

Research Article

Constant Sign Solutions for Variable Exponent System Neumann Boundary Value Problems with Singular Coefficient

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We deal with the existence of constant sign solutions for the following variable exponent system Neumann boundary value problem: $-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + \lambda|u|^{p(x)-2}u = F_u(x, u, v)$ in Ω , $-\operatorname{div}(|\nabla v|^{q(x)-2}\nabla v) + \lambda|v|^{q(x)-2}v = F_v(x, u, v)$ in Ω , $\partial u/\partial\gamma = 0 = \partial v/\partial\gamma$ on $\partial\Omega$. We give several sufficient conditions for the existence of the constant sign solutions, when $F(x, \cdot, \cdot)$ satisfies neither sub- $(p(x), q(x))$ growth condition, nor Ambrosetti-Rabinowitz condition (subcritical). In particular, we obtain the existence of eight constant sign solutions.

1. Introduction

In recent years, there is a lot of interest in the study of various mathematical problems with variable exponent (see [1–32]). We refer readers to [1, 23, 31, 33] for the background of these problems. On the existence of solutions for elliptic systems with variable exponent, we refer to [15, 25, 27]. In this paper, we consider the existence of constant sign solutions for the following problem:

$$\begin{aligned} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + \lambda|u|^{p(x)-2}u &= F_u(x, u, v) && \text{in } \Omega, \\ -\operatorname{div}(|\nabla v|^{q(x)-2}\nabla v) + \lambda|v|^{q(x)-2}v &= F_v(x, u, v) && \text{in } \Omega, \\ \frac{\partial u}{\partial\gamma} = 0 = \frac{\partial v}{\partial\gamma} &&& \text{on } \partial\Omega, \end{aligned} \quad (\text{P})$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain and $\partial\Omega$ possesses the cone property, $p, q \in C(\overline{\Omega})$ and $p(x), q(x) > 1$, $-\Delta_{p(x)}u := -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called the $p(x)$ -Laplacian, $\lambda \geq 1$ is a positive parameter, and γ is the outward unit normal to $\partial\Omega$. F satisfies

$$F(x, s, t) = a(x)G(x, s, t), \quad \forall (x, s, t) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}. \quad (1)$$

We make the following assumption.

(A₀) $a \in L^{r(\cdot)}(\Omega)$ is a nontrivial nonnegative singular coefficient; $G \in C^1(\overline{\Omega} \times \mathbb{R}^2 \rightarrow \mathbb{R})$ and satisfies

$$\begin{aligned} |G_s(x, s, t)| &\leq C \left(|s|^{\alpha(x)-1} + |t|^{\beta(x)/\alpha^0(x)} + 1 \right), \\ &\forall x \in \overline{\Omega}, \\ |G_t(x, s, t)| &\leq C \left(|t|^{\beta(x)-1} + |s|^{\alpha(x)/\beta^0(x)} + 1 \right), \\ &\forall x \in \overline{\Omega}, \end{aligned} \quad (2)$$

where $r, \alpha, \beta \in C(\overline{\Omega})$, $r(x) > 1$, $1 < \alpha(x) < p^*(x)$, $1 < \beta(x) < q^*(x)$ on $\overline{\Omega}$, $G_s = (\partial/\partial s)G$, $G_t = (\partial/\partial t)G$, and satisfies

$$r(x) > \max \left\{ \left(\frac{p^*(x)}{\alpha(x)} \right)^0, (p(x))^0, \left(\frac{q^*(x)}{\beta(x)} \right)^0, (q(x))^0 \right\}, \quad (3)$$

where the notation $f^0(x)$ means the conjugate function of $f(x)$, namely, $(1/f(x)) + (1/f^0(x)) \equiv 1$, and

$$p^*(x) = \begin{cases} \frac{Np(x)}{(N-p(x))}, & p(x) < N, \\ \infty, & p(x) \geq N. \end{cases} \quad (4)$$

When $p(\cdot) \equiv p$ (a constant), $p(x)$ -Laplacian is the usual p -Laplacian. The $p(x)$ -Laplacian is nonhomogeneity. Because of its nonhomogeneity, the $p(x)$ -Laplacian possesses more complicated nonlinearity than the p -Laplacian. Many results and methods for p -Laplacian problems do not hold for $p(x)$ -Laplacian problems anymore. For the following examples.

(1⁰) if $\Omega \subset \mathbb{R}^N$ is an open bounded domain, the Rayleigh quotient

$$\lambda_{p(\cdot)} = \inf_{u \in W_0^{1,p(\cdot)}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} (1/p(x)) |\nabla u|^{p(x)} dx}{\int_{\Omega} (1/p(x)) |u|^{p(x)} dx} \quad (5)$$

is zero in general. Only under some special conditions, we have $\lambda_{p(\cdot)} > 0$. For example, $\lambda_{p(\cdot)} > 0$ if and only if $p(\cdot)$ is monotone in one-dimensional case (i.e., $N = 1$) (see [11]). It is well known that the fact that $\lambda_p > 0$ is very important in the study of p -Laplacian problems. For instance, in [34], the first eigenvalue and first eigenfunction are used to discuss the existence of positive solutions of p -Laplacian problems successfully. But the $p(x)$ -Laplacian does not have the first eigenvalue and first eigenfunction in general.

(2⁰) The norm in $L^{p(\cdot)}(\Omega)$ is of Luxemburg type (we will explain later in Section 2). It is easy to see that $\int_{\Omega} |u|^{p(x)} dx = |u|_{p(\cdot)}^{p(\xi)}$ for some $\xi \in \bar{\Omega}$. Hence the integral and the norm cannot keep the constant exponent relationship. It implies that we will have more difficulties in the study of $p(x)$ -Laplacian problems. For example, it is very difficult to get the best Sobolev imbedding constant when we deal with the critical Sobolev exponent problems. Even if the best Sobolev imbedding constant could be obtained, it is also very hard to be applied to study the critical exponent problems.

(3⁰) In [35], the authors applied the homogeneous transformation method to discuss the existence of positive solutions for a class of superlinear semipositon systems. Nonetheless, the $p(x)$ -Laplacian is nonhomogeneity; this method is very hard to be used on the $p(x)$ -Laplacian problems.

On the existence of constant sign solutions of p -Laplacian problems, we refer to [35–40]. On the results of the constant sign solutions of variable exponent differential equations, we refer to [14, 19, 29].

Regarding the existence of solutions of (P), if $F(x, \cdot, \cdot)$ satisfies the sub- (p^-, q^-) growth condition, that is,

$$\max_{x \in \bar{\Omega}} \alpha(x) < \min_{x \in \bar{\Omega}} p(x), \quad \max_{x \in \bar{\Omega}} \beta(x) < \min_{x \in \bar{\Omega}} q(x), \quad (6)$$

then the corresponding functional of (P) is coercive; if $F(x, \cdot, \cdot)$ satisfies the super- (p^+, q^+) growth condition (subcritical), that is, the following Ambrosetti-Rabinowitz condition:

$$0 < G(x, s, t) \leq \frac{s}{\theta_1} \frac{\partial}{\partial s} G(x, s, t) + \frac{t}{\theta_2} \frac{\partial}{\partial t} G(x, s, t), \quad (7)$$

$\forall x \in \bar{\Omega},$

where positive constants θ_1 and θ_2 satisfy

$$\max_{x \in \bar{\Omega}} p(x) < \theta_1 < p^*(x), \quad \max_{x \in \bar{\Omega}} q(x) < \theta_2 < q^*(x), \quad (8)$$

then the corresponding functional of (P) satisfies Palais-Smale conditions (see [15, 25]). If $F(x, \cdot, \cdot)$ satisfies the subcritical growth condition, but it satisfies neither the sub- (p^-, q^-) growth condition nor the super- (p^+, q^+) growth condition, then it would be difficult to testify that the corresponding functional is coercive or satisfying Palais-Smale conditions; the results in this case are rare.

