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Research Article

A Strong Law of Large Numbers for Weighted Sums of i.i.d. Random Variables under Capacities

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With the notion of independent identically distributed (i.i.d.) random variables under sublinear expectations initiated by Peng, a strong law of large numbers for weighted sums of i.i.d. random variables under capacities induced by sublinear expectations is obtained.

1. Introduction

The strong law of large numbers plays important role in the development of probability theory and mathematical statistics; many studies about the extension of it have been completed by many authors. For example, Chow and Lai [1], Stout [2], Choi and Sung [3], Cuzick [4], Rosalsky and Sreehari [5], Wu [6], Bai and Cheng [7], Bai et al. [8], and so forth investigated the almost sure limiting behavior of weighted sums of i.i.d. random variables. In fact, the additivity of probability and expectations is not reasonable in many areas of applications because many uncertain phenomena cannot be well modeled using additive probabilities or linear expectations (see, e.g., Chen and Epstein [9], Huber and Strassen [10], and Wakker [11]). In the case of nonadditive probabilities, Marinacci [12] proved several limit laws for nonadditive probabilities and Maccheroni and Marinacci [13] obtained a strong law of large numbers for totally monotone capacities.

Recently, motivated by the risk measures, superhedge pricing, and modelling uncertainty in finance, Peng [14] introduced the notion of sublinear expectation space, which is a generalization of probability space. Together with the notion of sublinear expectation, Peng also introduced the notions about i.i.d., G-normal distribution, and G-Brownian motion. Under this framework, the weak law of large numbers and the central limit theorems under sublinear expectations were obtained in the studies by Peng in [15, 16]. Soon thereafter, Denis et al. [17] introduced the function spaces and capacity related to a sublinear expectation. Chen et al. [18] proved a strong law of large numbers for nonadditive probabilities.

A natural question is the following: can we investigate strong laws of large numbers for weighted sums of random variables under capacities? Indeed, the goal of this paper is to discuss the strong laws of large numbers for weighted sums of i.i.d. random variables under capacities. Under some assumptions, we obtain a strong law of large numbers for weighted sums of i.i.d. random variables under capacities.

The paper is organized as follows: in Section 2, we give some definitions and lemmas that are useful in this paper. In Section 3, we give our main results including the proofs.

2. Preliminaries

In this section, we present some preliminaries in the theory of sublinear expectations and capacities. More details of this section can be found in the studies by Chen et al. [18] and Peng [19].

Let $(\Omega, \mathscr{F})$ be a measurable space, and let $\mathscr{H}$ be the set of random variables on $(\Omega, \mathscr{F})$.

Definition 1. A sublinear expectation $\widehat{\mathbb{E}}$ is a functional $\widehat{\mathbb{E}} : \mathscr{H} \to R$ satisfying the following:

(i) monotonicity: $\widehat{\mathbb{E}}[X] \geq \widehat{\mathbb{E}}[Y]$ if $X \geq Y$;

(ii) constant preserving: $\widehat{\mathbb{E}}[C] = C$ for $C \in R$;

(iii) subadditivity: $\hat{\mathbb{E}}[X+Y] \le \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$;

(iv) positive homogeneity: $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$ for $\lambda \ge 0$.

Artzner et al. [20] showed that a sublinear expectation can be expressed as a supremum of linear expectations. That is, if $\hat{\mathbb{E}}$ is a sublinear expectation on $\mathscr{H}$, then there exists a set (say $\mathscr{P}$) of probability measures such that

$$\hat{\mathbb{E}}[X] = \sup_{P\in\mathscr{P}} E_P[X], \quad -\hat{\mathbb{E}}[-X] = \inf_{P\in\mathscr{P}} E_P[X], \quad \forall X \in \mathscr{H}. \tag{1}$$

For this $\mathscr{P}$, following Huber and Strassen [10], we define a pair $(\overline{C}, \underline{C})$ of capacities denoted by

$$\overline{C}(A) := \sup_{P\in\mathscr{P}} P(A), \quad \underline{C}(A) := \inf_{P\in\mathscr{P}} P(A), \quad \forall A \in \mathscr{F}. \tag{2}$$

Obviously,

$$\overline{C}(A) + \underline{C}(A^c) = 1, \tag{3}$$

where A^c is the complement set of A.

