Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 393147, 8 pages
http://dx.doi.org/10.1155/2014/393147

Research Article

L^p Bounds for the Commutators of Oscillatory Singular Integrals with Rough Kernels

Yanping Chen and Kai Zhu

Department of Applied Mathematics, University of Science and Technology Beijing, Beijing 100083, China

Correspondence should be addressed to Yanping Chen; yanpingch@126.com

Received 27 May 2014; Accepted 18 July 2014; Published 5 August 2014

Academic Editor: Abdelghani Bellouquid

We establish the L^p boundedness for some commutators of oscillatory singular integrals with the kernel condition which was introduced by Grafakos and Stefanov. Our theorems contain various conditions on the phase function.

1. Introduction

The homogeneous singular integral operator T_Ω is defined by

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\, dy, \tag{1}$$

where $\Omega \in L^1(S^{n-1})$ satisfies the following conditions.

(a) Ω is homogeneous function of degree zero on $\mathbb{R}^n \setminus \{0\}$; that is,

$$\Omega(tx) = \Omega(x) \tag{2}$$

for any $t > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$.

(b) Ω has mean zero on S^{n-1}, the unit sphere in $\mathbb{R}^n$; that is,

$$\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0. \tag{3}$$

The oscillatory singular integral we will consider here is defined by

$$T_\phi f(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{i\phi(y)} \frac{\Omega(y)}{|y|^n} f(x-y)\, dy. \tag{4}$$

If $\phi(x) \equiv 0$, the operator T_ϕ becomes the singular integral operator T_Ω.

When $\phi(x) = P(x)$ is a real polynomial, the L^p boundedness of T_ϕ was first studied by Ricci and Stein [1] with $\Omega \in C^1(S^{n-1})$, and Hu and Pan [2] obtained the weighted H^1 boundedness of T_ϕ. When $\Omega \in L^r(S^{n-1})$, $r > 1$, Lu and Zhang proved the L^p boundedness [3] and this was extended to the case of $\Omega \in L \ln^+ L(S^{n-1})$ by Ojanen [4] and the case of $\Omega \in H^1(S^{n-1})$ by Fan and Pan [5].

Grafakos and Stefanov [6] introduced a class of kernel functions $F_\alpha(S^{n-1})$ which contains all $\Omega(y) \in L^1(S^{n-1})$ satisfying (3) and

$$\sup_{\xi \in S^{n-1}} \int_{S^{n-1}} |\Omega(y)| \left(\ln |y \cdot \xi|^{-1}\right)^{1+\alpha} d\sigma(y) < \infty, \tag{5}$$

where $\alpha > 0$ is a fixed constant. This kernel condition has been considered by many authors [7–13].

The singular integral along surfaces which is defined by

$$T_{\phi,\Omega} f(x, x_{n+1}) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^n} f(x-y, x_{n+1} - \phi(|y|))\, dy \tag{6}$$

was also studied by many authors [14–18]. Under the condition $\Omega \in F_\alpha(S^{n-1})$, Pan et al. [16] established the following Theorem.

Theorem A (see [16]). *Let $\phi(t) \in C^1([0,\infty))$, $\phi(0) = \phi'(0) = 0$, and ϕ' is a convex increasing function for $t > 0$, $\Omega \in F_\alpha(S^{n-1})$ for some $\alpha > 0$; then, $T_{\phi,\Omega}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $(2+2\alpha)/(1+2\alpha) < p < 2+2\alpha$.*

Later, Cheng and Pan [14] improved the result for $n = 2$ by removing the condition $\phi'(0) = 0$.

Theorem B (see [14]). *Let $\phi(t) \in C^1([0,\infty))$, $\phi(0) = 0$, and ϕ' is a convex increasing function for $t > 0$, $\Omega \in F_\alpha(S^{n-1})$ for some $\alpha > 0$; then, $T_{\phi,\Omega}$ is bounded on $L^p(\mathbb{R}^3)$ for $(2+2\alpha)/(1+2\alpha) < p < 2+2\alpha$.*

It has been proved that the boundedness of T_ϕ on $L^p(\mathbb{R}^n)$ can be obtained from the $L^p(\mathbb{R}^{n+1})$ boundedness of $T_{\phi,\Omega}$ (see [5]).

For a function $b \in L_{\text{loc}}(\mathbb{R}^n)$, let A be a linear operator on some measurable function space; the commutator between A and b is defined by $[b, A]f(x) := b(x)Af(x) - A(bf)(x)$.

It has been proved by Hu [19] that $\Omega \in L(\log L)^2(S^{n-1})$ is a sufficient condition for the commutator to be bounded on $L^p(\mathbb{R}^n)$, which is defined by

$$[b, T_\Omega] f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} (b(x) - b(y)) f(y)\, dy. \tag{7}$$

Recently, Chen and Ding [20] established the L^p boundedness of the commutator of singular integrals with the kernel condition $\Omega \in F_\alpha(S^{n-1})$.

It is natural to ask whether the similar result holds for the commutators of oscillatory singular integrals, which is defined by

$$[b, T_\phi] f(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{i\phi(y)} \frac{\Omega(y)}{|y|^n} (b(x) - b(x-y)) \times f(x-y)\, dy. \tag{8}$$

In this paper, we will give a positive answer to the above question by imposing some conditions on ϕ.

We first prove the boundedness of the commutator of singular integral along surfaces, which is defined by

$$\begin{aligned}[b, T_{\phi,\Omega}] f(x, x_{n+1}) &= \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^n} (b(x, x_{n+1}) - b(x-y, x_{n+1} - \phi(|y|))) \\ &\quad \times f(x-y, x_{n+1} - \phi(|y|))\, dy.\end{aligned} \tag{9}$$

Theorem 1. *Let Ω be a function in $L^1(S^{n-1})$ satisfying* (2) *and* (3), *$b \in BMO(\mathbb{R}^{n+1})$, radial function $\phi \in C^1([0,\infty))$ with $\phi(0) = \phi'(0) = 0$, and ϕ' is a convex increasing function. If $\Omega \in F_\alpha(S^{n-1})$ for some $\alpha > 1$, then $[b, T_{\phi,\Omega}]$ is bounded on $L^2(\mathbb{R}^{n+1})$.*

Theorem 2. *Let Ω be a function in $L^1(S^1)$ satisfying* (2) *and* (3), *$b \in BMO(\mathbb{R}^3)$, radial function $\phi(|t|) = |t|$. If $\Omega \in F_\alpha(S^1)$ for some $\alpha > 1$, then $[b, T_{\phi,\Omega}]$ is bounded on $L^p(\mathbb{R}^3)$ for $(\alpha+1)/\alpha < p < \alpha+1$.*