In this paper, we deal with the existence of constant sign solutions of the problem (P), when the corresponding functional neither is coercive nor satisfies Ambrosetti-Rabinowitz condition. For example, we discuss the existence of solutions of (P), when F satisfies sub- $(p(x), q(x))$ growth condition near the origin in local; that is, the following condition

$$F(x, s, t) \geq \sigma (|s|^{\epsilon_1(x)} + |t|^{\epsilon_2(x)}),$$

$$\forall (x, s, t) \in \bar{\Omega}_0 \times (0, \delta) \times (0, \delta),$$

where $1 < \epsilon_1(x) < p(x), \quad 1 < \epsilon_2(x) < q(x) \quad \text{on } \bar{\Omega}_0,$ (9)

or F satisfies super- $(p(x), q(x))$ growth condition in local (subcritical growth); that is, the following condition:

$$0 < G(x, s, t) \leq \frac{s}{\theta_1(x)} \frac{\partial}{\partial s} G(x, s, t) + \frac{t}{\theta_2(x)} \frac{\partial}{\partial t} G(x, s, t),$$

$$x \in \bar{\Omega}, \quad s > 0, \quad t > 0, \quad |s| + |t| \geq M, \quad (10)$$

where positive functions $\theta_1(\cdot)$ and $\theta_2(\cdot)$ satisfy

$$p(x) < \theta_1(x) < p^*(x), \quad q(x) < \theta_2(x) < q^*(x), \quad (11)$$

$\forall x \in \bar{\Omega}.$

In particular, we get the existence of eight constant sign solutions of (P).

This paper is divided into four sections. In Section 2, we introduce some basic properties of the variable exponent Lebesgue-Sobolev spaces. In Section 3, several properties of $p(x)$ -Laplacian are presented. In Section 4, we give the existence results of constant sign solutions of problem (P).

2. Preliminary Results and Notations

Throughout this paper, the letters $c, c_i, C_i, i = 1, 2, \dots$, denote positive constants which may vary from line to line but are independent of the terms which will take part in any limit process.

In order to discuss the problem (P), we need some theories on space $W^{1,p(\cdot)}(\Omega)$ which we call variable exponent Sobolev space. Firstly, we state some basic properties of spaces

$W^{1,p(\cdot)}(\Omega)$ and $p(x)$ -Laplacian which we will use later (for details, see [6, 10, 12, 13]). Write

$$C_+(\overline{\Omega}) = \{h \mid h \in C(\overline{\Omega}), h(x) \geq 1 \text{ for } x \in \overline{\Omega}\},$$

$$h^+ = \operatorname{ess\,sup}_{x \in \Omega} h(x), \quad h^- = \operatorname{ess\,inf}_{x \in \Omega} h(x),$$

for any $h \in L^\infty(\Omega)$;

$$S(\Omega) = \{u \mid u \text{ is a measurable real-valued function in } \Omega\},$$

$$L^{p(\cdot)}(\Omega) = \left\{u \in S(\Omega) \mid \int_\Omega |u(x)|^{p(x)} dx < \infty\right\}. \tag{12}$$

We introduce the norm on $L^{p(\cdot)}(\Omega)$ by

$$|u|_{p(\cdot)} = \inf \left\{ \lambda > 0 \mid \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}, \tag{13}$$

and $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ becomes a Banach space; we call it variable exponent Lebesgue space.

Proposition 1 (see [6]). (i) *The space $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ is a separable, uniform convex Banach space, and its conjugate space is $L^{p^0(\cdot)}(\Omega)$, where $(1/p(x)) + (1/p^0(x)) \equiv 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p^0(\cdot)}(\Omega)$, one has*

$$\left| \int_\Omega uv \, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{(p^0)^-} \right) |u|_{p(\cdot)} |v|_{p^0(\cdot)}. \tag{14}$$

(ii) *If $p_1, p_2 \in C_+(\overline{\Omega})$, $p_1(x) \leq p_2(x)$ for any $x \in \overline{\Omega}$, then $L^{p_2(\cdot)}(\Omega) \subset L^{p_1(\cdot)}(\Omega)$, and the imbedding is continuous.*

Denote $Y = \prod_{i=1}^k L^{p_i(\cdot)}(\Omega)$ with the norm

$$\|y\|_Y = \sum_{i=1}^k |y^i|_{p_i(\cdot)}, \quad \forall y = (y^1, \dots, y^k) \in Y, \tag{15}$$

where $p_i(x) \in C_+(\overline{\Omega})$, $i = 1, \dots, m$; then Y is a Banach space.

Proposition 2. *If $f(x, y) : \Omega \times \mathbb{R}^k \rightarrow \mathbb{R}^m$ is a Caratheodory function, that is, f satisfies the following:*

- (i) *for a.e. $x \in \Omega$, $y \rightarrow f(x, y)$ is a continuous function from \mathbb{R}^k to \mathbb{R}^m ;*
- (ii) *for any $y \in \mathbb{R}^k$, $x \rightarrow f(x, y)$ is measurable.*

If there exist $\beta(x), p_1(x), \dots, p_k(x) \in C_+(\overline{\Omega})$, $\rho(x) \in L^{\beta(\cdot)}(\Omega)$, and positive constant $c > 0$ such that

$$|f(x, y)| \leq \rho(x) + c \sum_{i=1}^k |y_i|^{p_i(x)/\beta(x)} \tag{16}$$

for any $x \in \Omega$, $y \in \mathbb{R}^k$,

then the Nemytsky operator from Y to $(L^{\beta(\cdot)}(\Omega))^m$ defined by $(N_f u)(x) = f(x, u(x))$ is a continuous and bounded operator.

Proof. Similar to the proof of [41], we omit it here. □

Proposition 3 (see [6]). *If one denotes*

$$\rho(u) = \int_\Omega |u|^{p(x)} dx, \quad \forall u \in L^{p(\cdot)}(\Omega), \tag{17}$$

then

- (i) $|u|_{p(\cdot)} < 1 (= 1; > 1) \Leftrightarrow \rho(u) < 1 (= 1; > 1)$;
- (ii) $|u|_{p(\cdot)} > 1 \Rightarrow |u|_{p(\cdot)}^- \leq \rho(u) \leq |u|_{p(\cdot)}^+; |u|_{p(\cdot)} < 1 \Rightarrow |u|_{p(\cdot)}^- \geq \rho(u) \geq |u|_{p(\cdot)}^+$;
- (iii) $|u|_{p(\cdot)} \rightarrow 0 \Leftrightarrow \rho(u) \rightarrow 0; |u|_{p(\cdot)} \rightarrow \infty \Leftrightarrow \rho(u) \rightarrow \infty$.

Proposition 4 (see [6]). *If $u, u_n \in L^{p(\cdot)}(\Omega)$, $n = 1, 2, \dots$, then the following statements are equivalent to each other:*

- (1) $\lim_{n \rightarrow \infty} |u_n - u|_{p(\cdot)} = 0$;
- (2) $\lim_{n \rightarrow \infty} \rho(u_n - u) = 0$;
- (3) $u_n \rightarrow u$ in measure in Ω and $\lim_{n \rightarrow \infty} \rho(u_n) = \rho(u)$.