It is easy to check that $\overline{C}$ and $\underline{C}$ are two continuous capacities in the sense of the following definition.

Definition 2. A set function $C: \mathscr{F} \to [0,1]$ is called a continuous capacity if it satisfies

(1) $C(\phi) = 0$, $C(\Omega) = 1$;

(2) $C(A) \le C(B)$, whenever $A \subset B$ and $A, B \in \mathscr{F}$;

(3) $C(A_n) \uparrow C(A)$, if $A_n \uparrow A$;

(4) $C(A_n) \downarrow C(A)$, if $A_n \downarrow A$, where $A_n, A \in \mathscr{F}$.

Definition 3 (see Peng [19]).

Identical Distribution. Let X_1 and X_2 be two n-dimensional random vectors in $\mathscr{H}$. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if, for each measurable function φ on R^n such that $\varphi(X_1), \varphi(X_2) \in \mathscr{H}$, one has

$$\hat{\mathbb{E}}[\varphi(X_1)] = \hat{\mathbb{E}}[\varphi(X_2)]. \tag{4}$$

Independence. A random vector $Y := (Y_1, \ldots, Y_n)$, $Y_i \in \mathscr{H}$, is said to be independent of another random vector $X := (X_1, \ldots, X_m)$, $X_i \in \mathscr{H}$, under $\hat{\mathbb{E}}$ if, for each measurable function φ on $R^m \times R^n$ with $\varphi(X, Y) \in \mathscr{H}$ and $\varphi(x, Y) \in \mathscr{H}$ for each $x \in R^m$, one has

$$\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}\left[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=X}\right]. \tag{5}$$

Remark 4. A sequence of random variables $\{X_i, i \ge 1\}$ is said to be i.i.d., if $X_i \overset{d}{=} X_1$ and X_{i+1} is independent of $Y := (X_1, \ldots, X_i)$ for each $i \ge 1$.

The following lemma shows the relation between Peng's independence and pairwise independence in the study by Marinacci in [12].

Lemma 5 (see Chen et al. [18]). *Suppose that $X, Y \in \mathscr{H}$ are two random variables. $\hat{\mathbb{E}}$ is a sublinear expectation and $(\overline{C}, \underline{C})$ is the pair of capacities induced by $\hat{\mathbb{E}}$. If random variable X is independent of Y under $\hat{\mathbb{E}}$, then X is also pairwise independent of Y under capacities $\overline{C}$ and $\underline{C}$; that is, for all subsets D and $G \subset R$,*

$$C(X \in D, Y \in G) = C(X \in D)\, C(Y \in G) \tag{6}$$

holds for both capacities $\overline{C}$ and $\underline{C}$.

Borel-Cantelli lemma is still true for capacities $\overline{C}$ and $\underline{C}$ under some assumptions.

Lemma 6 (see Chen et al. [18]). *Let $\{A_n, n \ge 1\}$ be a sequence of events in $\mathscr{F}$.*

(1) *If $\sum_{n=1}^{\infty} \overline{C}(A_n) < \infty$, then $\overline{C}(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i) = 0$.*

(2) *Suppose that $\{A_n, n \ge 1\}$ are pairwise independent with respect to $\overline{C}$; that is,*

$$\overline{C}\left(\bigcap_{i=1}^{\infty} A_i^c\right) = \prod_{i=1}^{\infty} \overline{C}(A_i^c). \tag{7}$$

If $\sum_{n=1}^{\infty} \underline{C}(A_n) = \infty$, then $\underline{C}(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i) = 1$.

