corollary 12. *Assume that $f : \mathscr{H} \to \mathbb{R}$ is convex and differentiable with $(1/\kappa)$-Lipschitz-continuous gradient ∇f and that*

(C1) $\overline{\lim}_n \alpha_n > 0$;

(C2) $0 \le r_n \le 2\kappa$;

(C3) $0 \le \alpha_n \le 4\kappa/(2\kappa + r_n)$;

(C4) $\overline{\lim}_n (4\kappa/(2\kappa + r_n) - \alpha_n) > 0$.

Then the sequence (x_n) generated by

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n P_C (x_n - r_n \nabla f(x_n)) \tag{34}$$

converges weakly to $x^* := P_\Omega x_n$, whenever such point exists.

Corollary 13. *Assume that $f : \mathscr{H} \to \mathbb{R}$ is convex and differentiable with $(1/\kappa)$-Lipschitz-continuous gradient ∇f and that*

(C1) $\sum_n |r_{n+1} - r_n| < \infty$;

(C2) $0 < \lim_n r_n \le 2\kappa$;

(C3) $0 \le \alpha_n \le 4\kappa/(2\kappa + r_n)$;

(C4) $\sum_n \alpha_n(4\kappa/(2\kappa + r_n) - \alpha_n) = \infty$.

Then the sequence (x_n) *generated by*

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n P_C (x_n - r_n \nabla f (x_n)) \qquad (35)$$

converges weakly to $x^* := P_\Omega x_n$*, whenever such point exists.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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