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Research Article

Multipliers of Modules of Continuous Vector-Valued Functions

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In 1961, Wang showed that if A is the commutative C^*-algebra $C_0(X)$ with X a locally compact Hausdorff space, then $M(C_0(X)) \cong C_b(X)$. Later, this type of characterization of multipliers of spaces of continuous scalar-valued functions has also been generalized to algebras and modules of continuous vector-valued functions by several authors. In this paper, we obtain further extension of these results by showing that $\mathrm{Hom}_{C_0(X,A)}(C_0(X,E), C_0(X,F)) \simeq C_{s,b}(X, \mathrm{Hom}_A(E,F))$, where E and F are p-normed spaces which are also essential isometric left A-modules with A being a certain commutative F-algebra, not necessarily locally convex. Our results unify and extend several known results in the literature.

1. Introduction

Characterizations of multipliers on algebras and modules of continuous functions with values in a commutative Banach or C^*-algebra A have been obtained by several authors. In 1961, Wang [1] showed that if A is taken as the commutative C^*-algebra $C_0(X)$ with X being a locally compact Hausdorff space, then $M(C_0(X)) \cong C_b(X)$. This result has also been generalized to vector-valued functions by several authors (see, e.g., [2–6]). In 1985, Lai [6] showed that if X is a locally compact abelian group and A is a commutative Banach algebra with a bounded approximate identity, then $M(C_0(X,A)) \cong C_b(X, M(A)_u)$. In 1992, Candeal Haro and Lai [3] had obtained

$$\mathrm{Hom}_{C_0(X,A)}\left(C_0(X,E), C_0(X,F)\right) \simeq C_{s,b}\left(X, \mathrm{Hom}_A(E,F)\right), \tag{1}$$

in the case when A is a commutative Banach algebra and E and F are left Banach A-modules.

A natural question arises is to investigate the extent to which these characterizations can be made beyond Banach modules. We will focus mainly on the nonlocally convex case by considering A a commutative complete p-normed algebra, $0 < p \le 1$, having a minimal approximate identity and E and F being F-spaces which are also left A-modules.

We mention that the arguments of earlier authors relied heavily on the fact that, in the case of A, a Banach algebra, $C_0(X,A)$ is isometrically isomorphic to the completed tensor product $C_0(X)\otimes_\lambda A$ with respect to the smallest cross norm λ (see [2–5]). We will avoid the use of this technique as it need not work in our case. In fact, when A is not locally convex, $\otimes_\lambda$ is no longer appropriate; even for A a complete p-normed space, many complications arise (see [7, Section 10.4]; [8, p. 100]).

2. Preliminaries

In this section, we include some basic definitions and study various classes of topological algebras considered in this paper.

Definition 1 (see [9, 10]). Let E be a vector space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

(a) A function $q : E \to \mathbb{R}$ is called an *F-seminorm* on E if it satisfies the following:

(F_1) $q(u) \ge 0$ for all $u \in E$;
(F_2) $q(u) = 0$ if $u = 0$;
(F_3) $q(\alpha u) \le q(u)$ for all $u \in E$ and $\alpha \in \mathbb{K}$ with $|\alpha| \le 1$;
(F_4) $q(u+v) \le q(u) + q(v)$ for all $u, v \in E$;
(F_5) if $\alpha_n \to 0$ in $\mathbb{K}$, then $q(\alpha_n u) \to 0$ for all $u \in E$.

(b) An F-seminorm q on E is called an *F-norm* if, for any $u \in E$, $q(u) = 0$ implies $u = 0$.

(c) An F-seminorm (or F-norm) q on E is called a *p-seminorm* (resp., *p-norm*), $0 < p \leq 1$, if it also satisfies

$$q(\alpha u) = |\alpha|^p q(u) \quad \forall u \in E,\ \alpha \in \mathbb{K}. \ (p\text{-homogeneous}). \tag{2}$$

(d) If q is an F-norm (resp., a p-norm) on a vector space E, then the pair (E, q) is called an *F-normed* (resp., a *p-normed*) *space*.

(e) An F-norm (or a p-norm) q on an algebra A is called *submultiplicative* if

$$q(ab) \leq q(a) q(b) \quad \forall a, b \in A. \tag{3}$$

An algebra A with a submultiplicative F-norm (resp., p-norm) q is called an *F-normed* (resp., *p-normed*) *algebra*.