3. Main Results

In this section, we give our main results including the proofs.

Theorem 7. *Let $\{X_i, i \ge 1\}$ be a sequence of i.i.d. random variables in $\mathscr{H}$ satisfying $\overline{\mu} = \hat{\mathbb{E}}[X_1]$, $\underline{\mu} = -\hat{\mathbb{E}}[-X_1]$ and for any $h, r > 0$*

$$\hat{\mathbb{E}}\left[e^{(h|X_i|^r)}\right] < \infty. \tag{8}$$

Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of constants satisfying

$$A_\alpha = \limsup_{n\to\infty} A_{\alpha,n} < \infty, \qquad A_{\alpha,n}^{\alpha} = \frac{\sum_{i=1}^{n} |a_{ni}|^\alpha}{n}, \quad (1 < \alpha \le 2). \tag{9}$$

Then, for $0 < r \le 1$, if

$$\overline{\mu}\left(1 - \limsup_{n\to\infty} \frac{\sum_{i=1}^{n} a_{ni}}{b_n}\right) \ge 0, \tag{10}$$

we have

$$\underline{C}\left(\underline{\mu} \le \liminf_{n\to\infty} \frac{\sum_{i=1}^{n} a_{ni} X_i}{b_n} \le \limsup_{n\to\infty} \frac{\sum_{i=1}^{n} a_{ni} X_i}{b_n} \le \overline{\mu}\right) = 1, \tag{11}$$

where $b_n = n^{1/\alpha} \log^{1/r} n$.

Moreover, for $r > 1$, *if* $\overline{\mu}(1-\limsup_{n\to\infty}(\sum_{i=1}^n a_{ni}/b_n)) \geq 0$, *we have*

$$\underline{C}\left(\underline{\mu} \leq \liminf_{n\to\infty}\frac{\sum_{i=1}^n a_{ni}X_i}{b_n} \leq \limsup_{n\to\infty}\frac{\sum_{i=1}^n a_{ni}X_i}{b_n} \leq \overline{\mu}\right) = 1, \tag{12}$$

where $b_n = n^{1/\alpha}(\log n)^{(1+(\alpha-1)(r-1))/(1+\alpha(r-1))}$.

In order to prove Theorem 7, we need the following lemma.

Lemma 8. *Let* $\{X_i, i \geq 1\}$ *be a sequence of i.i.d. random variables satisfying* (8), *and let* $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ *be an array of constants. Truncate* $|X_i - \overline{\mu}|$ *at* Δ_n *and denote* $X_{ni} := (X_i - \overline{\mu})I_{\{|X_i-\overline{\mu}|\leq\Delta_n\}}$. *Suppose that the following conditions hold:*

(1) $|a_{ni}X_{ni}| \leq C|X_i|^\beta/\log n$ *a.s.* $\overline{C}$, *for some* $0 < \beta \leq r$ *and some constant* $C > 0$;

(2) $X_{ni}^2\sum_{i=1}^n a_{ni}^2 \leq u_n|X_i|^\delta/\log n$ *a.s.* $\overline{C}$, *for some* $\delta > 0$ *and some sequence* $\{u_n\}$ *of constants such that* $u_n \to 0$.

Then,

$$\overline{C}\left(\limsup_{n\to\infty}\sum_{i=1}^n a_{ni}X_{ni} > \varepsilon\right) = 0 \quad \forall\varepsilon > 0. \tag{13}$$