Remark 3. However, for $n \geq 3$, we can not prove the $L^p(\mathbb{R}^{n+1})$ boundedness of $[b, T_{\phi,\Omega}]$ by our method using Lemma 11, since the conditions imposed on ϕ in Theorem 1 conflict with Lemma 11. Only when $n = 2$ by removing the condition $\phi'(0) = 0$ in Theorem 1 can we eliminate the conflict, and $\phi(|t|) = |t|$ is a feasible function. Also, by another method, it is hard to give the boundedness of the maximal operator defined by

$$\begin{aligned}[b, M_{\phi,\Omega}] f(x, x_{n+1}) &= \sup_{j\in\mathbb{Z}} \left| \int_{2^j<|y|<2^{j+1}} \frac{\Omega(y)}{|y|^n} \right. \\ &\quad \times (b(x, x_{n+1}) - b \\ &\quad \times (x-y, x_{n+1} - \phi(|y|))) \\ &\quad \left. \times f(x-y, x_{n+1} - \phi(|y|))\, dy \right|.\end{aligned} \tag{10}$$

Then we give the boundedness of the commutators of oscillatory singular integral $[b, T_\phi]$.

Let $b(x) \in \text{BMO}(\mathbb{R}^n)$, $\tilde{x} = (x, x_{n+1}) \in \mathbb{R}^{n+1}$, $B(\tilde{x}) = b(x)$, and we have the following result.

Theorem 4. *If $[B, T_{\phi,\Omega}]$ is bounded on $L^p(\mathbb{R}^{n+1})$ with bound $C\|B\|_{BMO(\mathbb{R}^{n+1})}$, then $[b, T_\phi]$ is bounded on $L^p(\mathbb{R}^n)$ with bound $C\|b\|_{BMO(\mathbb{R}^n)}$.*

Combining Theorem 4 with Theorems 1 and 2, respectively, we can get the following two theorems immediately.

Theorem 5. *Let Ω be a function in $L^1(S^{n-1})$ satisfying* (2) *and* (3), *$b \in BMO(\mathbb{R}^n)$, radial function $\phi \in C^1([0,\infty))$ with $\phi(0) = \phi'(0) = 0$, and ϕ' is a convex increasing function. If $\Omega \in F_\alpha(S^{n-1})$ for some $\alpha > 1$, then $[b, T_\phi]$ is bounded on $L^2(\mathbb{R}^n)$.*

Theorem 6. *Let Ω be a function in $L^1(S^1)$ satisfying* (2) *and* (3), *$b \in BMO(\mathbb{R}^2)$, radial function $\phi(|t|) = |t|$. If $\Omega \in F_\alpha(S^1)$ for some $\alpha > 1$, then $[b, T_\phi]$ is bounded on $L^p(\mathbb{R}^2)$ for $(\alpha+1)/\alpha < p < \alpha+1$.*

In above theorems, the phase functions are radial. But when Ricci and Stein first studied the oscillatory singular integral T_ϕ, they take $\phi(x) = P(x)$, apparently nonradial. In Theorem 7, we will take $\phi(x) = P(x) = \sum_{|\alpha|/2=1}^{m} a_\alpha x^\alpha$, and this condition was mentioned in [21].

Theorem 7. *Let Ω be a function in $L^1(S^{n-1})$ satisfying* (2) *and* (3), *$b \in BMO(\mathbb{R}^n)$. If $\Omega \in F_\alpha(S^{n-1})$ is an odd kernel for some $\alpha > 1$, $\phi(x) = \sum_{|\alpha|/2=1}^{m} a_\alpha x^\alpha$ is an even phase; then, $[b, T_\phi]$ extends to a bounded operator from $L^p(\mathbb{R}^n)$ into itself for $(\alpha+1)/\alpha < p < \alpha+1$.*

2. Lemmas

We give some lemmas which will be used in the proof of Theorems 1 and 2.

Lemma 8. *Let* $m_\delta(\tilde{\xi}) \in C^1(\mathbb{R}^{n+1})$ $(0 < \delta < \infty)$ *be a family of multipliers such that* $\operatorname{supp} m_\delta \subset \{\tilde{\xi} : |\tilde{\xi}| \le \delta\}$, $\nabla_\xi m_\delta = (\partial m_\delta/\partial \xi_1, \ldots, \partial m_\delta/\partial \xi_n)$, *and for some constants* $C, 0 < A \le 1/2$, *and* $\alpha > 0$

$$\|m_\delta\|_\infty \le C \min\left\{A\delta, \log^{-(\alpha+1)}(2+\delta)\right\}, \quad \left\|\nabla_\xi m_\delta\right\|_\infty \le C. \tag{11}$$

Let T_δ *be the multiplier operator defined by* $\widehat{T_\delta f}(\tilde{\xi}) = m_\delta(\tilde{\xi})\hat{f}(\tilde{\xi})$, $\tilde{\xi} = (\xi, \xi_{n+1})$. *For* $b \in BMO(\mathbb{R}^{n+1})$, *denote by* $[b, T_\delta]$ *the commutator of* T_δ. *Then for any* $0 < \nu < 1$, *there exists a positive constant* $C = C(n, \nu)$ *such that*

$$\begin{aligned}\left\|[b, T_\delta] f\right\|_2 &\le C\|b\|_{BMO(\mathbb{R}^{n+1})}(A\delta)^\nu \log\left(\frac{1}{A}\right)\|f\|_2, \quad \text{if } \delta < \frac{10}{\sqrt{A}};\\ \left\|[b, T_\delta] f\right\|_2 &\le C\|b\|_{BMO(\mathbb{R}^{n+1})}\log^{-(\alpha+1)\nu+1}(2+\delta)\|f\|_2, \quad \text{if } \delta > \frac{1}{\sqrt{A}}.\end{aligned} \tag{12}$$