Definition 2. (1) A net $\{e_\lambda : \lambda \in I\}$ in a topological algebra A is called an *approximate identity* if

$$\lim_\lambda e_\lambda a = \lim_\lambda a e_\lambda = a \quad \forall a \in A. \tag{4}$$

(2) An approximate identity $\{e_\lambda : \lambda \in I\}$ in an F-normed algebra (A, q) is said to be *minimal* if $q(e_\lambda) \leq 1$ for all $\lambda \in I$.

If E and F are topological vector spaces over the field $\mathbb{K} \in \{\mathbb{R}$ or $\mathbb{C}\}$, then the set of all continuous linear mappings $T : E \to F$ is denoted by $CL(E, F)$. Clearly, $CL(E, F)$ is a vector space over $\mathbb{K}$ with the usual pointwise operations. Further, if $F = E, CL(E) = CL(E, E)$ is an algebra under composition (i.e., $(ST)(u) = S(T(u)), u \in E$) and has the identity $I : E \to E$ given by $I(u) = u$ $(u \in E)$.

Definition 3. Let (E, q_E) and (F, q_F) be p-normed spaces. For any linear map $T : E \to F$, define

$$\|T\|_{q_E, q_F} = \sup \{q_F(Tu) : u \in E, q_E(u) \leq 1\}. \tag{5}$$

Then, by ([10, p. 101-102]), $T \in CL(E, F)$ if and only if $\|T\|_{q_E, q_F} < \infty$. Further, $\|\cdot\|_{q_E, q_F}$ is an F-norm on $CL(E, F)$ and, for any $T \in CL(E, F)$,

$$q_F(Tu) \leq \|T\|_{q_E, q_F} \cdot q_E(u) \quad \forall u \in E. \tag{6}$$

In particular, if $T \in CL(E) = CL(E, E)$, we denote

$$\|T\|_{q_E} := \sup \{q_E(T(u)) : u \in E, q_E(u) \leq 1\}. \tag{7}$$

In this case, for any $S, T \in CL(E)$, $\|ST\|_{q_E} \leq \|S\|_{q_E} \|T\|_{q_E}$; hence $(CL(E), \|\cdot\|_{q_E})$ is a p-normed algebra.

Definition 4. Let E and F be topological vector spaces. The *uniform operator topology* σ (resp., the *strong operator topology* s) on $CL(E, F)$ is defined as the linear topology which has a base of neighborhoods of 0 consisting of all the sets of the form

$$N(D, W) = \{T \in CL(A) : T(D) \subseteq W\}, \tag{8}$$

where D is a bounded (resp., finite) subset of E and W is a neighborhood of 0 in F. Clearly, $s \leq \sigma$. In particular, if (A, q_A) is a p-normed algebra, then the σ-topology on $CL(A)$ is the one given by the p-norm $\|\cdot\|_{A_p}$. In this setting, the strong operator topology s on $CL(A)$ is given by the family of $\{P_a : a \in A\}$ of F-seminorms, where

$$P_a(T) = q_A(T(a)), \quad T \in CL(A). \tag{9}$$

Remark 5. If (E, q_E) is a general F-algebra, then $\|T\|_{q_E}$ need not exist since the set $\{u \in E : q_E(u) \leq 1\}$ may not be bounded (see ([10, p. 8]; [11, 12]) for counterexamples).

Definition 6. Let X be a Hausdorff topological space and E a Hausdorff topological vector space over the field $\mathbb{K}$ (= $\mathbb{R}$ or $\mathbb{C}$) with a base $\mathscr{W}$ of neighborhoods of 0 in E. A function $f : X \to E$ is said to *vanish at infinity* if, for each neighborhood W of 0 in E, there exists a compact set $K = K_W \subseteq X$ such that

$$f(x) \in W \quad \forall x \in X \setminus K. \tag{10}$$

We will denote by $C_b(X, E)$ the vector space of all continuous bounded E-valued functions on X and by $C_0(X, E)$ the subspace of $C_b(X, E)$ consisting of those functions which vanish at infinity. When $E = \mathbb{K}$ (= $\mathbb{R}$ or $\mathbb{C}$), these spaces will be denoted by $C_b(X)$ and $C_0(X)$. Let $C_b(X) \otimes E$ denote the vector subspace of $C_b(X, E)$ spanned by the set of all functions of the form $\varphi \otimes u$, where $\varphi \in C_b(X)$, $u \in E$, and

$$(\varphi \otimes u)(x) = \varphi(x) u, \quad x \in X. \tag{11}$$

We mention that, if X is not locally compact, then $C_0(X, E)$ may be the trivial vector space $\{0\}$. For example, if $X = \mathbb{Q}$, the space of rationals, and $E = \mathbb{R}$, then $C_0(\mathbb{Q}, \mathbb{R}) = \{0\}$.