Proof. From the inequality $e^x \leq 1 + x + (1/2)x^2e^{|x|}$ for all $x \in R$, we have

$$\widehat{\mathbb{E}}\left[e^{ta_{ni}X_{ni}}\right] \leq 1 + \frac{1}{2}t^2a_{ni}^2\widehat{\mathbb{E}}\left[X_{ni}^2e^{t|a_{ni}X_{ni}|}\right] \tag{14}$$

for any $t > 0$. Let $\varepsilon > 0$ be given. We set $t = 2\log n/\varepsilon$ and obtain by (1) and (2) in Lemma 8 and (8) that

$$\begin{aligned}\widehat{\mathbb{E}}\left[e^{ta_{ni}X_{ni}}\right] &\leq 1 + \frac{2\log^2 n}{\varepsilon^2}a_{ni}^2\widehat{\mathbb{E}}\left[X_{ni}^2e^{t|a_{ni}X_{ni}|}\right] \\ &\leq 1 + \frac{2u_n\log n}{\varepsilon^2}\frac{a_{ni}^2}{\sum_{i=1}^n a_{ni}^2}\widehat{\mathbb{E}}\left[|X_i|^\delta e^{(2/\varepsilon)C|X_i|^\beta}\right] \\ &\leq 1 + \frac{2C_1u_n\log n}{\varepsilon^2}\frac{a_{ni}^2}{\sum_{i=1}^n a_{ni}^2}\widehat{\mathbb{E}}\left[e^{C(\varepsilon)|X_i|^\beta}\right] \\ &\leq 1 + \frac{\log n}{2}\frac{a_{ni}^2}{\sum_{i=1}^n a_{ni}^2} \\ &\leq \exp\left\{\frac{\log n}{2}\frac{a_{ni}^2}{\sum_{i=1}^n a_{ni}^2}\right\}\end{aligned} \tag{15}$$

for all large n. For the large n, it follows by the Markov inequality and (15) that

$$\begin{aligned}\overline{C}\left(\sum_{i=1}^n a_{ni}X_{ni} > \varepsilon\right) &\leq e^{-t\varepsilon}\widehat{\mathbb{E}}\left[e^{t\sum_{i=1}^n a_{ni}X_{ni}}\right] \\ &\leq \frac{1}{n^2}\prod_{i=1}^n \exp\left\{\frac{\log n}{2}\frac{a_{ni}^2}{\sum_{i=1}^n a_{ni}^2}\right\} \\ &= n^{-3/2}.\end{aligned} \tag{16}$$

Using Lemma 6, we have

$$\overline{C}\left(\limsup_{n\to\infty}\sum_{i=1}^n a_{ni}X_{ni} > \varepsilon\right) = 0 \quad \forall\varepsilon > 0. \tag{17}$$

The proof is complete. □

Proof of Theorem 7. The proof of (11) is similar to that of (12); we only prove (12). We denote $X'_{ni} := (X_i - \overline{\mu})I_{\{|X_i-\overline{\mu}|\leq(\log n)^{\delta_1}\}}$, $X''_{ni} := (X_i - \overline{\mu})I_{\{|X_i-\overline{\mu}|\leq(\log n)^{1/r}\}}$, and $X'''_{ni} := (X_i - \overline{\mu})I_{\{(\log n)^{\delta_1}\leq|X_i-\overline{\mu}|\leq(\log n)^{1/r}\}}$ for $1 \leq i \leq n$ and $n \geq 1$, where $\delta_1 = 1/(1+\alpha(r-1))$. Denote $a'_{ni} := a_{ni}I_{\{|a_{ni}|\leq n^{1/\alpha}(\log n)^{\delta_2}\}}$ and $a''_{ni} := a_{ni} - a'_{ni}$ for $1 \leq i \leq n$ and $n \geq 1$, where $\delta_2 = (r-1)/(1+\alpha(r-1))$. Then,

$$\begin{aligned}\frac{\sum_{i=1}^n a_{ni}(X_i - \overline{\mu})}{b_n} &= \frac{\sum_{i=1}^n a'_{ni}X'_{ni}}{b_n} + \frac{\sum_{i=1}^n a''_{ni}X'_{ni}}{b_n} \\ &\quad + \frac{\sum_{i=1}^n a_{ni}X'''_{ni}}{b_n} + \frac{\sum_{i=1}^n a_{ni}X''_{ni}}{b_n} \\ &:= A_n + B_n + C_n + D_n.\end{aligned} \tag{18}$$