The spaces $W^{1,p(\cdot)}(\Omega)$ and $W^{1,q(\cdot)}(\Omega)$ are defined by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) \mid |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

$$W^{1,q(\cdot)}(\Omega) = \{v \in L^{q(\cdot)}(\Omega) \mid |\nabla v| \in L^{q(\cdot)}(\Omega)\}, \tag{18}$$

and endowed with the following norm:

$$\|u\|_{p(\cdot)} = \inf \left\{ \mu > 0 \mid \int_\Omega \left| \frac{\nabla u}{\mu} \right|^{p(x)} dx + \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\},$$

$$\|u\|_{q(\cdot)} = \inf \left\{ \mu > 0 \mid \int_\Omega \left| \frac{\nabla u}{\mu} \right|^{q(x)} dx + \int_\Omega \left| \frac{u(x)}{\mu} \right|^{q(x)} dx \leq 1 \right\}. \tag{19}$$

We denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

Proposition 5 (see [6]). (i) *$W^{1,p(\cdot)}(\Omega)$ is a separable reflexive Banach space.*

(ii) *If $\beta \in C_+(\overline{\Omega})$ and $\beta(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the imbedding from $W^{1,p(\cdot)}(\Omega)$ to $L^{\beta(\cdot)}(\Omega)$ is compact and continuous.*

Let $\beta \in C_+(\overline{\Omega})$, $\mu \in S(\Omega)$, and $\mu(x) > 0$ for a.e. $x \in \Omega$. Define

$$L_{\mu(\cdot)}^{\beta(\cdot)}(\Omega) = \left\{ u \mid u \in S(\Omega), \int_\Omega \mu(x) |u(x)|^{\beta(x)} dx < \infty \right\}, \tag{20}$$

with the norm

$$|u|_{L_{\mu(\cdot)}^{\beta(\cdot)}(\Omega)} = |u|_{(\beta(\cdot), \mu(\cdot))}$$

$$= \inf \left\{ \lambda > 0 \mid \int_\Omega \mu(x) \left| \frac{u(x)}{\lambda} \right|^{\beta(x)} dx \leq 1 \right\}; \tag{21}$$

then $L_{\mu(\cdot)}^{\beta(\cdot)}(\Omega)$ is a Banach space.

Proposition 6 (see [12]). Assume that the boundary of Ω possesses the cone property and $1 < p \in C(\bar{\Omega})$. Suppose that $\mu \in L^{r(\cdot)}(\Omega)$, $\mu(x) > 0$ for a.e. $x \in \Omega$, $r \in C(\bar{\Omega})$, and $r^- > 1$. If $\beta \in C_+(\bar{\Omega})$ and

$$1 \leq \beta(x) < \frac{r(x) - 1}{r(x)} p^*(x), \quad \forall x \in \bar{\Omega}; \quad (22)$$

then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{\beta(\cdot)}(\Omega)$ is compact.

Denote $X = W^{1,p(\cdot)}(\Omega) \times W^{1,q(\cdot)}(\Omega)$. The norm $\|\cdot\|$ on X is defined by

$$\|(u, v)\| = \max \{ \|u\|_{p(\cdot)}, \|v\|_{q(\cdot)} \}. \quad (23)$$

For any (u, v) and (φ, ψ) in X , let

$$\begin{aligned} \Phi_1(u) &= \int_{\Omega} \frac{1}{p(x)} (|\nabla u|^{p(x)} dx + \lambda |u|^{p(x)}) dx, \\ \Phi_2(v) &= \int_{\Omega} \frac{1}{q(x)} (|\nabla v|^{q(x)} dx + \lambda |v|^{q(x)}) dx, \end{aligned} \quad (24)$$

$$\Phi(u, v) = \Phi_1(u) + \Phi_2(v),$$

$$\Psi(u, v) = \int_{\Omega} F(x, u, v) dx.$$

Then

$$\begin{aligned} \Phi'(u, v)(\varphi, \psi) &= D_1\Phi(u, v)(\varphi) + D_2\Phi(u, v)(\psi), \\ \Psi'(u, v)(\varphi, \psi) &= D_1\Psi(u, v)(\varphi) + D_2\Psi(u, v)(\psi), \end{aligned} \quad (25)$$

where

$$\begin{aligned} D_1\Phi(u, v)(\varphi) &= \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla \varphi dx + \int_{\Omega} \lambda |u|^{p(x)-2} u \varphi dx \\ &= \Phi'_1(u)(\varphi), \\ D_2\Phi(u, v)(\psi) &= \int_{\Omega} |\nabla v|^{q(x)-2} \nabla v \nabla \psi dx + \int_{\Omega} \lambda |v|^{q(x)-2} v \psi dx \\ &= \Phi'_2(v)(\psi), \end{aligned}$$

$$\begin{aligned} D_1\Psi(u, v)(\varphi) &= \int_{\Omega} \frac{\partial}{\partial u} F(x, u, v) \varphi dx, \\ D_2\Psi(u, v)(\psi) &= \int_{\Omega} \frac{\partial}{\partial v} F(x, u, v) \psi dx. \end{aligned} \quad (26)$$

The integral functional associated with the problem (P) is

$$J(u, v) = \Phi(u, v) - \Psi(u, v). \quad (27)$$

Without loss of generality, we may assume that $G(x, 0, 0) = 0, \forall x \in \bar{\Omega}$. Obviously, We have

$$\begin{aligned} G(x, u, v) &= \int_0^1 [u \partial_2 G(x, tu, tv) + v \partial_3 G(x, tu, tv)] dt, \\ &\forall x \in \bar{\Omega}, \end{aligned} \quad (28)$$

where ∂_j denotes the partial derivative of G with respect to its j th variable; then the condition (A_0) holds

$$|G(x, u, v)| \leq c (|u|^{\alpha(x)} + |v|^{\beta(x)} + 1), \quad \forall x \in \bar{\Omega}. \quad (29)$$

From Propositions 2 and 6 and condition (A_0) , it is easy to see that $J \in C^1(X, \mathbb{R})$ and satisfies

$$J'(u, v)(\varphi, \psi) = D_1J(u, v)(\varphi) + D_2J(u, v)(\psi), \quad (30)$$

where

$$\begin{aligned} D_1J(u, v)(\varphi) &= D_1\Phi(u, v)(\varphi) - D_1\Psi(u, v)(\varphi), \\ D_2J(u, v)(\psi) &= D_2\Phi(u, v)(\psi) - D_2\Psi(u, v)(\psi). \end{aligned} \quad (31)$$

We say $(u, v) \in X$ is a critical point of J if

$$J'(u, v)(\varphi, \psi) = 0, \quad \forall (\varphi, \psi) \in X. \quad (32)$$

The dual space of X will be denoted by X^* ; then for any $H \in X^*$, there exists $f \in (W^{1,p(\cdot)}(\Omega))^*$, $g \in (W^{1,q(\cdot)}(\Omega))^*$ such that $H(u, v) = f(u) + g(v)$. We denote $\|\cdot\|_*$, $\|\cdot\|_{*,p(\cdot)}$, and $\|\cdot\|_{*,q(\cdot)}$ to be the norms of X^* , $(W^{1,p(\cdot)}(\Omega))^*$, and $(W^{1,q(\cdot)}(\Omega))^*$, respectively. It is well known,

$$\|H\|_* = \|f\|_{*,p(\cdot)} + \|g\|_{*,q(\cdot)}, \quad (33)$$

and $X^* = (W^{1,p(\cdot)}(\Omega))^* \times (W^{1,q(\cdot)}(\Omega))^*$. Therefore

$$\|J'(u, v)\|_* = \|D_1J(u, v)\|_{*,p(\cdot)} + \|D_2J(u, v)\|_{*,q(\cdot)}. \quad (34)$$