Proof. We assume that $\|b\|_{\mathrm{BMO}(\mathbb{R}^{n+1})} = 1$. Let $\tilde{x} = (x, x_{n+1})$ and let $\Psi(\tilde{x})$ be a radial function such that $\operatorname{supp}\Psi \subset \{\tilde{x} : 1/4 \le |\tilde{x}| \le 4\}$, and

$$\sum_{l\in\mathbb{Z}} \Psi\left(2^{-l}\tilde{x}\right) = 1 \tag{13}$$

for $|\tilde{x}| > 0$. Set $\Psi_0(\tilde{x}) = \sum_{l=-\infty}^{0} \Psi(2^{-l}\tilde{x})$ and $\Psi_l(\tilde{x}) = \Psi(2^{-l}\tilde{x})$ for positive integer l. Let $K_\delta(\tilde{x}) = m_\delta^\vee(\tilde{x})$ the inverse Fourier transform of m_δ. Split K_δ as

$$K_\delta(\tilde{x}) = K_\delta(\tilde{x})\Psi_0(\tilde{x}) + \sum_{l=1}^{\infty} K_\delta(\tilde{x})\Psi_l(\tilde{x}) = \sum_{l=0}^{\infty} K_{\delta,l}(\tilde{x}). \tag{14}$$

Let $T_{\delta,l}$ be the convolution operator whose kernel is $K_{\delta,l}$; that is, $T_{\delta,l} f = K_{\delta,l} * f$. Recall that $\operatorname{supp} m_\delta \subset \{\tilde{\xi} : |\tilde{\xi}| \le \delta\}$. Trivial computation shows that $\|K_{\delta,l}\|_\infty \le \|K_\delta\|_\infty \le \|m_\delta\|_1 \le C\delta^{n+1}$. This via the Young inequality says that

$$\left\|T_{\delta,l} f\right\|_\infty \le C\delta^{n+1}\|f\|_1. \tag{15}$$

Note that $\int_{\mathbb{R}^{n+1}} \widehat{\Psi}(\tilde{\eta}) d\tilde{\eta} = 0$. Thus

$$\begin{aligned}\left\|\widehat{K_{\delta,l}}\right\|_\infty &= \left\|\int_{\mathbb{R}^{n+1}} \left(m_\delta\left(\xi - 2^{-l}\eta, \xi_{n+1} - 2^{-l}\eta_{n+1}\right) - m_\delta \times \left(\xi, \xi_{n+1} - 2^{-l}\eta_{n+1}\right)\right)\widehat{\Psi}(\tilde{\eta})\, d\tilde{\eta}\right\|_\infty\\ &\le C2^{-l}\left\|\nabla_\xi m_\delta\right\|_\infty \int_{\mathbb{R}^{n+1}} |\eta|\left|\widehat{\Psi}(\tilde{\eta})\right| d\tilde{\eta}\\ &\le C2^{-l}\left\|\nabla_\xi m_\delta\right\|_\infty \int_{\mathbb{R}^{n+1}} |\tilde{\eta}|\left|\widehat{\Psi}(\tilde{\eta})\right| d\tilde{\eta} \le C2^{-l}.\end{aligned} \tag{16}$$

On the other hand, by the Yong inequality, we have

$$\left\|\widehat{K_{\delta,l}}\right\|_\infty \le \left\|\widehat{K_\delta}\right\|_\infty \left\|\widehat{\Psi_l}\right\|_1 \le C \min\left\{A\delta, \log^{-(\alpha+1)}(2+\delta)\right\}. \tag{17}$$

Then, using the same argument of the proof of Lemma 2 in [22] we can prove Lemma 8. □

Let the measure σ_j on $\mathbb{R}^{n+1}$ be defined by

$$\int_{\mathbb{R}^{n+1}} f(y, y_{n+1})\, d\sigma_j = \int_{\mathbb{R}^n} f(y, \phi(|y|)) \frac{\Omega(y')}{|y|^n} \chi_{\{2^j < |y| \le 2^{j+1}\}} dy \tag{18}$$

for all $j \in \mathbb{Z}$. Define the maximal operator in $\mathbb{R}^{n+1}$ by $\sigma^* f = \sup_{j\in\mathbb{Z}} |\sigma_j| * |f|$.

Lemma 9 (see [18]). *Suppose* σ^* *is bounded on* $L^q(\mathbb{R}^{n+1})$ *for all* $1 < q < \infty$. *Then, for arbitrary functions* g_j, *the following vector valued inequality:*

$$\left\|\left(\sum_j \left|\sigma_j * g_j\right|^2\right)^{1/2}\right\|_{L^q(\mathbb{R}^{n+1})} \le C\left\|\left(\sum_j \left|g_j\right|^2\right)^{1/2}\right\|_{L^q(\mathbb{R}^{n+1})} \tag{19}$$

holds with any $1 < q < \infty$.

The maximal function in $\mathbb{R}^2$ is defined by

$$\left(M_\phi f\right)(x_1, x_2) = \sup_{k\in\mathbb{Z}} \frac{1}{2^k} \int_{2^k}^{2^{k+1}} \left|f\left(x_1 - t, x_2 - \phi(t)\right)\right| dt. \tag{20}$$

We know that the $L^q(\mathbb{R}^{n+1})$ boundedness of σ^* is deduced from the $L^q(\mathbb{R}^2)$ boundedness of M_ϕ by method of rotations, and if ϕ is as in Theorem 1 or Theorem 2, M_ϕ is a bounded operator on $L^q(\mathbb{R}^2)$ for all $1 < q < \infty$ (see [23, 24]).

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a radial function satisfying $0 \le \varphi \le 1$ with its support in the unit ball and $\varphi(\xi) = 1$ for $|\xi| \le 1/2$. The function $\varphi_0(\xi) = \varphi(\xi/2) - \varphi(\xi) \in \mathcal{S}(\mathbb{R}^n)$ satisfies $\sum_{j\in\mathbb{Z}} \varphi_0(2^{-j}\xi) = 1$ for $\xi \ne 0$. For $j \in \mathbb{Z}$, denote by Δ_j and G_j the convolution operators whose symbols are $\varphi_0(2^{-j}\xi)$ and $\varphi(2^{-j}\xi)$, respectively.

Lemma 10 (see [20]). *For the multiplier* G_k $(k \in \mathbb{Z})$, $b \in BMO(\mathbb{R}^n)$, *and any fixed* $0 < \tau < 1/2$, *we have*

$$\left|G_k b(x) - G_k b(y)\right| \le C\frac{2^{k\tau}}{\tau}|x - y|^\tau \|b\|_{BMO}, \tag{21}$$

where C *is independent of* k *and* τ.

Let $\tilde{\xi} = (\xi, \xi_{n+1}) \in \mathbb{R}^{n+1}$ and let $\psi(\tilde{\xi}) \in C_0^\infty(\mathbb{R}^{n+1})$ be a radial function such that $0 \le \psi \le 1$, $\operatorname{supp}\psi \subset \{1/2 \le |\tilde{\xi}| \le 2\}$, and $\sum_{l\in\mathbb{Z}} \psi^3(2^{-l}\tilde{\xi}) = 1$, $|\tilde{\xi}| \ne 0$. Define the multiplier operator S_l by $\widehat{S_l f}(\tilde{\xi}) = \psi(2^{-l}|\tilde{\xi}|)\hat{f}(\tilde{\xi})$.