For A_n, we will apply Lemma 8 to the random variable X'_{ni} and weight $b_n^{-1}a'_{ni}$. Note that

$$\begin{aligned}\left|b_n^{-1}a'_{ni}X'_{ni}\right| &\leq \frac{n^{1/\alpha}}{b_n(\log n)^{\delta_2}}|X_i| = \frac{|X_i|}{\log n} \quad \text{a.s. } \overline{C}, \\ X'^2_{ni}\sum_{i=1}^n b_n^{-2}a'^2_{ni} &\leq \frac{n^{(2-\alpha)/\alpha}\sum_{i=1}^n |a_{ni}|^\alpha}{b_n^2(\log n)^{(2-\alpha)\delta_2}}X'^2_{ni} \\ &\leq \frac{A_{\alpha,n}^\alpha}{(\log n)^{(2+\alpha(r-1))/(1+\alpha(r-1))}}X_i^2 \quad \text{a.s. } \overline{C}.\end{aligned} \tag{19}$$

Hence, by Lemma 8, we have

$$\overline{C}\left(\limsup_{n\to\infty}A_n > \frac{\varepsilon}{2}\right) = 0 \quad \forall\varepsilon > 0. \tag{20}$$

For B_n, we observe that

$$\begin{aligned}B_n \leq \frac{(\log)^{\delta_1}}{b_n}\sum_{i=1}^n\left|a''_{ni}\right| &\leq \frac{(\log n)^{(1+(\alpha-1)(r-1))/(1+\alpha(r-1))}}{b_nn^{(\alpha-1)/\alpha}} \\ &\quad \times\sum_{i=1}^n|a_{ni}|^\alpha = A_{\alpha,n}^\alpha \quad \text{a.s. } \overline{C}.\end{aligned} \tag{21}$$

Namely,

$$\overline{C}\left(\limsup_{n\to\infty}B_n \leq A_{\alpha,n}^\alpha\right) = 1. \tag{22}$$

By replacing X_i by θX_i $(\theta > 0)$, we have

$$\limsup_{n\to\infty}B_n \leq \frac{A_{\alpha,n}^\alpha}{\theta} \quad \text{a.s. } \overline{C}. \tag{23}$$

Letting $\theta \to \infty$, we have

$$\overline{C}\left(\limsup_{n\to\infty} B_n \leq 0\right) = 1. \quad (24)$$

For C_n, note that

$$\left|\frac{a_{ni}X_{ni}'''}{b_n}\right| \leq \frac{A_{\alpha,n}}{\log n}|X_i|^r \quad \text{a.s. } \overline{C},$$
$$\frac{X_{ni}'''^2 \sum_{i=1}^n a_{ni}^2}{b_n^2} \leq \frac{A_{\alpha,n}^2}{\log^2 n}|X_i|^{2r} \quad \text{a.s. } \overline{C}. \quad (25)$$

Then, by Lemma 8, we have

$$\overline{C}\left(\limsup_{n\to\infty} C_n > \frac{\varepsilon}{2}\right) = 0 \quad \forall \varepsilon > 0. \quad (26)$$

For D_n, assumption (8) implies $\sum_{i=1}^n \overline{C}(|X_n| > \log^{1/r}) < \infty$. Hence, by Lemma 6, $\sum_{i=1}^n |X_{ni}''|$ is bounded a.s. $\overline{C}$. It follows that

$$D_n \leq \frac{\sum_{i=1}^n |a_{ni}| \sum_{i=1}^n |X_{ni}''|}{b_n} \leq \frac{A_{\alpha,n}}{(\log n)^{(1+(\alpha-1)(r-1))/(1+\alpha(r-1))}} \times \sum_{i=1}^n |X_{ni}''| \longrightarrow 0 \quad \text{a.s. } \overline{C} \quad (27)$$

as $n \to \infty$.