Proposition 7. (i) If G satisfies

$$\begin{aligned} G(x, s, t) &\geq \frac{1}{\theta_1} s G_s(x, s, t) + \frac{1}{\theta_2} t G_t(x, s, t) \geq 0 \\ &\text{for } x \in \bar{\Omega}, \quad |s|^{\theta_1} + |t|^{\theta_2} \geq 2M, \end{aligned} \quad (35)$$

then $G(x, s, t) \leq c_1 [(|s|^{\theta_1} + |t|^{\theta_2}) + 1], \forall (x, s, t) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}$.

(ii) If G satisfies

$$\begin{aligned} 0 < G(x, s, t) &\leq \frac{1}{\theta_1} s G_s(x, s, t) + \frac{1}{\theta_2} t G_t(x, s, t) \\ &\text{for } x \in \bar{\Omega}, \quad |s|^{\theta_1} + |t|^{\theta_2} \geq 2M, \end{aligned} \quad (36)$$

then $G(x, s, v) \geq c_2 [(|s|^{\theta_1} + |t|^{\theta_2}) - 1], \forall (x, s, t) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}$.

Proof. (i) Similar to the proof of [15], we omit it here. \square

3. Properties of Operators

In this section, we will discuss the properties of $p(x)$ -Laplacian and Nemytsky operator.

From Propositions 2 and 6, we can easily see that $\Phi \in C^1(X, \mathbb{R})$.

Proposition 8 (see [25]). (i) Φ is a convex functional;
 (ii) Φ' is strictly monotone; that is, for any $(u_1, v_1), (u_2, v_2) \in X$ with $(u_1, v_1) \neq (u_2, v_2)$, we have

$$(\Phi'(u_1, v_1) - \Phi'(u_2, v_2))(u_1 - u_2, v_1 - v_2) > 0; \quad (37)$$

(iii) Φ' is a mapping of type (S_+) ; that is, if $(u_n, v_n) \rightarrow (u, v)$ in X and

$$\lim_{n \rightarrow \infty} [\Phi'(u_n, v_n) - \Phi'(u, v)](u_n - u, v_n - v) \leq 0, \quad (38)$$

then $(u_n, v_n) \rightarrow (u, v)$ in X ;

(iv) $\Phi' : X \rightarrow X^*$ is a bounded homeomorphism.

Theorem 9. If the assumption (A_0) is satisfied, then $\Psi \in C^1(X, \mathbb{R})$. Moreover, Ψ and Ψ' are weakly-strongly continuous; that is, $(u_n, v_n) \rightarrow (u, v)$ in X implies $\Psi(u_n, v_n) \rightarrow \Psi(u, v)$ and $\Psi'(u_n, v_n) \rightarrow \Psi'(u, v)$ in X^* .

Proof. Suppose $\{(u_n, v_n)\}$ is a weak convergent sequence in X . From Proposition 6, we can conclude that $\{(u_n, v_n)\}$ is strong convergent in $L^{\gamma(\cdot)} \times L^{\mu(\cdot)}$, where $\gamma, \mu \in C(\bar{\Omega})$ and satisfies

$$\begin{aligned} 1 \leq \gamma(x) &< \frac{r(x) - 1}{r(x)} p^*(x), \\ 1 \leq \mu(x) &< \frac{r(x) - 1}{r(x)} q^*(x), \end{aligned} \quad (39)$$

$$\forall x \in \bar{\Omega}.$$

□

From Proposition 2, we can see that Ψ is weakly-strongly continuous, and $\Psi'(u_n, v_n) \rightarrow \Psi'(u, v)$ in X^* .

Since X is a separable and reflexive Banach space, there are $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$\begin{aligned} X &= \overline{\text{span}} \{e_j, j = 1, 2, \dots\}, \\ X^* &= \overline{\text{span}}^{w^*} \{e_j^*, j = 1, 2, \dots\}, \end{aligned} \quad (40)$$

$$\langle e_i^*, e_j \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

For convenience, we write

$$X_j = \text{span} \{e_j\}, \quad Y_k = \bigoplus_{j=1}^k X_j, \quad Z_k = \overline{\bigoplus_{j=1}^k X_j}. \quad (41)$$

Definition 10. One says that J satisfies (PS) condition in X , if any sequence $\{(u_n, v_n)\} \subset X$ such that $\{J(u_n, v_n)\}$ is bounded and $\|J'(u_n, v_n)\|_* \rightarrow 0$, as $n \rightarrow \infty$, has a convergent subsequence.

One assumes G satisfies the following condition:

(B) there exist functions $\theta_1, \theta_2 \in C^1(\bar{\Omega})$ satisfying $p(x) < \theta_1(x) < p^*(x), q(x) < \theta_2(x) < q^*(x)$ on $\bar{\Omega}$ and

$$0 \leq G(x, s, t) \leq \frac{s}{\theta_1} \frac{\partial}{\partial s} G(x, s, t) + \frac{t}{\theta_2} \frac{\partial}{\partial t} G(x, s, t), \quad (42)$$

$$\forall x \in \Omega, \quad |s|^{\theta_1} + |t|^{\theta_2} \geq 2M.$$

Lemma 11. If $(\lambda \geq 1)$ is large enough, (A_0) and (B) are satisfied; then J satisfies (PS) condition on X .

Proof. It follows from (B) that

$$F(x, s, t) \leq \frac{1}{\theta_1} s F_s(x, s, t) + \frac{1}{\theta_2} t F_t(x, s, t) + c_1 a(x), \quad (43)$$

$$\forall (x, s, t) \in \bar{\Omega} \times \mathbb{R}^2.$$

Denote

$$l_1 = \min_{x \in \bar{\Omega}} \left(\frac{1}{p(x)} - \frac{1}{\theta_1(x)} \right), \quad (44)$$

$$l_2 = \min_{x \in \bar{\Omega}} \left(\frac{1}{q(x)} - \frac{1}{\theta_2(x)} \right).$$

It is easy to see that $l_1, l_2 > 0$.