Remarks 7. (i) If $E = A$ is an algebra, then $C_b(X, A)$ is also an algebra with respect to the pointwise multiplication defined by

$$(fg)(x) = f(x) g(x), \quad x \in X. \tag{12}$$

(ii) If $E = A$ is a commutative algebra, then $C_b(X, A)$ is also commutative; in particular, $C_b(X)$ is a commutative algebra.

(iii) If E is only a vector space, then $C_b(X, E)$ is a $C_b(X)$-bimodule with respect to the module multiplications $(\varphi, f) \to \varphi \cdot f$ and $(f, \varphi) \to f \cdot \varphi$ defined by

$$(\varphi \cdot f)(x) = \varphi(x) f(x) = (f \cdot \varphi)(x), \quad x \in X. \tag{13}$$

(iv) If E is a vector space and A is algebra, then $C_b(X, E)$ is a left A-module with respect to the module multiplication $(a, f) \to a \cdot f$ as pointwise action:

$$(a \cdot f)(x) = af(x), \quad a \in A,\ f \in C_b(X, A),\ x \in X. \tag{14}$$

In particular, $C_0(X, E)$ is a left A-module.

Definition 8. Let X be a Hausdorff space and E a Hausdorff topological vector space (TVS) over $\mathbb{K}$ (= $\mathbb{R}$ or $\mathbb{C}$). The *uniform topology* u on $C_b(X, E)$ is the linear topology which has a base of neighborhoods of 0 consisting of all sets of the form

$$N(X,G) = \{f \in C_b(X,E) : f(X) \subseteq W\}, \quad (15)$$

where W is a neighborhood of 0 in E. In particular, if $E = (E, q_E)$ is an F-normed space, the u-topology on $C_b(X,E)$ is given by the F-norm

$$\|f\|_{q_E,\infty} = \sup_{x\in X} q_E(f(x)), \quad f \in C_b(X,E). \quad (16)$$

3. Main Results

In this section we extend some results of [2–6] from Banach modules to the more general setting of topological modules.

Definition 9 (cf. [13, 14]). Let (A, q_A) be a commutative p-normed algebra, and let (E, q_E) be a p-normed space which is also an A-module in the usual algebraic sense. Then E is called an *isometric A-module* if

$$q_F(au) \le q_A(a)\, q_F(u) \quad \text{for any } a \in A,\ u \in E. \quad (17)$$

If (A, q_A) has a minimal approximate identity $\{e_\lambda : \lambda \in I\}$, then E is called an *essential A-module* if $\lim_\lambda e_\lambda u = \lim_\lambda u e_\lambda = u$ for all $u \in E$.

Definition 10. Let (A, q_A) be a commutative p-normed algebra, and let $E = (E, q_E)$ and $F = (F, q_F)$ be p-normed spaces which are also A-modules. One writes

$$\operatorname{Hom}_A(E,F) = \{T \in CL(E,F) : T(a\cdot u) = a\cdot T(u) \text{ for any } a \in A, u \in E\}. \quad (18)$$

If E is an A-bimodule, then defining $a * T$ by

$$(a * T)(u) = T(u\cdot a) \quad (a \in A, u \in E), \quad (19)$$

$\operatorname{Hom}_A(E,F)$ becomes a left A-module. In fact, for any $a, b \in A$, $u \in E$,

$$\begin{aligned}(a * T)(b\cdot u) &= T((b\cdot u)\cdot a) = T(b\cdot(u\cdot a)) \\ &= b\cdot T(u\cdot a) = b\cdot(a * T)(u).\end{aligned} \quad (20)$$

In particular, $\operatorname{Hom}_A(A,F)$ is a left A-module. If $E = F = A$, then $\operatorname{Hom}_A(A,A) = M(A)$ is the usual multiplier algebra of A:

$$\begin{aligned}M(A) = \{T \in CL(A,A) : T(ab) = aT(b) = T(a)b \\ \forall a, b \in A\},\end{aligned} \quad (21)$$

which is a commutative algebra (without A being commutative) and has the identity $I : A \to A$, $I(x) = x$ $(x \in A)$.