Thus,

$$\overline{C}\left(\limsup_{n\to\infty} D_n \leq 0\right) = 1. \quad (28)$$

On the other hand, $\forall \varepsilon > 0$; and we note that

$$\begin{aligned}\overline{C}&\left(\limsup_{n\to\infty}\frac{\sum_{i=1}^n a_{ni}X_i}{b_n} \geq \overline{\mu}+\varepsilon\right)\\ &\leq \overline{C}\left(\limsup_{n\to\infty}\frac{\sum_{i=1}^n a_{ni}(X_i-\overline{\mu})}{b_n} \geq \overline{\mu}\left(1-\limsup_{n\to\infty}\frac{\sum_{i=1}^n a_{ni}}{b_n}\right)+\varepsilon\right)\\ &= \overline{C}\left(\limsup_{n\to\infty}(A_n+B_n+C_n+D_n) \geq \overline{\mu}\left(1-\limsup_{n\to\infty}\frac{\sum_{i=1}^n a_{ni}}{b_n}\right)+\varepsilon\right),\end{aligned} \quad (29)$$

and the condition $\overline{\mu}(1-\limsup_{n\to\infty}\sum_{i=1}^n a_{ni}/b_n) \geq 0$: from (20), (24), (26), and (28), we conclude that

$$\overline{C}\left(\limsup_{n\to\infty}\frac{\sum_{i=1}^n a_{ni}X_i}{b_n} \geq \overline{\mu}+\varepsilon\right) = 0 \quad \forall \varepsilon > 0, \quad (30)$$

which implies

$$\overline{C}\left(\limsup_{n\to\infty}\frac{\sum_{i=1}^n a_{ni}X_i}{b_n} > \overline{\mu}\right) = 0. \quad (31)$$

Also,

$$\underline{C}\left(\limsup_{n\to\infty}\frac{\sum_{i=1}^n a_{ni}X_i}{b_n} \leq \overline{\mu}\right) = 1. \quad (32)$$

Similarly, considering the sequence $\{-X_i, i \geq 1\}$, from (29), we have

$$\underline{C}\left(\limsup_{n\to\infty}\frac{\sum_{i=1}^n a_{ni}(-X_i)}{b_n} \leq \hat{\mathbb{E}}[-X_1]\right) = 1. \quad (33)$$

Note that $\underline{\mu} = -\hat{\mathbb{E}}[-X_1]$. So

$$\underline{C}\left(\limsup_{n\to\infty}\frac{\sum_{i=1}^n a_{ni}X_i}{b_n} \geq \underline{\mu}\right) = 1. \quad (34)$$

Therefore, the proof of Theorem 7 is complete. □

The following theorem shows that if the norming constant b_n is stronger than that of Theorem 7, then condition (8) in Theorem 7 can be replaced by a weaker condition.

Theorem 9. *Let $\{X_i, i \geq 1\}$ be a sequence of i.i.d. random variables in $\mathscr{H}$ satisfying $\overline{\mu} = \hat{\mathbb{E}}[X_1]$, $\underline{\mu} = -\hat{\mathbb{E}}[-X_1]$ and for some $h, r > 0$*

$$\hat{\mathbb{E}}\left[e^{(h|X_i|^r)}\right] < \infty. \quad (35)$$

Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying (9) in Theorem 7. Then, for $0 < r \leq 1$ and $\beta > 0$, if $\overline{\mu}(1-\limsup_{n\to\infty}(\sum_{i=1}^n a_{ni}/b_n)) \geq 0$, we have

$$\underline{C}\left(\underline{\mu} \leq \limsup_{n\to\infty}\frac{\sum_{i=1}^n a_{ni}X_i}{b_n} \leq \limsup_{n\to\infty}\frac{\sum_{i=1}^n a_{ni}X_i}{b_n} \leq \overline{\mu}\right) = 1, \quad (36)$$

where $b_n = n^{1/\alpha}(\log n)^{1/r+\beta}$.