Let $\{(u_n, v_n)\}$ be a (PS) sequence. By computation,

$$\begin{aligned} &c + (\|u_n\|_{p(\cdot)} + \|v_n\|_{q(\cdot)}) \\ &\geq J(u_n, v_n) - J'(u_n, v_n) \left(\frac{1}{\theta_1(x)} u_n, \frac{1}{\theta_2(x)} v_n \right) \\ &= \int_{\Omega} \frac{1}{p(x)} (|\nabla u_n|^{p(x)} + \lambda |u_n|^{p(x)}) dx \\ &\quad + \int_{\Omega} \frac{1}{q(x)} (|\nabla v_n|^{q(x)} + \lambda |v_n|^{q(x)}) dx \\ &\quad - \int_{\Omega} F(x, u_n, v_n) dx \\ &\quad - \int_{\Omega} \frac{1}{\theta_1(x)} (|\nabla u_n|^{p(x)} + \lambda |u_n|^{p(x)}) dx \\ &\quad + \int_{\Omega} \frac{1}{\theta_1(x)} u_n F_u(x, u_n, v_n) dx \\ &\quad + \int_{\Omega} \frac{1}{\theta_1^2(x)} u_n |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \theta_1(x) dx \\ &\quad - \int_{\Omega} \frac{1}{\theta_2(x)} (|\nabla v_n|^{q(x)} + \lambda |v_n|^{q(x)}) dx \\ &\quad + \int_{\Omega} \frac{1}{\theta_2(x)} v_n F_v(x, u_n, v_n) dx \\ &\quad + \int_{\Omega} \frac{1}{\theta_2^2(x)} v_n |\nabla v_n|^{q(x)-2} \nabla v_n \nabla \theta_2(x) dx \end{aligned}$$

$$\begin{aligned}
&\geq \int_{\Omega} \left(\frac{1}{p(x)} - \frac{1}{\theta_1(x)} \right) (|\nabla u_n|^{p(x)} + \lambda |u_n|^{p(x)}) dx \\
&\quad + \int_{\Omega} \left(\frac{1}{q(x)} - \frac{1}{\theta_2(x)} \right) (|\nabla v_n|^{q(x)} + \lambda |v_n|^{q(x)}) dx \\
&\quad - \int_{\Omega} \frac{|\nabla \theta_1(x)|}{\theta_1^2(x)} |u_n| |\nabla u_n|^{p(x)-1} dx \\
&\quad - \int_{\Omega} \frac{|\nabla \theta_2(x)|}{\theta_2^2(x)} |v_n| |\nabla v_n|^{q(x)-1} dx - c_2 \\
&\geq l_1 \int_{\Omega} (|\nabla u_n|^{p(x)} + \lambda |u_n|^{p(x)}) dx \\
&\quad + l_2 \int_{\Omega} (|\nabla v_n|^{q(x)} + \lambda |v_n|^{q(x)}) dx \\
&\quad - \int_{\Omega} \frac{|\nabla \theta_1(x)|}{\theta_1^2(x)} |u_n| |\nabla u_n|^{p(x)-1} dx \\
&\quad - \int_{\Omega} \frac{|\nabla \theta_2(x)|}{\theta_2^2(x)} |v_n| |\nabla v_n|^{q(x)-1} dx - c_2.
\end{aligned} \tag{45}$$

From Young inequality, we have

$$\begin{aligned}
&\frac{|\nabla \theta_1(x)|}{\theta_1^2(x)} |u_n| |\nabla u_n|^{p(x)-1} \\
&\leq C_1 \frac{1}{p(x)} \left(\frac{1}{\varepsilon_1} |u_n| \right)^{p(x)} \\
&\quad + C_1 \frac{p(x)-1}{p(x)} \left(\varepsilon_1 |\nabla u_n|^{p(x)-1} \right)^{p(x)/(p(x)-1)} \\
&= C_1 \frac{1}{p(x)} \frac{1}{\varepsilon_1^{p(x)}} |u_n|^{p(x)} \\
&\quad + C_1 \frac{p(x)-1}{p(x)} \varepsilon_1^{p(x)/(p(x)-1)} |\nabla u_n|^{p(x)}.
\end{aligned} \tag{46}$$

Similarly, we have

$$\begin{aligned}
&\frac{|\nabla \theta_2(x)|}{\theta_2^2(x)} |v_n| |\nabla v_n|^{q(x)-1} \\
&\leq C_2 \frac{1}{q(x)} \frac{1}{\varepsilon_1^{q(x)}} |v_n|^{q(x)} \\
&\quad + C_2 \frac{q(x)-1}{q(x)} \varepsilon_1^{q(x)/(q(x)-1)} |\nabla v_n|^{q(x)}.
\end{aligned} \tag{47}$$

Assume ε_1 is a small enough positive constant and $\lambda \geq 1$ is large enough; we have

$$\begin{aligned}
&c + (\|u_n\|_{p(\cdot)} + \|v_n\|_{q(\cdot)}) \\
&\geq J(u_n, v_n) - J'(u_n, v_n) \left(\frac{1}{\theta_1(x)} u_n, \frac{1}{\theta_2(x)} v_n \right) \\
&\geq \frac{2l_1}{3} \int_{\Omega} (|\nabla u_n|^{p(x)} + \lambda |u_n|^{p(x)}) dx \\
&\quad + \frac{2l_2}{3} \int_{\Omega} (|\nabla v_n|^{q(x)} + \lambda |v_n|^{q(x)}) dx - C.
\end{aligned} \tag{48}$$

Thus $\{\|u_n\|_{p(\cdot)}\}$ and $\{\|v_n\|_{q(\cdot)}\}$ are bounded. Therefore $\{(u_n, v_n)\}$ has a weak convergent subsequence (which we still denote by $\{(u_n, v_n)\}$) such that $(u_n, v_n) \rightharpoonup (u, v)$ as $n \rightarrow +\infty$. According to Theorem 9, we have $\Psi'(u_n, v_n) \rightarrow \Psi'(u, v)$ as $n \rightarrow +\infty$. Since $J'(u_n, v_n) = \Phi'(u_n, v_n) - \Psi'(u_n, v_n) \rightarrow 0$ as $n \rightarrow +\infty$, we have $\Phi'(u_n, v_n) \rightarrow \Psi'(u, v)$ as $n \rightarrow +\infty$. Since Φ' is a homeomorphism, we have that $\{(u_n, v_n)\}$ is strong convergent in X . \square

4. Existence and Multiplicity of Solutions

In this section, using the critical point theory, we will discuss the existence and multiple existence of constant sign solutions of problem (P).

Definition 12. One calls $(u, v) \in X$ is a weak solution of (P) if

$$\begin{aligned}
&\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi dx + \int_{\Omega} \lambda |u|^{p(x)-2} u \cdot \varphi dx \\
&= \int_{\Omega} F_u(x, u, v) \varphi dx, \quad \forall \varphi \in W^{1,p(\cdot)}(\Omega), \\
&\int_{\Omega} |\nabla v|^{q(x)-2} \nabla v \cdot \nabla \psi dx + \int_{\Omega} \lambda |v|^{q(x)-2} v \cdot \psi dx \\
&= \int_{\Omega} F_v(x, u, v) \psi dx, \quad \forall \psi \in W^{1,q(\cdot)}(\Omega).
\end{aligned} \tag{49}$$

It is easy to see that the critical point of J is a solution of (P).

Denote $G^+(x, u, v) = G(x, S(u), S(v))$, $F^+(x, u, v) = F(x, S(u), S(v))$, where $S(t) = \max\{0, t\}$. For any $(u, v) \in X$, we say (u, v) belong to the first, the second, the third, or the fourth quadrant of X , if $u \geq 0$ and $v \geq 0$, $u \leq 0$ and $v \geq 0$, $u \leq 0$ and $v \leq 0$, and $u \geq 0$ and $v \leq 0$, respectively.

Definition 13. (i) One calls that F satisfies sub- $(p(x), q(x))$ growth condition near the origin in the first quadrant of X , if it satisfies the following.

$$\begin{aligned}
&(A_1) \\
&F(x, s, t) \geq \sigma (|s|^{\varepsilon_1(x)} + |t|^{\varepsilon_2(x)}), \\
&\forall (x, s, t) \in \overline{\Omega}_0 \times (0, \delta) \times (0, \delta),
\end{aligned} \tag{50}$$

where $1 \leq \varepsilon_1(x) < p(x)$, $1 \leq \varepsilon_2(x) < q(x)$ on $\overline{\Omega}_0$.