Lemma 11. *For any $j \in \mathbb{Z}$, define the operator T_j by $T_j f = \sigma_j * f$, and ϕ is monotonic and satisfies condition (1) or (2):*

(1) $|\phi(|y|)| \le C|y|$;

(2) $|\phi(|y|)| \ge C|y|$, $|\phi(a)\phi(b)| \le C|\phi(ab)|$ *for* $\forall a, b > 0$, *and* $|\phi(|y|)| \le C|y|^{k_1}$, $k_1 > 1$ *if* $|y| > 1$, $|\phi(|y|)| \le C|y|^{k_2}$, $0 < k_2 < 1$ *if* $|y| \le 1$.

Let $b \in BMO(\mathbb{R}^{n+1})$, and denote by $[b, S_{l-j}T_jS_{l-j}^2]$ the commutator of $S_{l-j}T_jS_{l-j}^2$. Suppose $\Omega \in L^1(S^{n-1})$ satisfying (2). Then for any fixed $0 < \tau < 1/2$, $1 < p < \infty$,

$$\left\| \sum_{j\in\mathbb{Z}} \left[b, S_{l-j}T_jS_{l-j}^2\right] f(\tilde{x}) \right\|_{L^p} \le C\|b\|_{BMO} \max\left\{\frac{2^{\tau l}}{\tau}, \frac{2^{\tau k_1 l}}{\tau}, \frac{2^{\tau k_2 l}}{\tau}, 2\right\} \|f\|_{L^p}. \tag{22}$$

Proof. We prove it by using arguments which are essentially the same as those in the proof of Lemma 3.7 in [20]. Two things must be modified:

(i) instead of Lemma 3.6 in [20], we use Lemma 9;

(ii) In [20], $M_1 = \|\sum_{j\in\mathbb{Z}} S_{l-j}[\pi_{(T_jS_{l-j}^2 f)}(b) - T_j(\pi_{(S_{l-j}^2 f)}(b))]\|_{L^p}$, and $\pi_f(g) = \sum_{j\in\mathbb{Z}}(\Delta_j f)(G_{j-3}g)$ is the paraproduct of Bony [25] between two functions f and g. In the estimate of M_1, we will use the following formulas:

$$\begin{aligned}
&\left|\left[G_{i-3}b, T_j\right]\left(\Delta_i S_{l-j}^2 f\right)(x, x_{n+1})\right| \\
&\quad = \Big|G_{i-3}b(x, x_{n+1}) T_j\left(\Delta_i S_{l-j}^2 f\right)(x, x_{n+1}) \\
&\qquad - T_j\left((G_{i-3}b)\left(\Delta_i S_{l-j}^2 f\right)\right)(x, x_{n+1})\Big| \\
&\quad = \Big|\int_{2^j<|y|\le 2^{j+1}} \frac{\Omega(y)}{|y|^n} \times \left(G_{i-3}b(x, x_{n+1}) - G_{i-3}b(x-y, x_{n+1} - \phi(|y|))\right) \\
&\qquad \cdot \Delta_i S_{l-j}^2 f(x-y, x_{n+1} - \phi(|y|))\, dy\Big| \\
&\quad \le C\int_{2^j<|y|\le 2^{j+1}} \frac{|\Omega(y)|}{|y|^n} \times \left|G_{i-3}b(x, x_{n+1}) - G_{i-3}b(x-y, x_{n+1} - \phi(|y|))\right| \\
&\qquad \cdot \left|\Delta_i S_{l-j}^2 f(x-y, x_{n+1} - \phi(|y|))\right| dy,
\end{aligned} \tag{23}$$

by Lemma 10,

$$\begin{aligned}
&\left|G_{i-3}b(x, x_{n+1}) - G_{i-3}b(x-y, x_{n+1} - \phi(|y|))\right| \\
&\quad \le C\frac{2^{i\tau}}{\tau}\left|(y, \phi(|y|))\right|^\tau \|b\|_{\mathrm{BMO}} \\
&\quad = C\frac{2^{i\tau}}{\tau}\sqrt{|y|^2 + \phi^2(|y|)}^{\,\tau} \|b\|_{\mathrm{BMO}}.
\end{aligned} \tag{24}$$

If ϕ satisfies condition (1), we have

$$\begin{aligned}
&\left|G_{i-3}b(x, x_{n+1}) - G_{i-3}b(x-y, x_{n+1} - \phi(|y|))\right| \\
&\quad \le C\frac{2^{i\tau}}{\tau}|y|^\tau \|b\|_{\mathrm{BMO}}.
\end{aligned} \tag{25}$$

Thus

$$\begin{aligned}
&\left|\left[G_{i-3}b, T_j\right]\left(\Delta_i S_{l-j}^2 f\right)(x, x_{n+1})\right| \\
&\quad \le C\frac{2^{i\tau}}{\tau}\|b\|_{\mathrm{BMO}} \times \int_{2^j<|y|\le 2^{j+1}} \frac{|\Omega(y)|}{|y|^n}|y|^\tau \times \left|\Delta_i S_{l-j}^2 f(x-y, x_{n+1} - \phi(|y|))\right| dy \\
&\quad \le C\frac{2^{(i+j)\tau}}{\tau}\|b\|_{\mathrm{BMO}} \times \int_{2^j<|y|\le 2^{j+1}} \frac{|\Omega(y)|}{|y|^n} \times \left|\Delta_i S_{l-j}^2 f(x-y, x_{n+1} - \phi(|y|))\right| dy \\
&\quad = C\frac{2^{(i+j)\tau}}{\tau}\|b\|_{\mathrm{BMO}} T_{|\Omega|,j}\left(\left|\Delta_i S_{l-j}^2 f\right|\right)(x, x_{n+1}).
\end{aligned} \tag{26}$$

If ϕ satisfies condition (2), we have

$$\begin{aligned}
&\left|G_{i-3}b(x, x_{n+1}) - G_{i-3}b(x-y, x_{n+1} - \phi(|y|))\right| \\
&\quad \le C\frac{2^{i\tau}}{\tau}\left|\phi^\tau(|y|)\right| \|b\|_{\mathrm{BMO}} \\
&\quad \le C\frac{\left|\phi^\tau(2^i)\right|}{\tau}\left|\phi^\tau(|y|)\right| \|b\|_{\mathrm{BMO}}.
\end{aligned} \tag{27}$$