Lemma 11. *Let (A, q_A) a commutative p-normed algebra having a minimal approximate identity, and let (F, q_F) be p-normed space which is an essential isometric A-bimodule. Then, for any $v \in F$,*

$$\|L_v\|_{q_F} = \|R_v\|_{q_F} = q_F(v), \quad (22)$$

where $L_v, R_v : A \to F$ are the maps given by $L_v(a) = v\cdot a$ and $R_v(a) = a\cdot v$, $a \in A$.

Proof. Let $v \in F$. Then

$$\begin{aligned}\|L_v\|_{q_A,q_F} &= \sup\{q_F(L_v(a)) : q_A(a) \le 1\} \\ &= \sup\{q_F(v\cdot a) : q_A(a) \le 1\} \\ &\le \sup\{q_A(a)\, q_F(v) : q_A(a) \le 1\} = q_F(v).\end{aligned} \quad (23)$$

On the other hand,

$$\begin{aligned}\|L_v\|_{q_A,q_F} &= \sup\{q_F(v\cdot a) : q_A(a) \le 1\} \\ &\ge q_F(v\cdot e_\lambda) \quad \forall \lambda \in I,\end{aligned} \quad (24)$$

so

$$\|L_v\|_{q_A,q_F} \ge \lim_\lambda q_F(v\cdot e_\lambda) = q_F\left(\lim_\lambda v\cdot e_\lambda\right) = q_F(v). \quad (25)$$

Hence $\|L_v\|_{q_A,q_F} = q_F(v)$. Similarly, $\|R_v\|_{q_E} = q_E(v)$. □

Lemma 12. *Let (A, q_A) a commutative p-normed algebra, and let (F, q_F) be an essential isometric A-bimodule. If A has an identity e, then $\operatorname{Hom}_A(A,F) \cong F$ and $M(A) \cong A$.*

Proof. We claim that

$$\begin{aligned}\operatorname{Hom}_A(A,F) &\cong \{L_{T(e)} : T \in \operatorname{Hom}_A(A,F)\} \\ &= \{L_v : v \in F\} \cong F.\end{aligned} \quad (26)$$

Clearly,

$$\{L_{T(e)} : T \in \operatorname{Hom}_A(A,F)\} \subseteq \{L_v : v \in F\} \subseteq \operatorname{Hom}_A(A,F). \quad (27)$$

On the other hand, if $T \in \operatorname{Hom}_A(A,F)$, then, for any $a \in A$,

$$T(a) = T(ea) = T(e)\cdot a = L_{T(e)}(a). \quad (28)$$

Hence $T = L_{T(e)}$. Further, by Lemma 11, $\|L_{T(e)}\|_{q_A,q_F} = q_F(T(e))$. Thus $\operatorname{Hom}_A(A,F) \cong F$. In particular, $M(A) \cong A$. □

Density Assumption. In the sequel, we will always assume that, for X a locally compact Hausdorff space and E a topological vector space, $C_0(X) \otimes E$ *is u-dense in* $C_0(X,E)$. This assumption is crucial for the proof of our main results. For its justification, we mention that as a consequence of the vector-valued versions of Stone-Weierstrass theorem [8, 12, 15], $C_0(X) \otimes E$ is u-dense in $C_0(X,E)$ in each of the following cases.

(a) E is locally convex.

(b) Every compact subset of X has a finite covering dimension and E is any topological vector space.

(c) E is an F-space with a basis (e.g., $E = \ell^p$ for $p > 0$).

(d) E has the approximation property.