(ii) One calls that F satisfies super- $(p(x), q(x))$ growth condition near the infinity in the first quadrant of X , if it satisfies

$$(A_2) \quad 0 < G_t(x, s, t) \leq (1/\theta_1)sG_s(x, s, t) + (1/\theta_2)tG_t(x, s, t) \text{ for } x \in \bar{\Omega} \text{ and } |s|^{\theta_1} + |t|^{\theta_2} \geq 2M > 0 \text{ with } s, t > 0.$$

Remark. (i) Similarly, we can give the definitions of F satisfying sub- $(p(x), q(x))$ growth condition near the origin, super- $(p(x), q(x))$ growth condition near the origin or near the infinity in the second, the third, and the fourth quadrant of X , respectively.

(ii) We say F satisfies sub- $(p(x), q(x))$ growth condition near the origin, super- $(p(x), q(x))$ growth condition near the infinity in X , if F satisfies corresponding growth condition in every quadrant of X .

(iii) We say F satisfies some growth condition in local, if it satisfies some growth condition in a quadrant.

We will discuss the existence of solutions in the following three cases:

Case (I): F satisfies sub- $(p(x), q(x))$ growth condition near the origin in local;

Case (II): F satisfies super- $(p(x), q(x))$ growth condition near the infinity in local;

Case (III): F satisfies sub- $(p(x), q(x))$ growth condition near the origin and super- $(p(x), q(x))$ growth condition near the infinity in local.

4.1. Case (I). We assume

(S) F satisfies $F_s(x, s, t) = F_t(x, s, t) = 0, \forall x \in \bar{\Omega}, \forall s, t \in \mathbb{R}$ with $st = 0$.

Theorem 14. *If $\lambda > 0$ is large enough and F satisfies (A_0) , (S), and sub- $(p(x), q(x))$ growth condition near the origin in the first quadrant of X , then the problem (P) has a nontrivial constant sign solution in the first quadrant of X .*

Proof. It is easy to check that $G^+(x, s, t) \in C^1(\bar{\Omega} \times \mathbb{R}^2, \mathbb{R})$, and

$$\begin{aligned} F_u^+(x, u, v) &= F_u(x, S(u), S(v)), \\ F_v^+(x, u, v) &= F_v(x, S(u), S(v)). \end{aligned} \tag{51}$$

Let us consider the following auxiliary problem:

$$\begin{aligned} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + \lambda|u|^{p(x)-2}u &= F_u^+(x, u, v) \quad \text{in } \Omega, \\ -\operatorname{div}(|\nabla v|^{q(x)-2}\nabla v) + \lambda|v|^{q(x)-2}v &= F_v^+(x, u, v) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \gamma} = 0 = \frac{\partial v}{\partial \gamma} &\quad \text{on } \partial\Omega. \end{aligned} \tag{P^+}$$

The corresponding functional is

$$J^+(u, v) = \Phi(u, v) - \Psi^+(u, v), \quad \forall (u, v) \in X, \tag{52}$$

where

$$\begin{aligned} \Psi^+(u, v) &= \int_{\Omega} F^+(x, u, v) dx \\ &= \int_{\Omega} F(x, S(u), S(v)) dx, \quad \forall (u, v) \in X. \end{aligned} \tag{53}$$

We divide Ω into several disjoint subsets $\Omega_1, \dots, \Omega_{n_0}$ which satisfy

$$\begin{aligned} \min_{x \in \Omega_j} \frac{p^*(x)}{r^0(x)} &> \max_{x \in \Omega_j} \alpha(x) > \max_{x \in \Omega_j} p(x), \quad j = 1, \dots, n_0, \\ \min_{x \in \Omega_j} \frac{q^*(x)}{r^0(x)} &> \max_{x \in \Omega_j} \beta(x) > \max_{x \in \Omega_j} q(x), \quad j = 1, \dots, n_0. \end{aligned} \tag{54}$$

In the following, we denote

$$\begin{aligned} f_j^- &= \min_{x \in \Omega_j} f(x), \quad f_j^+ = \max_{x \in \Omega_j} f(x) \\ j &= 1, \dots, n_0, \quad \forall f \in C(\bar{\Omega}), \\ \Phi_{\Omega_j}(u, v) &= \int_{\Omega_j} \frac{1}{p(x)} (|\nabla u|^{p(x)} + \lambda|u|^{p(x)}) dx \\ &\quad + \int_{\Omega_j} \frac{1}{q(x)} (|\nabla v|^{q(x)} + \lambda|v|^{q(x)}) dx, \\ &\quad \forall (u, v) \in X. \end{aligned} \tag{55}$$

Suppose $\|(u, v)\| = \delta$ is small enough. By (A_0) and Proposition 6, we have

$$\begin{aligned} J^+(u, v) &= \sum_{j=1}^{n_0} \left\{ \Phi_{\Omega_j}(u, v) - \int_{\Omega_j} F^+(x, u, v) dx \right\} \\ &\geq \sum_{j=1}^{n_0} \left\{ \Phi_{\Omega_j}(u, v) \right. \\ &\quad \left. - \int_{\Omega_j} a(x) [(|u| + |v|) \right. \\ &\quad \left. + C(|u|^{\alpha_j^+} + |v|^{\beta_j^+})] dx \right\} \\ &\geq \sum_{j=1}^{n_0} \left\{ \Phi_{\Omega_j}(u, v) - \int_{\Omega_j} a(x) (|u| + |v|) dx \right. \\ &\quad \left. - C(\|u\|^{\alpha_j^+} + \|v\|^{\beta_j^+}) \right\}. \end{aligned} \tag{56}$$

Note that $\alpha_j^+ > p_j^+$ and $\beta_j^+ > q_j^+$. When $\|(u, v)\| = \delta$ is small enough, it follows from Proposition 3 that

$$\begin{aligned} J^+(u, v) &\geq \sum_{j=1}^{n_0} \left\{ \frac{1}{2} \Phi_{\Omega_j}(u, v) - \int_{\Omega_j} a(x) (|u| + |v|) dx \right\} \\ &\geq \frac{1}{2} \Phi(u, v) - \int_{\Omega} a(x) (|u| + |v|) dx \\ &\geq \frac{1}{2p^+} \|u\|_{p(\cdot)}^{p^+} + \frac{1}{2q^+} \|v\|_{q(\cdot)}^{q^+} + \frac{\lambda - 1}{2p^+} |u|_{p(\cdot)}^{p^+} \\ &\quad + \frac{\lambda - 1}{2q^+} |v|_{q(\cdot)}^{q^+} - \int_{\Omega} a(x) (|u| + |v|) dx. \end{aligned} \tag{57}$$

Obviously,

$$\begin{aligned} \int_{\Omega} |a| (|u| + |v|) dx &\leq (|a|_{p^0(\cdot)} |u|_{p(\cdot)} + |a|_{q^0(\cdot)} |v|_{q(\cdot)}) \\ &\leq (|a|_{p^0(\cdot)} + |a|_{q^0(\cdot)}) (|u|_{p(\cdot)} + |v|_{q(\cdot)}) \\ &=: C_* (|u|_{p(\cdot)} + |v|_{q(\cdot)}). \end{aligned} \tag{58}$$

Noting that $\|(u, v)\| = \max\{\|u\|_{p(\cdot)}, \|v\|_{q(\cdot)}\} = \delta$, without loss of generality, we may assume that $\|u\|_{p(\cdot)} = \delta$.

