

Research Article

Global Asymptotic Stability of a Nonautonomous Difference Equation

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We study the following nonautonomous difference equation: $x_{n+1} = (x_n x_{n-1} + p_n) / (x_n + x_{n-1})$, $n = 0, 1, \dots$, where $p_n > 0$ is a period-2 sequence and the initial values $x_{-1}, x_0 \in (0, \infty)$. We show that the unique prime period-2 solution of the equation above is globally asymptotically stable.

1. Introduction

In [1], Xianyi and Deming studied the global asymptotic stability of positive solutions of the difference equation

$$x_{n+1} = \frac{x_n x_{n-1} + p}{x_n + x_{n-1}}, \quad n = 0, 1, \dots, \quad (1)$$

where $p \in [0, \infty)$ and the initial conditions are positive real numbers.

The mathematical modeling of a physical, physiological, or economical problem very often leads to difference equations (for partial review of the theory of difference equations and their applications). Moreover, a lot of difference equations with periodic coefficients have been applied in mathematical models in biology. In addition, between others in [2–10], we can see some more difference equations with periodic coefficient that have been studied. For constant coefficient, see [1, 11–15] and the references cited therein.

In the present paper, we consider the following nonautonomous difference equation:

$$x_{n+1} = \frac{x_n x_{n-1} + p_n}{x_n + x_{n-1}}, \quad n = 0, 1, \dots, \quad (2)$$

where $\{p_n\}$ is a two periodic sequence of nonnegative real numbers and the initial conditions x_{-1}, x_0 are arbitrary positive numbers.

The autonomous case of (2) is (1).

It is natural problem to investigate the behavior of the solutions of (1) where we replace the constant p by a nonnegative periodic sequence p_n . That is, we will consider the recursive sequence (2) where $\{p_n\}$ is a two-periodic sequence of nonnegative real numbers. What do the solutions of (2) with positive initial conditions x_{-1} and x_0 do?

Our aim in this paper is to investigate the question above and extend some results obtained in [1]. More precisely, we study the existence of a unique positive periodic solution of (2) and we investigate the boundedness character and the convergence of the positive solutions to the unique two periodic solution of (2). Finally, we study the global asymptotic stability of the positive periodic solution (2).

As far as we examine, there is no paper dealing with (2). Therefore, in this paper, we focus on (2) in order to fill the gap.

Now, assume that

$$p_n = \begin{cases} \alpha & \text{if } n \text{ is even} \\ \beta & \text{if } n \text{ is odd} \end{cases} \quad (3)$$

in (2). Then,

$$\begin{aligned} x_{2n+1} &= \frac{x_{2n}x_{2n-1} + \alpha}{x_{2n} + x_{2n-1}}, \quad n = 0, 1, \dots, \\ x_{2n+2} &= \frac{x_{2n+1}x_{2n} + \beta}{x_{2n+1} + x_{2n}}, \quad n = 0, 1, \dots, \end{aligned} \quad (4)$$

where $x_{-1}, x_0 \in (0, \infty)$.

Here, we review some results which will be useful in our investigation of the behavior of positive solutions of (2).

Let I be some interval of real numbers and let

$$f : I \times I \longrightarrow I \quad (5)$$

be a continuously differentiable function. Then for every pair of initial values $x_{-1}, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \dots, \quad (6)$$

has a unique solution $\{x_n\}$.

Definition 1. Let (\bar{x}, \bar{y}) be an equilibrium point of the map

$$B = (f, u_n, u_{n-1}, \dots, u_{n-k}, g, z_n, \dots, z_{n-k}), \quad (7)$$

where f and g are continuously differentiable functions at (\bar{x}, \bar{y}) . The linearized system of

$$\begin{aligned} u_{n+1} &= f(u_n, u_{n-1}, \dots, u_{n-k}, z_n, \dots, z_{n-k}), \\ z_{n+1} &= g(u_n, u_{n-1}, \dots, u_{n-k}, z_n, \dots, z_{n-k}) \end{aligned} \quad (8)$$

about the equilibrium point (\bar{x}, \bar{y}) is $v_{n+1} = B(v_n) = B_j v_n$, where

$$v_n = \begin{pmatrix} u_n \\ \vdots \\ u_{n-k} \\ v_n \\ \vdots \\ v_{n-k} \end{pmatrix} \quad (9)$$

and B_j is the Jacobian matrix of system (8) about the equilibrium point (\bar{x}, \bar{y}) .