Recall that if $T \in M(C_0(X,A))$, then $T(a\cdot f) = a\cdot T(f)$ for $f \in C_0(X,A)$ and $a \in A$ ([16, Lemma 4.5]). We also mention that if (A,q_A) is an p-normed algebra having a minimal approximate identity, then, by ([16, Lemma 4.4]), $C_0(X,A)$ has an approximate identity and hence it is a faithful topological A-module. Consequently, for any $T \in M(C_0(X,A))$, $T(fg) = fT(g) = T(f)g$ for all $f,g \in C_0(X,A)$; we will write

$$\|T\|_{q_A} := \sup\left\{q_A(T(f)) : f \in C_0(X,A), \|f\|_{q_A,\infty} \le 1\right\}. \tag{29}$$

If $T \in \mathrm{Hom}_{C_0(X,A)}(C_0(X,E),C_0(X,F))$, we let

$$\|T\|_{q_E,q_F} := \sup\left\{q_F(T(f)) : f \in C_0(X,E), \|f\|_{q_E,\infty} \le 1\right\}. \tag{30}$$

Definition 13. Now, let $E = (E,q_E)$ and $F = (F,q_F)$ be F-normed spaces. For any closed subspace $U = U_s(E,F)$ of $CL(E,F)$ endowed with the strong operator topology s, we define

$$C_{s,b}(X,U) = \{G : X \longrightarrow U : G \text{ is strongly continuous and bounded}\}. \tag{31}$$

We now define an F-norm on $C_{s,b}(X,U)$ by

$$\|G\|_{C_{s,b}} = \sup_{x\in X}\|G(x)\|_{q_E,q_F} = \sup_{x\in X}\sup_{u\in E, q_E(u)\le 1} q_F(G(x)(u)). \tag{32}$$

Then $C_{s,b}(X,U)$ is a complete p-normed space under the p-norm $\|\cdot\|_{q,\infty}$ defined in (24).

Recall that a left A-module E is called *faithful* (or *without order*) if, for any $u \in E$, $a\cdot u = 0$ for all $a \in A$ implies that $x = 0$ (cf. [13, 14]).

Lemma 14. *Let $A = (A,q_A)$ be a commutative complete p-normed algebra, and let E and F be A-modules. Then, for any $T \in \mathrm{Hom}_{C_0(X,A)}(C_0(X,E),C_0(X,F))$,*

(a) *$T(a\cdot f) = a\cdot T(f)$ for $a \in A$ and $f \in C_0(X,E)$,*

(b) *$T(\varphi\cdot f) = \varphi\cdot T(f)$ for $\varphi \in C_0(X)$ and $f \in C_0(X,E)$.*

Proof. (a) We first note that $C_0(X)$ is a Banach algebra with a bounded approximate identity, $\{\psi_\alpha\}$ (say). Then, for any $a \in A$, $u \in E$, and $\varphi \in C_0(X)$,

$$\begin{aligned}\lim_\alpha[(\psi_\alpha\otimes a)\cdot(\varphi\otimes u)] &= \lim_\alpha(\psi_\alpha\varphi\otimes a\cdot u)\\ &= \varphi\otimes a\cdot u = a(\varphi\otimes u).\end{aligned} \tag{33}$$

Since $T \in \mathrm{Hom}_{C_0(X,A)}(C_0(X,E),C_0(X,F))$ and $\psi_\alpha\otimes a \in C_0(X,A)$, $\varphi\otimes u \in C_0(X,E)$, we have

$$\begin{aligned}T(a\cdot(\varphi\otimes u)) &= \lim_\alpha T[(\psi_\alpha\otimes a)\cdot(\varphi\otimes u)]\\ &= \lim_\alpha(\psi_\alpha\otimes a)\cdot T(\varphi\otimes u)\\ &= a\cdot T(\varphi\otimes u).\end{aligned} \tag{34}$$

By T being linear and $C_0(X)\otimes E$ being assumed to be u-dense in $C_0(X,E)$, it follows that $T(a\cdot f) = a\cdot T(f)$ holds for all $f \in C_0(X,A)$ and $a \in A$.

(b) Similar to the above part. □

We now give the following characterization in the pseudoscaler case by considering both $C_0(X)$ and $C_0(X,F)$ as $C_0(X)$-modules.