Assume $C_* |u|_p \leq (1/4p^+) \|u\|_{p(\cdot)}^{p^+}$. Obviously

$$\frac{1}{2p^+} \|u\|_{p(\cdot)}^{p^+} - C_* |u|_p \geq \frac{1}{4p^+} \|u\|_{p(\cdot)}^{p^+}. \tag{59}$$

Assume $C_* |u|_{p(\cdot)} > (1/4p^+) \|u\|_{p(\cdot)}^{p^+} = (1/4p^+) \delta^{p^+}$. When λ is large enough, then we have

$$\frac{\lambda - 1}{2p^+} |u|_{p(\cdot)}^{p^+} \geq \frac{\lambda - 1}{2p^+} \left(\frac{1}{4p^+ C_*} \delta^{p^+} \right)^{p^+} \geq 4C_* \delta \geq 4C_* |u|_{p(\cdot)}. \tag{60}$$

Thus, when λ is large enough, we have

$$\frac{1}{2p^+} \|u\|_{p(\cdot)}^{p^+} + \frac{\lambda - 1}{2p^+} |u|_{p(\cdot)}^{p^+} - C_* |u|_p \geq \frac{1}{4p^+} \|u\|_{p(\cdot)}^{p^+} = \frac{1}{4p^+} \delta^{p^+}. \tag{61}$$

Assume $C_* |v|_{q(\cdot)} \leq (1/8p^+) \|u\|_{p(\cdot)}^{p^+} = (1/8p^+) \delta^{p^+}$. We have

$$\begin{aligned} &\frac{1}{2p^+} \|u\|_{p(\cdot)}^{p^+} + \frac{1}{2q^+} \|v\|_{q(\cdot)}^{q^+} + \frac{\lambda - 1}{2p^+} |u|_{p(\cdot)}^{p^+} + \frac{\lambda - 1}{2q^+} |v|_{q(\cdot)}^{q^+} \\ &\quad - C_* (|u|_p + |v|_q) \\ &\geq \frac{1}{4p^+} \delta^{p^+} - C_* |v|_{q(\cdot)} \geq \frac{1}{8p^+} \delta^{p^+}. \end{aligned} \tag{62}$$

Assume $C_* |v|_{q(\cdot)} \geq (1/8p^+) \|u\|_{p(\cdot)}^{p^+} = (1/8p^+) \delta^{p^+}$. Similar to the above discussion, when λ is large enough, we have

$$\begin{aligned} &\frac{1}{2p^+} \|u\|_{p(\cdot)}^{p^+} + \frac{1}{2q^+} \|v\|_{q(\cdot)}^{q^+} + \frac{\lambda - 1}{2p^+} |u|_{p(\cdot)}^{p^+} + \frac{\lambda - 1}{2q^+} |v|_{q(\cdot)}^{q^+} \\ &\quad - C_* (|u|_p + |v|_q) \\ &\geq \frac{1}{4q^+} \|u\|_{p(\cdot)}^{p^+}. \end{aligned} \tag{63}$$

Therefore, when λ is large enough, we have

$$J^+(u, v) \geq \max \left\{ \frac{1}{4p^+} \|u\|_{p(\cdot)}^{p^+}, \frac{1}{4q^+} \|v\|_{q(\cdot)}^{q^+} \right\} > C > 0, \tag{64}$$

when $\|(u, v)\| = \delta$.

It is easy to see that J^+ is weak lower semicontinuous. Therefore, $J^+(u, v)$ can archive its infimum at some point (u_*, v_*) . Obviously, $(u_*^-, v_*^-) := (S(-u_*), S(-v_*)) \in X$. Taking (u_*^-, v_*^-) as a test function for (P^+) , it is easy to see that $(u_*^-, v_*^-) = (0, 0)$. Thus (u_*, v_*) is nonnegative, and then it is a constant sign solution of (P) .

Note that (50) is satisfied. Without loss of generality, we may assume that

$$\max_{x \in \bar{\Omega}_0} \epsilon_1(x) < \min_{x \in \bar{\Omega}_0} p(x), \quad \max_{x \in \bar{\Omega}_0} \epsilon_2(x) < \min_{x \in \bar{\Omega}_0} q(x). \tag{65}$$

Take $u_0, v_0 \in C_0^2(\bar{\Omega}_0)$ which are nontrivial nonnegative. It is easy to see that

$$\begin{aligned} J^+(tu_0, tv_0) &= \Phi(tu_0, tv_0) - \Psi^+(tu_0, tv_0) \\ &\leq \Phi(tu_0, tv_0) - \sigma \int_{\Omega} (|tu_0|^{\epsilon_1(x)} + |tv_0|^{\epsilon_2(x)}) dx \\ &< 0 \quad \text{as } t \rightarrow 0^+. \end{aligned} \tag{66}$$

Thus $J(u, v)$ has at least one nontrivial critical point (u_*, v_*) in the first quadrant of X with $J(u_*, v_*) < 0$. Thus, (u_*, v_*) is a nontrivial constant sign solution of (P) . According to condition (S), it is easy to see that u_* and v_* are all nontrivial. \square

Theorem 15. *If $\lambda > 0$ is large enough and F satisfies (A_0) , (S) , and sub- $(p(x), q(x))$ growth condition near the origin in X , then problem (P) has at least four nontrivial constant sign solutions.*

Proof. (i) Similar to the proof of Theorem 14, we can see that (P) has a nontrivial constant sign (u_i, v_i) in the i th quadrant of X , such that $J(u_i, v_i) < 0, i = 1, 2, 3, 4$. According to condition (S), u_i and v_i are both nontrivial. Thus (P) has at least four constant sign solutions. \square

4.2. Case (II)

Theorem 16. *If $\lambda > 0$ is large enough and F satisfies (A_0) , (S) , and the super- $(p(x), q(x))$ growth condition near the infinity*

in the first quadrant of X , then (P) has a nontrivial constant sign solution in the first quadrant of X .

Proof. It is easy to check that $G^+(x, s, t) \in C^1(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$ and

$$\begin{aligned} F_u^+(x, u, v) &= F_u(x, S(u), S(v)), \\ F_v^+(x, u, v) &= F_v(x, S(u), S(v)). \end{aligned} \tag{67}$$

Let us consider the auxiliary problem (P^+) . The corresponding functional is

$$J^+(u, v) = \Phi(u, v) - \Psi^+(u, v), \quad \forall (u, v) \in X, \tag{68}$$

where

$$\begin{aligned} \Psi^+(u, v) &= \int_{\Omega} F^+(x, u, v) dx = \int_{\Omega} F(x, S(u), S(v)) dx, \\ &\quad \forall (u, v) \in X. \end{aligned} \tag{69}$$

We will prove that J^+ satisfies the conditions of Mountain Pass Lemma.

It is easy to see that G^+ satisfies

$$0 \leq G^+(x, s, t) \leq \frac{1}{\theta_1} s G_s^+(x, s, t) + \frac{1}{\theta_2} t G_t^+(x, s, t) + M_* \tag{70}$$

for $x \in \overline{\Omega}$ and $|s|^{\theta_1} + |t|^{\theta_2} \geq 2M > 0$, where

$$\begin{aligned} M_* &= \max_{x \in \overline{\Omega}} \max_{|s|^{\theta_1} + |t|^{\theta_2} \leq 2M} [|G(x, s, t)| + |s G_s(x, s, t)| \\ &\quad + |t G_t(x, s, t)|]. \end{aligned} \tag{71}$$

From Lemma 11, we can see that $J^+(u, v)$ satisfies (PS) condition in X .