Theorem 2 (linearized stability). *Consider the following.*

- (1) *If both roots of the quadratic equation*

$$\lambda^2 - p\lambda - q = 0 \quad (10)$$

lie in the open unit disk $|\lambda| < 1$, then the equilibrium \bar{x} of (6) is locally asymptotically stable.

- (2) *If at least one of the roots of (10) has absolute value greater than one, then the equilibrium \bar{x} of (6) is unstable.*

- (3) *A necessary and sufficient condition for both roots of (10) to lie in the open unit disk $|\lambda| < 1$ is*

$$|p| < 1 - q < 2. \quad (11)$$

- (4) *A necessary and sufficient condition for both roots of (10) to have absolute value greater than one is*

$$|q| > 1, \quad |p| < |1 - q|. \quad (12)$$

- (5) *A necessary and sufficient condition for one root of (10) to have absolute value greater than one and for the other to have absolute value less than one is*

$$p^2 + 4q > 0, \quad |p| > |1 - q|. \quad (13)$$

- (6) *A necessary and sufficient condition for a root of (10) to have absolute value equal to one is*

$$|p| = |1 - q| \quad (14)$$

or

$$q = -1, \quad |p| \leq 2. \quad (15)$$

2. Boundedness of Solutions to (2)

In this section, we mainly investigate the boundedness character of (2), assuming (3) is satisfied.

Theorem 3. *Let $\alpha, \beta > 0$. Then, every positive solution of (2) is bounded.*

Proof. Suppose for the sake of contradiction that $\{x_n\}$ is an unbounded solution of (2). Then, there exist the subsequences $\{x_{2n+1}\}$ and $\{x_{2n+2}\}$ of the solution $\{x_n\}$ such that one of the following statements is true.

- (i) $x_{2n+1} \rightarrow \infty$ and $x_{2n+2} \rightarrow a$ (a is arbitrary positive real number) for all $n \geq 0$.
- (ii) $x_{2n+1} \rightarrow b$ (b is arbitrary positive real number) and $x_{2n+2} \rightarrow \infty$ for all $n \geq 0$.
- (iii) $x_{2n+1} \rightarrow \infty$ and $x_{2n+2} \rightarrow \infty$ for all $n \geq 1$.

Now, we will show that the above statements are never satisfied.

From (2), we have the following inequalities:

$$\begin{aligned} x_{2n+1} &< x_{2n-1} + \frac{\alpha}{x_{2n}}, \quad \text{for } n \geq 0, \\ x_{2n+1} &< x_{2n} + \frac{\alpha}{x_{2n-1}}, \quad \text{for } n \geq 0, \\ x_{2n+2} &< x_{2n} + \frac{\beta}{x_{2n+1}}, \quad \text{for } n \geq 0, \\ x_{2n+2} &< x_{2n+1} + \frac{\beta}{x_{2n}}, \quad \text{for } n \geq 0. \end{aligned} \quad (16)$$

If we get limits on both sides of the above inequalities, they contradict cases (i) and (ii). From (16) and using the induction, we obtain that

$$\begin{aligned} x_{2n+1} &< x_{-1} + \alpha \left(\frac{1}{x_0} + \dots + \frac{1}{x_{2n}} \right), \\ x_{2n+2} &< x_0 + \beta \left(\frac{1}{x_1} + \dots + \frac{1}{x_{2n+1}} \right). \end{aligned} \quad (17)$$

Also, if we consider the limiting case of the above inequalities, they contradict case (iii) and complete the proof. \square

3. Periodicity and Stability of Solutions to (2)

In this section, we show that (2) has the unique two-periodic solution and the unique prime period-2 solution of this equation is locally asymptotically stable.

Theorem 4. Suppose that $p_n > 0$ is a two periodic sequence such that

$$p_{n+2} = p_n, \quad n = 0, 1, \dots \quad (18)$$

Then (2) has the unique two-periodic solution.