Theorem 15. *Let X be a locally compact Hausdorff space and $F = (F,q_F)$ a p-normed space. Then*

$$\mathrm{Hom}_{C_0(X)}(C_0(X), C_0(X,F)) \cong C_b(X,F). \tag{35}$$

Proof. Let $T \in \mathrm{Hom}_{C_0(X)}(C_0(X),C_0(X,F))$ and $x \in X$. If $\varphi,\psi \in C_0(X)$ with $\varphi(x)\ne 0$ and $\psi(x)\ne 0$, then there is a neighborhood $N(x)$ of x in X such that

$$\varphi(t)\ne 0,\quad \psi(t)\ne 0\quad \text{for any } t \in N(x). \tag{36}$$

Since $C_0(X)$ is commutative and $C_0(X,F)$ is a $C_0(X)$-module, following as in ([1, p. 1135]), we have

$$\begin{aligned}\psi(t)(T\varphi)(t) &= T(\psi\cdot\varphi)(t) = T(\varphi\cdot\psi)(t)\\ &= \varphi(t)(T\psi)(t)\end{aligned} \tag{37}$$

and then

$$\frac{T(\psi)(t)}{\psi(t)} = \frac{(T\varphi)(t)}{\varphi(t)}\quad \text{for any } t \in N(x). \tag{38}$$

Now, for each $x \in X$ with $\varphi(x)\ne 0$, define $g_T : X \to F$ by

$$g_T(x) = \frac{(T\varphi)(x)}{\varphi(x)}. \tag{39}$$

By the above argument, the function $g_T(x)$ defined in this way is independent of the choice of $\varphi \in C_0(X)$; hence g_T is well-defined.

Clearly if $\varphi(x)\ne 0$, then $(T\varphi)(x) = g_T(x)\varphi(x)$. The equality also holds when $\varphi(x) = 0$. [To see this, choose $\psi \in C_0(X)$ such that $\psi(x)\ne 0$. Then

$$\psi(x)(T\varphi)(x) = T(\psi\varphi)(x) = \varphi(x)(T\psi)(x) = 0, \tag{40}$$

and so $T\varphi(x) = 0$.]

Next, $g_T \in C_b(X,F)$, as follows. For any $x \in X$ with $\varphi(x)\ne 0$, by Urysohn's lemma, we can choose a $\varphi \in C_0(X)$ such that $\|\varphi\|_\infty = |\varphi(x)|$. So

$$q_F[g_T(x)] = \frac{q_F[T\varphi(x)]}{|\varphi(x)|} \le \frac{\|T\|_{q_F}\|\varphi\|_\infty}{|\varphi(x)|} = \|T\|_{q_F} \tag{41}$$

for all $x \in X$. Hence $\|g_T\|_{q,\infty} \le \|T\|_{q_F}$, and so $g_T \in C_b(X,F)$.

On the other hand, since

$$q_F[(T\varphi)(x)] = q_F[g_T(x)\varphi(x)] \le \|g_T\|_{q,\infty}\|\varphi\|_\infty, \tag{42}$$

we have $\|T\|_{q_F} \le \|g_T\|_{q,\infty}$. Consequently $\|g_T\|_{q_F,\infty} = \|T\|_{q_F}$. This shows that $\mathrm{Hom}_{C_0(X)}(C_0(X),C_0(X,F))$ is isometrically embedded in $C_b(X,F)$.

Conversely, for any $g \in C_b(X,F)$, we define $T_g : C_0(X) \to C_0(X,F)$ by

$$T_g(\varphi) = g\cdot\varphi, \varphi \in C_0(X). \tag{43}$$

Then one can easily show that T_g is a multiplier from $C_0(X)$ to $C_0(X,F)$ and that $\|g\|_{q,\infty} = \|T_g\|_{q_F}$. □

Now we can establish the main theorem by considering both $C_0(X,E)$ and $C_0(X,F)$ as $C_0(X,A)$-modules.

Theorem 16. *Let $A=(A,q_A)$ be a commutative complete p-normed algebra, and let $E=(E,q_E)$ and $F=(F,q_F)$ be p-normed spaces which are also essential isometric A-modules. Then*

$$\text{Hom}_{C_0(X,A)}(C_0(X,E), C_0(X,F)) \cong C_{s,b}(X, \text{Hom}_A(E,F)). \tag{44}$$

The correspondence between the multiplier T and the function G is given by the following relation:

$$(Tf)(x) = G(x)\cdot f(x) \quad \text{for } x\in X \text{ and any } f \in C_0(X,E). \tag{45}$$