From (29) and Proposition 7, we have

$$\begin{aligned} G^+(x, u, v) &\geq c_1 [(|u|^{\theta_1} + |v|^{\theta_2}) - 1], \\ \forall (x, u, v) &\in \overline{\Omega} \times [1, +\infty) \times [1, +\infty). \end{aligned} \tag{72}$$

In the following, we denote

$$\begin{aligned} f_j^- &= \min_{x \in \Omega_j} f(x), & f_j^+ &= \max_{x \in \overline{\Omega}_j} f(x), \\ j &= 1, \dots, n_0, & \forall f &\in C(\overline{\Omega}), \\ \Phi_{\Omega_j}(u, v) &= \int_{\Omega_j} \frac{1}{p(x)} (|\nabla u|^{p(x)} + \lambda |u|^{p(x)}) dx \\ &\quad + \int_{\Omega_j} \frac{1}{q(x)} (|\nabla v|^{q(x)} + \lambda |v|^{q(x)}) dx, \\ &\quad \forall (u, v) \in X. \end{aligned} \tag{73}$$

Without loss of generality, we may assume that

$$\theta_{1,j}^+ < \min_{x \in \overline{\Omega}_j} \frac{p^*(x)}{r^0(x)}, \quad \theta_{2,j}^+ < \min_{x \in \overline{\Omega}_j} \frac{q^*(x)}{r^0(x)}. \tag{74}$$

Similar to the proof of Theorem 14, when $\lambda > 0$ is large enough, we can get

$$\begin{aligned} J^+(u, v) &\geq \sum_{j=1}^{n_0} \frac{1}{4} \Phi_{\Omega_j}(u, v) \geq \max \left\{ \frac{1}{4p^+} \|u\|_{p(\cdot)}^{p^+}, \frac{1}{4q^+} \|v\|_{q(\cdot)}^{q^+} \right\} \\ &> C > 0, \quad \text{when } \|(u, v)\| = \delta. \end{aligned} \tag{75}$$

For fixed $(u_0, v_0) \in X$ with $u_0, v_0 \geq 1$ and $t > 1$, we have

$$\begin{aligned} J^+(tu_0, tv_0) &= \Phi(tu_0, tv_0) - \int_{\Omega} F^+(x, tu_0, tv_0) dx \\ &\leq \Phi(tu_0, tv_0) \\ &\quad - \int_{\Omega} |a(x)| (t^{\theta_1} |u_0|^{\theta_1} + t^{\theta_2} |v_0|^{\theta_2}) dx + c \\ &\leq \sum_{j=1}^{n_0} \left\{ \Phi_j(tu_0, tv_0) \right. \\ &\quad \left. - \int_{\Omega_j} |a(x)| (t^{\theta_1} |u_0|^{\theta_1} + t^{\theta_2} |v_0|^{\theta_2}) dx \right\} \\ &\quad + c. \end{aligned} \tag{76}$$

Since $p_j^+ < \theta_{1,j}^-$ and $q_j^+ < \theta_{2,j}^-$ on $\overline{\Omega}_j$, $J^+(tu_0, tv_0) \rightarrow -\infty$ as $t \rightarrow +\infty$. Obviously, $J^+(0, 0) = 0$; then J^+ satisfies the conditions of Mountain Pass Lemma (see [42, 43]). So, we can conclude that $J^+(u, v)$ has at least one nontrivial critical point (u_*, v_*) with $J(u_*, v_*) > 0$. Obviously, $(u_*^-, v_*^-) := (S(-u_*), S(-v_*)) \in X$. Taking (u_*^-, v_*^-) as a test function, it is easy to see that $(u_*^-, v_*^-) = (0, 0)$. Thus (u_*, v_*) is nontrivial nonnegative; then it is a constant sign solution of (P). According to condition (S), it is easy to see that u_* and v_* are both nontrivial and satisfy $J(u_*, v_*) > 0$. \square

Theorem 17. *If $\lambda > 0$ is large enough and F satisfies (A_0) , (S), and the super- $(p(x), q(x))$ growth condition near infinity in X , then (P) has four nontrivial constant sign solutions.*

Proof. As F satisfies the super- $(p(x), q(x))$ growth condition in X , then F satisfies the super- $(p(x), q(x))$ growth condition in every quadrant of X . By Theorem 16, we can see that (P) has a solution (u_i, v_i) in the i th quadrant of X and satisfies $J(u_i, v_i) > 0$ and u_i and v_i ($i = 1, 2, 3, 4$) are both nontrivial. \square

4.3. Case (III)

Theorem 18. *If $\lambda > 0$ is large enough and F satisfies (A_0) , (S), sub- $(p(x), q(x))$ growth condition near the origin, and super- $(p(x), q(x))$ growth condition near the infinity in the first quadrant of X , then (P) has two nontrivial constant sign solutions in the first quadrant of X .*

Proof. Similar to the proof of Theorem 16, we can get the existence of solution (u_1, v_1) in the first quadrant in X , which

satisfies $J(u_1, v_1) > 0$, and u_1 and v_1 both are nontrivial. By Theorem 14, (P) has the second solution (u_1^*, v_1^*) in the first quadrant in X , which satisfies $J(u_1^*, v_1^*) < 0$, and u_1^* and v_1^* both are nontrivial. Therefore, (P) has at least two constant sign solutions. \square

Note. Let

$$F(x, u, v) = \begin{cases} |u|^{\alpha_1(x)} |v|^{\beta_1(x)}, & u > 0, v > 0 \\ |u|^{\alpha_2(x)} |v|^{\beta_2(x)}, & \text{rest,} \end{cases} \quad (77)$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in C_+(\overline{\Omega})$ and satisfy

$$\begin{aligned} p(x) < \alpha_1(x), \quad q(x) < \beta_1(x), \\ \frac{\alpha_1(x)}{p^*(x)} + \frac{\beta_1(x)}{q^*(x)} < 1, \end{aligned} \quad (78)$$

$\alpha_2(x) - p(x)$ and $\beta_2(x) - q(x)$ are all changed sign functions, and

$$\frac{\alpha_2(x)}{p^*(x)} + \frac{\beta_2(x)}{q^*(x)} < 1; \quad (79)$$

then we can see that the functional J^+ satisfies (PS) condition and (P) has a nontrivial constant sign solution, but J does not satisfy the Ambrosetti-Rabinowitz condition, and it is not coercive.

Theorem 19. *If $\lambda > 0$ is large enough and F satisfies (A_0) , (S), sub- $(p(x), q(x))$ growth condition near the origin, and super- $(p(x), q(x))$ growth condition near the infinity in X , then (P) has eight nontrivial constant sign solutions in X .*

Proof. Similar to the proof of Theorem 18, we can see that (P) has nontrivial solutions (u_i, v_i) and (u_i^*, v_i^*) in the i th quadrant in X , which satisfy $J(u_i, v_i) > 0$ and $J(u_i^*, v_i^*) < 0$, and u_i, v_i, u_i^* and v_i^* are all nontrivial. Thus (P) has at least eight nontrivial constant sign solutions. \square

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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