Moreover, for $r > 1$ and $\beta > 0$, if $\overline{\mu}(1-\limsup_{n\to\infty}(\sum_{i=1}^n a_{ni}/b_n)) \geq 0$, we have

$$\underline{C}\left(\underline{\mu} \leq \limsup_{n\to\infty}\frac{\sum_{i=1}^n a_{ni}X_i}{b_n} \leq \limsup_{n\to\infty}\frac{\sum_{i=1}^n a_{ni}X_i}{b_n} \leq \overline{\mu}\right) = 1, \quad (37)$$

where $b_n = n^{1/\alpha}(\log n)^{(1+(\alpha-1)(r-1))/(1+\alpha(r-1))+\beta}$.

Proof. We can prove that Lemma 8 is also true except that (8) and (1) of Lemma 8 are replaced by (35) and the following condition.

$|a_{ni}X_{ni}| \leq \nu_n|X_i|^{\beta}/\log n$ a.s. $\overline{C}$, for some $0 < \beta \leq r$ and some sequence $\{\nu_n\}$ of constants such that $\nu_n \to 0$.

For the case $0 < r \leq 1$, we let $X'_{ni} = X_i I_{\{|X_i| \leq (h^{-1}\log n)^{1/r}\}}$ and $X''_{ni} = X_i - X'_{ni}$ for $1 \leq i \leq n$ and $n \geq 1$. For the case $r > 1$, we let $X'_{ni} = X_i I_{\{|X_i| \leq (\log n)^{1/(1+\alpha(r-1))}\}}$, $X''_{ni} = X_i I_{\{|X_i| > h^{-1}\log^{1/r} n\}}$, and $X'''_{ni} = X_i I_{\{(\log n)^{1/(1+\alpha(r-1))} < |X_i| \leq (h^{-1}\log n)^{1/r}\}}$. The rest of the proof is similar to that of Theorem 7 and is omitted. □

Remark 10. If $\overline{\mu} = \underline{\mu} = 0$ in Theorem 9, then, for $0 < r \leq 1$ and $\beta > 0$, we have

$$\frac{\sum_{i=1}^{n} a_{ni}X_i}{b_n} \longrightarrow 0 \quad \text{a.s.,} \tag{38}$$

where $b_n = n^{1/\alpha}(\log n)^{1/r+\beta}$. Moreover, if $r > 1$, we have

$$\frac{\sum_{i=1}^{n} a_{ni}X_i}{b_n} \longrightarrow 0 \quad \text{a.s.,} \tag{39}$$

where $b_n = n^{1/\alpha}(\log n)^{(1+(\alpha-1)(r-1))/(1+\alpha(r-1))+\beta}$. Since if $\alpha > 1$ and $r \geq 1$, then $r(\alpha-1)/\alpha(1+r) > 0$ and $(1+(\alpha-1)(r-1))/(1+\alpha(r-1)) < 1/r + r(\alpha-1)/\alpha(1+r)$, we also have

$$\frac{\sum_{i=1}^{n} a_{ni}X_i}{b_n} \longrightarrow 0 \quad \text{a.s.,} \tag{40}$$

where $b_n = n^{1/\alpha}(\log n)^{1/r+r(\alpha-1)/\alpha(1+r)}$. The result is similar to Theorem 2.2 of Bai and Cheng [7].

Remark 11. If $\overline{\mu} = \underline{\mu} = 0$ in Theorem 7, we can get the classical strong law of large numbers for weighted sums of i.i.d. random variables as follows:

$$P\left(\lim_{n\to\infty}\frac{\sum_{i=1}^{n} a_{ni}X_i}{b_n} = 0\right) = 1. \tag{41}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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