Proof. Let $T \in \text{Hom}_{C_0(X,A)}(C_0(X,E), C_0(X,F))$. Then we can define a map $\Psi_T : E \to \text{Hom}_{C_0(X)}(C_0(X), C_0(X,F))$ by

$$\Psi_T(u)(\varphi) = T(\varphi\otimes u) \quad \text{for } u\in E,\ \varphi\in C_0(X). \tag{46}$$

To see that this map is well-defined, first note that $\Psi_T(u)(\varphi) \in C_0(X,F)$. For a fixed $u\in E$, the operator $\Phi_T(u)$ defines a bounded linear operator from $C_0(X)$ into $C_0(X,F)$, since by (46),

$$\begin{aligned}\|\Psi_T(u)(\varphi)\|_{q_E,\infty} &= \|T(\varphi\otimes u)\|_{q_E,\infty}\\ &\le \|T\|_{q_E}\cdot\|\varphi\otimes a\|_{q_E,\infty};\end{aligned} \tag{47}$$

further, it is a multiplier since, for any $\varphi,\psi \in C_0(X)$,

$$\Psi_T(u)(\varphi\psi) = T(\varphi\psi\otimes u) = \varphi\cdot T(\psi\otimes u). \tag{48}$$

Hence $\Psi_T(u) \in \text{Hom}_{C_0(X)}(C_0(X), C_0(X,F))$. By Theorem 15, there exists an element, say g_u, in $C_b(X,F)$ such that

$$\Psi_T(u)(\varphi) = g_u\cdot\varphi, \quad \text{for } u\in E,\ \varphi\in C_0(X). \tag{49}$$

Now, we can define a map $G : X \to \text{Hom}_A(E,F)$ by

$$G(x)(u) = g_u(x) \quad \text{for } x\in X,\ u\in E. \tag{50}$$

To see that this map is well-defined, first note that, for a fixed $x\in X$, $G(x)$ is a linear operator from E into F. Moreover, for $a\in A$ and $\varphi \in C_0(X)$, we have

$$\begin{aligned}G(x)(a\cdot u)\cdot\varphi(u) &= g_{au}(x)\varphi(x) = T(\varphi\otimes a\cdot u)(x)\\ &= a\cdot T(\varphi\otimes u)(x) = a\cdot g_u(x)\varphi(x)\\ &= a\cdot G(x)(u)\varphi(x),\end{aligned} \tag{51}$$

or

$$G(x)(a\cdot u) = a\cdot G(x)(u). \tag{52}$$

This implies that $G(x) \in \text{Hom}_A(E,F)$, and hence $G \in C_{s,b}(X, \text{Hom}_A(E,F))$. Next we establish isometry between T and G. For $x\in X$ and $\varphi\otimes u \in C_0(X)\otimes E$ with $\|\varphi\otimes u\|_{q_E,\infty} \le 1$,

$$\begin{aligned}\|G(x)\|_{q_E,q_F} &= \sup_{q_E(u)\le 1} q_F[G(x)(u)] = \sup_{q_E(u)\le 1} q_F[g_u(x)]\\ &\le \sup_{q_E(u)\le 1}\|g_u\|_{q_F,\infty} = \sup_{\substack{q_E(u)\le 1\\ \|\varphi\|_\infty\le 1}}\|g_u\cdot\varphi\|_{q_F,\infty}\\ &= \sup_{\|\varphi\otimes u\|_{q_E,\infty}\le 1}\|T(\varphi\otimes u)\|_{q_F,\infty} = \|T\|_{q_E,q_F},\end{aligned} \tag{53}$$

since $C_0(X)\otimes E$ is u-dense in $C_0(X,E)$. So $\|G\|_{C_{s,b}} \le \|T\|_{q_E,q_F}$. But

$$\begin{aligned}\|T(\varphi\otimes u)\|_{q_F,\infty} &= \|g_u\cdot\varphi\|_{q_F,\infty} \le \|g_u\|_{q_F,\infty}\|\varphi\|_\infty\\ &\le \|G\|_{C_{s,b}}\|u\|\,\|\varphi\|_\infty = \|G\|_{C_{s,b}}\|\varphi\otimes u\|_{q_E,\infty}\end{aligned} \tag{54}$$

for all $\varphi\otimes u \in C_0(X)\otimes E$. Consequently, $\|T\|_{q_E,q_F} \le \|G\|_{C_{s,b}}$.