Thus if $|y| > 1$,

$$\begin{aligned} &\left|\left[G_{i-3}b, T_j\right]\left(\Delta_i S_{l-j}^2 f\right)(x, x_{n+1})\right| \\ &\quad \le C\frac{\left|\phi^{\tau}\left(2^{(i+j)}\right)\right|}{\tau}\|b\|_{\mathrm{BMO}} \\ &\qquad \times \int_{2^j<|y|\le 2^{j+1}} \frac{|\Omega(y)|}{|y|^n} \\ &\qquad\qquad \times \left|\Delta_i S_{l-j}^2 f\left(x-y, x_{n+1}-\phi(|y|)\right)\right| dy \\ &\quad \le C\frac{2^{(i+j)k_1\tau}}{\tau}\|b\|_{\mathrm{BMO}} T_{|\Omega|,j}\left(\left|\Delta_i S_{l-j}^2 f\right|\right)(x, x_{n+1}), \end{aligned} \tag{28}$$

and if $|y| \le 1$,

$$\begin{aligned} &\left|\left[G_{i-3}b, T_j\right]\left(\Delta_i S_{l-j}^2 f\right)(x, x_{n+1})\right| \\ &\quad \le C\frac{2^{(i+j)k_2\tau}}{\tau}\|b\|_{\mathrm{BMO}} T_{|\Omega|,j}\left(\left|\Delta_i S_{l-j}^2 f\right|\right)(x, x_{n+1}). \end{aligned} \tag{29}$$

□

3. The Proof of Theorems 1 and 2

Proof of Theorem 1. Let $\tilde{\xi} = (\xi, \xi_{n+1}) \in \mathbb{R}^{n+1}$ and let $\psi(\tilde{\xi}) \in C_0^{\infty}(\mathbb{R}^{n+1})$ be a radial function such that $0 \le \psi \le 1$, $\operatorname{supp}\psi \subset \{1/2 \le |\tilde{\xi}| \le 2\}$, and

$$\sum_{l\in Z}\psi^3\left(2^{-l}\tilde{\xi}\right) = 1, \quad \left|\tilde{\xi}\right| \ne 0. \tag{30}$$

Define the multiplier operator S_l by

$$\widehat{S_l f}\left(\tilde{\xi}\right) = \psi\left(2^{-l}\left|\tilde{\xi}\right|\right)\hat{f}\left(\tilde{\xi}\right). \tag{31}$$

Let the measure σ_j on $\mathbb{R}^{n+1}$ be defined by

$$\begin{aligned} &\int_{\mathbb{R}^{n+1}} f(y, y_{n+1})\, d\sigma_j \\ &\quad = \int_{\mathbb{R}^n} f\left(y, \phi(|y|)\right)\frac{\Omega(y')}{|y|^n}\chi_{\{2^j<|y|\le 2^{j+1}\}}dy \end{aligned} \tag{32}$$

for all $j \in \mathbb{Z}$. Since

$$\begin{aligned} \sigma_j * f &= \int_{\mathbb{R}^n} f\left(x-y, x_{n+1}-\phi(|y|)\right)\frac{\Omega(y')}{|y|^n}\chi_{\{2^j<|y|\le 2^{j+1}\}}dy, \\ T_{\phi,\Omega}f &= \int_{\mathbb{R}^n} f\left(x-y, x_{n+1}-\phi(|y|)\right)\frac{\Omega(y')}{|y|^n}dy, \end{aligned} \tag{33}$$

we get

$$T_{\phi,\Omega}f = \sum_{j\in\mathbb{Z}}\sigma_j * f. \tag{34}$$

Define the operator $T_j f(\tilde{x}) = \sigma_j * f(\tilde{x})$, where $\tilde{x} = (x, x_{n+1}) \in \mathbb{R}^{n+1}$ and the multiplier

$$\widehat{T_j^l f}\left(\tilde{\xi}\right) = \widehat{T_j S_{l-j} f}\left(\tilde{\xi}\right) = \psi\left(2^{j-l}\left|\tilde{\xi}\right|\right)\widehat{\sigma_j}\left(\tilde{\xi}\right)\hat{f}\left(\tilde{\xi}\right). \tag{35}$$

From the above notation, it is easy to see that

$$\begin{aligned} \left[b, T_{\phi,\Omega}\right]f(\tilde{x}) &= \sum_{l\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\left[b, S_{l-j}T_j S_{l-j}^2\right]f(\tilde{x}) \\ &= \sum_{l\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\left[b, S_{l-j}T_j^l S_{l-j}\right]f(\tilde{x}) \\ &:= \sum_{l\in\mathbb{Z}}V_l f(\tilde{x}), \end{aligned} \tag{36}$$

where

$$V_l f(\tilde{x}) = \sum_{j\in\mathbb{Z}}\left[b, S_{l-j}T_j^l S_{l-j}\right]f(\tilde{x}). \tag{37}$$

Then by the Minkowski inequality, we get

$$\begin{aligned} &\left\|\left[b, T_{\phi,\Omega}\right]f\right\|_{L^2(\mathbb{R}^{n+1})} \\ &\quad \le \left\|\sum_{l=-\infty}^{[\log\sqrt{2}]}V_l f\right\|_{L^2(\mathbb{R}^{n+1})} + \left\|\sum_{l=[\log\sqrt{2}]+1}^{\infty}V_l f\right\|_{L^2(\mathbb{R}^{n+1})}. \end{aligned} \tag{38}$$

For $\|\sum_{l=-\infty}^{[\log\sqrt{2}]}V_l f\|_{L^2(\mathbb{R}^{n+1})}$, we recall

$$\widehat{\sigma_j}(\xi, \xi_{n+1}) = \int_{S^{n-1}}\Omega(\theta)\int_{2^j}^{2^{j+1}}e^{-i(s\theta\cdot\xi+\phi(|s|)\xi_{n+1})}\frac{ds}{s}d\sigma(\theta). \tag{39}$$

By Lemma 2.3 of [16], we have

$$\left|\widehat{\sigma_j}(\xi, \xi_{n+1})\right| \le C\|\Omega\|_{L^1}\left|2^j\xi\right|. \tag{40}$$

Denote by $\nabla_\xi\widehat{\sigma_j}$ the before n components truncation of $\nabla\widehat{\sigma_j}$; that is,

$$\nabla_\xi\widehat{\sigma_j} = \left(\frac{\partial\widehat{\sigma_j}}{\partial\xi_1}, \dots, \frac{\partial\widehat{\sigma_j}}{\partial\xi_n}\right). \tag{41}$$