Proof. Let x_n be a solution of (2). Using (18), x_n is two-periodic solution if and only if x_{-1} and x_0 satisfy

$$x_{-1} = x_1 = \frac{x_0 x_{-1} + \alpha}{x_0 + x_{-1}}, \quad x_0 = x_2 = \frac{x_0 x_1 + \beta}{x_0 + x_1}. \quad (19)$$

We set $x_{-1} = x$ and $x_0 = y$; then from (19) we obtain the system of equations

$$x = \frac{xy + \alpha}{x + y}, \quad y = \frac{xy + \beta}{x + y}. \quad (20)$$

We prove that (2) has a solution (\bar{x}, \bar{y}) , $\bar{x}, \bar{y} > 0$. From (20) we get

$$x^2 + xy = xy + \alpha, \quad y^2 + xy = xy + \beta. \quad (21)$$

It follows that (2) has a prime period-2 solution if and only if $\alpha, \beta \in (0, \infty)$, $\alpha \neq \beta$, in which case (2) has a unique two-periodic solution

$$\sqrt{\alpha}, \sqrt{\beta}, \dots, \sqrt{\alpha}, \sqrt{\beta}, \dots \quad (22)$$

\square

Theorem 5. Assume that $p_n > 0$ is a two periodic sequence and

$$\alpha, \beta \in (0, \infty). \quad (23)$$

Then (2) has the unique prime period-2 solution which is locally asymptotically stable.

Proof. Let x_n be a positive solution of (2) and let

$$\dots, \sqrt{\alpha}, \sqrt{\beta}, \sqrt{\alpha}, \sqrt{\beta}, \dots, \sqrt{\alpha}, \sqrt{\beta}, \dots \quad (24)$$

be the unique prime period-2 solution of (2). For any $n \geq 0$, set

$$u_n = x_{2n-1}, \quad v_n = x_{2n}. \quad (25)$$

Then, (2) is equivalent to the system

$$u_{n+1} = \frac{u_n v_n + \alpha}{u_n + v_n}, \quad v_{n+1} = \frac{u_n (v_n^2 + \beta) + (\alpha + \beta) v_n}{v_n (2u_n + v_n) + \alpha}. \quad (26)$$

So, in order to study (2) we investigate system (26).

Let H be the function on $(0, \infty)^2$ defined by

$$H \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{uv + \alpha}{u + v} \\ \frac{u(v^2 + \beta) + (\alpha + \beta)v}{v(2u + v) + \alpha} \end{pmatrix} = \begin{pmatrix} g(u, v) \\ h(u, v) \end{pmatrix}. \quad (27)$$

Then,

$$\begin{pmatrix} \sqrt{\alpha} \\ \sqrt{\beta} \end{pmatrix} \quad (28)$$

is a fixed point of H . Clearly the unique prime period-2 solution is locally asymptotically stable when the eigenvalues of the jacobian matrix J_H , evaluated at $(\frac{\sqrt{\alpha}}{\sqrt{\beta}})$, lie inside the unit disk. We have

$$J_H \begin{pmatrix} \sqrt{\alpha} \\ \sqrt{\beta} \end{pmatrix} = \begin{pmatrix} \frac{\partial g}{\partial u}(\sqrt{\alpha}, \sqrt{\beta}) & \frac{\partial g}{\partial v}(\sqrt{\alpha}, \sqrt{\beta}) \\ \frac{\partial h}{\partial u}(\sqrt{\alpha}, \sqrt{\beta}) & \frac{\partial h}{\partial v}(\sqrt{\alpha}, \sqrt{\beta}) \end{pmatrix}. \quad (29)$$

The linearized system of (26) about $(\sqrt{\alpha}, \sqrt{\beta})$ is the system

$$z_{n+1} = B z_n, \quad (30)$$

where

$$B = \begin{pmatrix} \frac{\beta - \alpha}{\alpha + \beta + 2\sqrt{\alpha\beta}} & 0 \\ 0 & \frac{\alpha - \beta}{\alpha + \beta + 2\sqrt{\alpha\beta}} \end{pmatrix}, \quad z_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix}. \quad (31)$$

The characteristic equation of B is

$$\lambda^2 - \left(\frac{\beta - \alpha}{\alpha + \beta + 2\sqrt{\alpha\beta}} + \frac{\alpha - \beta}{\alpha + \beta + 2\sqrt{\alpha\beta}} \right) \lambda - \frac{(\alpha - \beta)^2}{(\alpha + \beta + 2\sqrt{\alpha\beta})^2} = 0 \quad (32)$$

and thus

$$\lambda^2 - \frac{(\alpha - \beta)^2}{(\alpha + \beta + 2\sqrt{\alpha\beta})^2} = 0. \quad (33)$$