Conversely, let $G \in C_{s,b}(X, \text{Hom}_A(E,F))$ and $\varphi \in C_0(X)$. Then $G\cdot\varphi$ is a continuous function on X given by

$$(G\cdot\varphi)(x)(u) = (G(x)u)\varphi(x), \quad x\in X,\ u\in E. \tag{55}$$

It is easy to see that $G\cdot\varphi$ vanishes at infinity, and so $G\cdot\varphi \in C_0(X, \text{Hom}_A(E,F))$. For any $u\in E$ and $\varphi\in C_0(X)$, G determines a bounded linear operator T from $C_0(X,E)$ to $C_0(X,F)$ given by

$$T(\varphi\otimes u)(x) = (G(x)u)\varphi(x). \tag{56}$$

Again, since $C_0(X)\otimes E$ is u-dense in $C_0(X,E)$, it follows that $\|T\|_{q_E,q_F} = \|G\|_{C_{s,b}}$.

Since E and F are A-modules, for any $h\otimes a \in C_0(X)\otimes A$ and $\varphi\otimes u \in C_0(X)\otimes E$,

$$\begin{aligned}T((h\otimes a)\cdot(\varphi\otimes u)) &= T(h\varphi\otimes au)\\ &= G(\cdot)(a\cdot u)(h\varphi)(\cdot)\\ &= a\cdot h(\cdot)G(\cdot)(u)\varphi(\cdot)\\ &= (h\otimes a)\cdot T(\varphi\otimes u).\end{aligned} \tag{57}$$

Hence T is a multiplier on $C_0(X,E)$ since $C_0(X)\otimes E$ is u-dense in $C_0(X,E)$. The isometry between G and T now implies that

$$\text{Hom}_{C_0(X,A)}(C_0(X,E), C_0(X,F)) \cong C_{s,b}(X, \text{Hom}_A(E,F)). \tag{58}$$

□

4. Applications

As an application of the above results, in particular of Theorem 16, we can deduce several known results, as follows.

Corollary 17 (see [3]). *Let X be a locally compact Hausdorff space and $A = (A, \|\cdot\|)$ a commutative Banach algebra, and let E and F be Banach A-modules. Then*

$$\mathrm{Hom}_{C_0(X,A)}\left(C_0(X,E), C_0(X,F)\right) \cong C_{s,b}\left(X, \mathrm{Hom}_A(E,F)\right). \quad (59)$$

Corollary 18 (see [3, 5]). *Let X be a locally compact Hausdorff space and $A = (A, \|\cdot\|)$ be a commutative Banach algebra with identity of norm 1, and let E be a Banach A-module. Then*

$$\mathrm{Hom}_{C_0(X,A)}\left(C_0(X,E), C_0(X,E)\right) \cong C_b(X,E). \quad (60)$$

Corollary 19 (see [16]). *Let X be a locally compact Hausdorff space and $A = (A, q)$ a commutative complete p-normed algebra with a minimal approximate identity. Then*

$$M\left(C_0(X,A)\right) \cong C_{s,b}\left(X, M(A)_u\right). \quad (61)$$

Proof. This follows from the fact that $\mathrm{Hom}_A(A,A) = M(A)$. □

Corollary 20 (see [1]). *Let X be a locally compact Hausdorff space. Then*

$$M\left(C_0(X)\right) \cong C_b(X). \quad (62)$$

Proof. This follows from the fact that $\mathrm{Hom}_{C_0(X)}(C_0(X), C_0(X)) \cong C_b(X)$. □

Example 21. Let A_p, $0 < p \le 1$, denote the algebra of all holomorphic functions in the unit disc $D = \{z \in \mathbb{C} : |z| \le 1\}$:

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in D, \quad (63)$$

for which

$$\|\varphi\|_p = \sum_{n=0}^{\infty} |a_n|^p < \infty. \quad (64)$$

This is a commutative complete p-normed algebra with the pointwise multiplication and has an identity ([7, p. 135]; [17, p. 8]). In this case,

$$M\left(C_0\left(X, A_p\right)\right) \simeq C_b\left(X, M\left(A_p\right)_s\right) \simeq C_b\left(X, A_p\right). \quad (65)$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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