Since

$$\widehat{\sigma_j}(\xi, \xi_{n+1}) = \int_{\mathbb{R}^n}\frac{\Omega(y')}{|y|^n}\chi_{\{2^j<|y|\le 2^{j+1}\}}e^{-i(y\cdot\xi+\phi(|y|)\xi_{n+1})}dy, \tag{42}$$

we get

$$\left|\nabla_\xi\widehat{\sigma_j}\right| \le C2^j\|\Omega\|_{L^1}. \tag{43}$$

Set $m_j(\tilde{\xi}) = \widehat{\sigma_j}(\tilde{\xi})$, $m_j^l(\tilde{\xi}) = m_j(\tilde{\xi})\psi(2^{j-l}|\tilde{\xi}|)$. Recall that T_j^l by $\widehat{T_j^l f}(\tilde{\xi}) = m_j^l(\tilde{\xi})\hat{f}(\tilde{\xi})$. Straightforward computations lead to

$$\left\|m_j^l(2^{-j}\tilde{\xi})\right\|_{L^\infty} \le C\|\Omega\|_{L^1}2^l. \tag{44}$$

Since

$$\operatorname{supp}\left\{m_j^l\left(2^{-j}\widetilde{\xi}\right)\right\}\subset\left\{\left|\widetilde{\xi}\right|\le 2^{l+2}\right\},\tag{45}$$

we get

$$\left\|\nabla_\xi m_j^l\left(2^{-j}\widetilde{\xi}\right)\right\|_{L^\infty}\le C\|\Omega\|_{L^1}.\tag{46}$$

Let $\widetilde{T}_j^l$ be the operator defined by $\widehat{\widetilde{T}_j^l f}(\widetilde{\xi}) = m_j^l(2^{-j}\widetilde{\xi})\widehat{f}(\widetilde{\xi})$. Denote by $T_{j,b,1}^l f = [b,T_j^l]f$ and $T_{j,b,0}^l f = T_j^l f$. Similarly, denote by $\widetilde{T}_{j,b,1}^l f = [b,\widetilde{T}_j^l]f$ and $\widetilde{T}_{j,b,0}^l f = \widetilde{T}_j^l f$. Thus via the Plancherel theorem and Lemma 8 it is stated that for any fixed $0<\nu<1$, $k\in\{0,1\}$,

$$\left\|\widetilde{T}_{j,b,k}^l f\right\|_{L^2}\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^{n+1})}^k\|\Omega\|_{L^1}2^{\nu l}\|f\|_{L^2},\quad l\le\left[\log\sqrt{2}\right].\tag{47}$$

Dilation-invariance says that

$$\left\|T_{j,b,k}^l f\right\|_{L^2}\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^{n+1})}^k\|\Omega\|_{L^1}2^{\nu l}\|f\|_{L^2},\quad l\le\left[\log\sqrt{2}\right].\tag{48}$$

By the proof of Theorem 1 in [20], we can get

$$\left\|V_l f\right\|_{L^2}\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^{n+1})}2^{\nu l}\|\Omega\|_{L^1}\|f\|_{L^2},\quad l\le\left[\log\sqrt{2}\right].\tag{49}$$

So, we have

$$\begin{aligned}\left\|\sum_{l=-\infty}^{[\log\sqrt{2}]}V_l f\right\|_{L^2(\mathbb{R}^{n+1})}&\le C\sum_{l=-\infty}^{[\log\sqrt{2}]}2^{\nu l}\|b\|_{\mathrm{BMO}(\mathbb{R}^{n+1})}\|f\|_{L^2(\mathbb{R}^{n+1})}\\&\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^{n+1})}\|f\|_{L^2(\mathbb{R}^{n+1})}.\end{aligned}\tag{50}$$

For $\|\sum_{l=1+[\log\sqrt{2}]}^{\infty}V_l f\|_{L^2(\mathbb{R}^{n+1})}$, by Lemma 2.3 of [16], if ϕ satisfies the hypotheses in Theorem 1, we have

$$\left|\widehat{\sigma_j}(\xi,\xi_{n+1})\right|\le C\log^{-\alpha-1}\left(\left|2^j\xi\right|+2\right),\qquad\left|\nabla_\xi\widehat{\sigma_j}\right|\le C2^j.\tag{51}$$

When $\phi(|t|)=|t|$, if $n=2$, we also have the above estimates (see [14]). Set $m_j(\widetilde{\xi})=\widehat{\sigma_j}(\widetilde{\xi})$, $m_j^l(\widetilde{\xi})=m_j(\widetilde{\xi})\psi(2^{j-l}|\widetilde{\xi}|)$. Recall T_j^l by $\widehat{T_j^l f}(\widetilde{\xi}) = m_j^l(\widetilde{\xi})\widehat{f}(\widetilde{\xi})$. Straightforward computations lead to

$$\begin{gathered}\left\|m_j^l\left(2^{-j}\widetilde{\xi}\right)\right\|_{L^\infty}\le C\log^{-\alpha-1}\left(2+2^l\right),\\\left\|\nabla_\xi m_j^l\left(2^{-j}\widetilde{\xi}\right)\right\|_{L^\infty}\le C,\\\operatorname{supp}\left\{m_j^l\left(2^{-j}\widetilde{\xi}\right)\right\}\subset\left\{\left|\widetilde{\xi}\right|\le 2^{l+2}\right\}.\end{gathered}\tag{52}$$

Let $\widetilde{T}_j^l$ be the operator defined by $\widehat{\widetilde{T}_j^l f}(\widetilde{\xi}) = m_j^l(2^{-j}\widetilde{\xi})\widehat{f}(\widetilde{\xi})$. Denote by $T_{j,b,1}^l f = [b,T_j^l]f$ and $T_{j,b,0}^l f = T_j^l f$. Similarly, denote by $\widetilde{T}_{j,b,1}^l f = [b,\widetilde{T}_j^l]f$ and $\widetilde{T}_{j,b,0}^l f = \widetilde{T}_j^l f$. Thus via the Plancherel theorem and Lemma 8 it is stated that for any fixed $0<\nu<1$, $k\in\{0,1\}$,

$$\left\|\widetilde{T}_{j,b,k}^l f\right\|_{L^2}\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^{n+1})}^k\log^{(-\alpha-1)\nu+1}\left(2+2^l\right)\|f\|_{L^2},\quad l\ge 1+\left[\log\sqrt{2}\right].\tag{53}$$