From Theorem 2 and (23) all the roots of (33) are of modulus less than 1. So the system (26) is locally asymptotically stable. Therefore, (2) has the unique prime period-2 solution which is locally asymptotically stable. \square

4. Global Asymptotic Stability for (2)

In this section, we give our main result. Before doing so, in the next theorem, we will provide an alternative proof for the global stability of (1). In the proof of the main theorem, we will use a similar technique.

Theorem 6. *The positive equilibrium point \bar{x} of (1) is globally asymptotically stable.*

Proof. We know from Theorem 3 that every positive solution of (1) is bounded for $p \in (0, \infty)$ and we know from [1] that the positive equilibrium point \bar{x} of (1) is locally asymptotically stable for $p \in (0, \infty)$.

Let $\{x_n\}$ be a positive solution of (1) and set

$$\limsup x_n = K, \quad \liminf x_n = k. \quad (34)$$

Then by the boundedness of $\{x_n\}$; k, K are positive real numbers.

Thus, it is easy to see from (1) that

$$k \geq \frac{k^2 + p}{2K}, \quad K \leq \frac{K^2 + p}{2k}, \quad (35)$$

$$k^2 + p \leq 2Kk \leq K^2 + p.$$

Thus, we have

$$k^2 \leq K^2 \quad (36)$$

and so

$$k \leq K. \quad (37)$$

From (37), if $k = K$, the proof is obvious. If $k < K$, this case contradicts the unique equilibrium solution of (1) which is locally asymptotically stable. Consequently, it must be $K = k$. This completes the proof. \square

Now, we are ready to give the proof of the main result.

Theorem 7. *Assume that $p_n > 0$ is a two-periodic sequence and $\alpha, \beta \in (0, \infty)$; then (2) has the unique prime period-2 solution which is globally asymptotically stable.*

Proof. We must prove that the unique prime period-2 solution of (2) is both locally asymptotically stable and globally attractive. We know from Theorem 5 that the unique prime period-2 solution of (2) is locally asymptotically stable for $\alpha, \beta \in (0, \infty)$. It remains to verify that every positive solution of (2) converges the unique prime period-2 solution. Since $\alpha, \beta \in (0, \infty)$, by Theorem 3 every positive solution of (2) is bounded. Let $\{x_n\}$ be a positive solution of (2) and set

$$\begin{aligned} \limsup x_{2n+1} &= M, & \limsup x_{2n} &= S, \\ \liminf x_{2n+1} &= m, & \liminf x_{2n} &= s. \end{aligned} \quad (38)$$

Then by the boundedness of $\{x_n\}$, m, M, s , and S are positive real numbers. Moreover, it is obvious that since $\alpha \neq \beta$, then from (2)

$$\lim_{n \rightarrow \infty} x_{2n} \neq \lim_{n \rightarrow \infty} x_{2n+1}, \quad (39)$$

and it is clear that if $\lim x_{2n+1}$ exists, then so does $\lim_{n \rightarrow \infty} x_{2n}$.

Thus, it is easy to see from (2) that

$$m \geq \frac{ms + \alpha}{M + S}, \quad M \leq \frac{MS + \alpha}{m + s}, \quad (40)$$

$$s \geq \frac{ms + \beta}{M + S}, \quad S \leq \frac{MS + \beta}{m + s}. \quad (41)$$

From (40), we have

$$mM + mS - ms \geq \alpha \geq mM + sM - MS,$$

$$mM + mS - ms \geq mM + sM - MS, \quad (42)$$

$$S \geq s.$$

From (41), we have

$$sM + sS - ms \geq \beta \geq mS + sS - MS,$$

$$sM + sS - ms \geq mS + sS - MS, \quad (43)$$

$$M \geq m.$$

With cases (42) and (43), if $s = S$ and $m = M$, the proof is obvious. If $s < S$ and $m < M$, this case contradicts the unique prime period-2 solution of (2) which is locally asymptotically stable. Consequently, it must be $s = S$ and $M = m$. The proof is complete. \square

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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