Dilation-invariance says that

$$\left\|T_{j,b,k}^l f\right\|_{L^2}\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^{n+1})}^k\log^{(-\alpha-1)\nu+1}\left(2+2^l\right)\|f\|_{L^2},\quad l\ge 1+\left[\log\sqrt{2}\right].\tag{54}$$

By the proof of Theorem 1 in [20], we can get

$$\left\|V_l f\right\|_{L^2}\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^{n+1})}\log^{(-\alpha-1)\nu+1}\left(2+2^l\right)\|f\|_{L^2},\quad l\ge 1+\left[\log\sqrt{2}\right].\tag{55}$$

So take $\nu\to 1$, and we have

$$\begin{aligned}&\left\|\sum_{l=1+[\log\sqrt{2}]}^{\infty}V_l f\right\|_{L^2(\mathbb{R}^{n+1})}\\&\quad\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^{n+1})}\sum_{l=1+[\log\sqrt{2}]}^{\infty}l^{(-\alpha-1)\nu+1}\|f\|_{L^2(\mathbb{R}^{n+1})}\\&\quad\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^{n+1})}\|f\|_{L^2(\mathbb{R}^{n+1})}.\end{aligned}\tag{56}$$

Then, by (50) and (56) we obtain Theorem 1. □

Proof of Theorem 2. By (36), we have

$$\begin{aligned}&\left\|\left[b,T_{\phi,\Omega}\right]f\right\|_{L^p(\mathbb{R}^3)}\\&\quad\le\left\|\sum_{l=-\infty}^{[\log\sqrt{2}]}V_l f\right\|_{L^p(\mathbb{R}^3)}+\left\|\sum_{l=[\log\sqrt{2}]+1}^{\infty}V_l f\right\|_{L^p(\mathbb{R}^3)}.\end{aligned}\tag{57}$$

For $\|\sum_{l=-\infty}^{[\log\sqrt{2}]}V_l f\|_{L^p(\mathbb{R}^3)}$, recall $T_j^l f(\widetilde{x}) = T_j S_{l-j}f(\widetilde{x})$; then, $V_l f(\widetilde{x}) = \sum_{j\in\mathbb{Z}}[b,S_{l-j}T_j S_{l-j}^2]f(\widetilde{x})$. $\phi(|t|)=|t|$, and applying Lemma 11, we get for $1<p<\infty$

$$\left\|V_l f\right\|_{L^p}\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^{n+1})}\|f\|_{L^p},\quad l\le\left[\log\sqrt{2}\right].\tag{58}$$

Interpolating between (49) and (58) with $n=2$, as the proof of Theorem 1 in [20], we can get

$$\left\|\sum_{l=-\infty}^{[\log\sqrt{2}]}V_l f\right\|_{L^p(\mathbb{R}^3)}\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^3)}\|f\|_{L^p(\mathbb{R}^3)}.\tag{59}$$

For $\|\sum_{l=1+[\log\sqrt{2}]}^{\infty}V_l f\|_{L^p(\mathbb{R}^3)}$, $\phi(|t|)=|t|$, and applying Lemma 11, we get for any fixed $0<\tau<1/2$, $1<p<\infty$,

$$\left\|V_l f\right\|_{L^p}\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^{n+1})}\frac{2^{\tau l}}{\tau}\|f\|_{L^p},\quad l\ge 1+\left[\log\sqrt{2}\right].\tag{60}$$

Take $\tau = 1/l$; then, we get

$$\|V_l f\|_{L^p} \leq Cl\|b\|_{\mathrm{BMO}(\mathbb{R}^{n+1})}\|f\|_{L^p}, \quad l \geq 1 + \left[\log \sqrt{2}\right]. \tag{61}$$

For $\phi(|t|) = |t|$, (55) can be established only when $n = 2$, so interpolating between (55) and (61) with $n = 2$, as the proof of Theorem 1 in [20], we get

$$\left\| \sum_{l=1+[\log \sqrt{2}]}^{\infty} V_l f \right\|_{L^p(\mathbb{R}^3)} \leq C\|b\|_{\mathrm{BMO}(\mathbb{R}^3)}\|f\|_{L^p(\mathbb{R}^3)}. \tag{62}$$

Then, by (59) and (62) we obtain Theorem 2. □

4. The proof of Theorems 4 and 7

We begin with a lemma, which plays an important role in proving Theorem 4.

Lemma 12. *Let $b(x) \in BMO(\mathbb{R}^n)$, $\tilde{x} = (x, x_{n+1}) \in \mathbb{R}^{n+1}$, and $B(\tilde{x}) = b(x)$; then, $B(\tilde{x}) \in BMO(\mathbb{R}^{n+1})$ and $\|B\|_{BMO(\mathbb{R}^{n+1})} = \|b\|_{BMO(\mathbb{R}^n)}$.*

Proof. We know

$$\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} = \sup_{Q\subset\mathbb{R}^n} \frac{1}{|Q|}\int_Q \left|b(x) - b_Q\right| dx, \tag{63}$$

where $b_Q = (1/|Q|)\int_Q b(x)dx$ and Q is the square in $\mathbb{R}^n$ whose edges are parallel to the axis. So

$$\| B\|_{\mathrm{BMO}(\mathbb{R}^{n+1})} = \sup_{\widetilde{Q}\subset\mathbb{R}^{n+1}} \frac{1}{|\widetilde{Q}|}\int_{\widetilde{Q}} \left|B(\tilde{x}) - B_{\widetilde{Q}}\right| d\tilde{x}, \tag{64}$$

where $\widetilde{Q}$ is the square in $\mathbb{R}^{n+1}$ whose edges are parallel to the axis. Consider

$$\begin{aligned} B_{\widetilde{Q}} &= \frac{1}{|\widetilde{Q}|}\int_{\widetilde{Q}} B(\tilde{x})\, d\tilde{x} \\ &= \frac{1}{a|Q|}\int_m^{m+a}\int_Q b(x)\,dx\,dx_{n+1} \\ &= \frac{1}{a|Q|}\int_Q b(x)\,dx\int_m^{m+a} dx_{n+1} \\ &= \frac{1}{|Q|}\int_Q b(x)\,dx = b_Q, \end{aligned} \tag{65}$$

where Q is the projection on $\mathbb{R}^n$ of $\widetilde{Q}$ and a is the side length of $\widetilde{Q}$. Then

$$\begin{aligned} \|B\|_{\mathrm{BMO}(\mathbb{R}^{n+1})} &= \sup_{\widetilde{Q}\subset\mathbb{R}^{n+1}} \frac{1}{|\widetilde{Q}|}\int_{\widetilde{Q}} \left|B(\tilde{x}) - B_{\widetilde{Q}}\right| d\tilde{x} \\ &= \sup_{\widetilde{Q}\subset\mathbb{R}^{n+1}} \frac{1}{a|Q|}\int_m^{m+a}\int_Q \left|b(x) - b_Q\right| dx\,dx_{n+1} \\ &= \sup_{\widetilde{Q}\subset\mathbb{R}^{n+1}} \frac{1}{a|Q|}\int_Q \left|b(x) - b_Q\right| dx\int_m^{m+a} dx_{n+1} \\ &= \sup_{\widetilde{Q}\subset\mathbb{R}^{n+1}} \frac{1}{|Q|}\int_Q \left|b(x) - b_Q\right| dx \\ &= \sup_{Q\subset\mathbb{R}^n} \frac{1}{|Q|}\int_Q \left|b(x) - b_Q\right| dx = \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}. \end{aligned} \tag{66}$$

□

Proof of Theorem 4. By Lemma 12, $B \in \mathrm{BMO}(\mathbb{R}^{n+1})$. Using the method in [5], for $f \in L^p(\mathbb{R}^n)$ and $N \in \mathbb{N}$, let $F_N(x, x_{n+1}) = f(x)e^{-ix_{n+1}}\chi_{[-N,N]}(x_{n+1})$. Then by mean value theorem of integrals and Lemma 12, we have

$$\begin{aligned} 2N\int_{\mathbb{R}^n} \Bigg| b(x)\int_{\mathbb{R}^n} &\frac{\Omega(y)}{|y|^n} f(x-y)\, e^{i\phi(|y|)}\chi_{[-N,N]} \\ &\times (x_{n+1} - \phi(|y|))\,dy \\ &- \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^n} b(x-y)\, f(x-y)\, e^{i\phi(|y|)} \\ &\times \chi_{[-N,N]}(x_{n+1} - \phi(|y|))\,dy\Bigg|^p dx \\ \leq \int_{\mathbb{R}}\int_{\mathbb{R}^n} \Bigg| b(x)\int_{\mathbb{R}^n} &\frac{\Omega(y)}{|y|^n} f(x-y)\, e^{-i(x_{n+1}-\phi(|y|))} \\ &\times \chi_{[-N,N]}(x_{n+1} - \phi(|y|))\,dy \\ &- \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^n} b(x-y)\, f(x-y) \\ &\times e^{-i(x_{n+1}-\phi(|y|))}\chi_{[-N,N]} \\ &\times (x_{n+1} - \phi(|y|))\,dy\Bigg|^p dx\,dx_{n+1} \\ = \left\|\left[B, T_{\phi,\Omega}\right] F_N\right\|^p_{L^p(\mathbb{R}^{n+1})} & \\ \leq C\|B\|^p_{\mathrm{BMO}(\mathbb{R}^{n+1})}\|F_N\|^p_{L^p(\mathbb{R}^{n+1})} & \\ = C2N\|b\|^p_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|^p_{L^p(\mathbb{R}^n)}. & \end{aligned} \tag{67}$$

Dividing both sides by $2N$ and letting $N \to \infty$, we obtain

$$\left\|\left[b, T_\phi\right] f\right\|_{L^p(\mathbb{R}^n)} \leq C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}. \tag{68}$$

Thus, we obtain Theorem 4. □

Proof of Theorem 7. Theorem 7 can be proved by using arguments which are essentially the same as the proof of Theorem 1 in [20]. Only the following two things must be modified.

(i) Instead of $K_j(x)$ and $T_j f(x)$, we use

$$K_{j,\phi}(x) = e^{i\phi(x)} \frac{\Omega(x)}{|x|^n} \chi_{\{2^j<|x|\le 2^{j+1}\}},$$
$$T_{j,\phi} f(x) = K_{j,\phi} * f(x) = \int_{2^j<|y|\le 2^{j+1}} \frac{\Omega(x)}{|x|^n} e^{i\phi(x)} f(x-y)\, dy. \quad (69)$$

(ii) Since $\Omega(\theta)$ is odd and $\phi(\theta t)$ is even with respect to θ, we get $\Omega(\theta)e^{i\phi(\theta t)}$ is odd and $\int_{S^{n-1}} \Omega(\theta)e^{i\phi(\theta t)} d\sigma(\theta) = 0$. So we use the estimates in [21]: Consider

$$\left|\widehat{K}_{j,\phi}(\xi)\right| \le C\|\Omega\|_{L^1} \left|2^j\xi\right|,$$
$$\left|\widehat{K}_{j,\phi}(\xi)\right| \le C\log^{-\alpha-1} \left|2^j\xi + 2\right| \quad (70)$$

in the proof, and we omit the details. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The research was supported by NCET of China (Grant: NCET-11-0574), the Fundamental Re-search Funds for the Central Universities (FRF-TP-12-006B), NSF of China (Grant: 10901017), and CSC Funding of China.

References

[1] F. Ricci and E. M. Stein, "Harmonic analysis on nilpotent groups and singular integrals. I. Oscillatory integrals," *Journal of Functional Analysis*, vol. 73, no. 1, pp. 179–194, 1987.

[2] Y. Hu and Y. Pan, "Boundedness of oscillatory singular integrals on Hardy spaces," *Arkiv för Matematik*, vol. 30, no. 2, pp. 311–320, 1992.

[3] S. Lu and Y. Zhang, "Criterion on L^p-boundedness for a class of oscillatory singular integrals with rough kernels," *Revista Matemática Iberoamericana*, vol. 8, pp. 201–218, 1992.

[4] H. Ojanen, "Weighted estimates for rough oscillatory singular integrals," *Journal of Fourier Analysis and Applications*, vol. 6, no. 4, pp. 427–436, 2000.

[5] D. Fan and Y. Pan, "Singular integral operators with rough kernels supported by subvarieties," *American Journal of Mathematics*, vol. 119, no. 4, pp. 799–839, 1997.

[6] L. Grafakos and A. Stefanov, "L^p bounds for singular integrals and maximal singular integrals with rough kernels," *Indiana University Mathematics Journal*, vol. 47, no. 2, pp. 455–469, 1998.

[7] H. Al-Qassem and A. Al-Salman, "Rough Marcinkiewicz integral operators," *International Journal of Mathematics and Mathematical Sciences*, vol. 27, no. 8, pp. 495–503, 2001.

[8] A. Al-Salman and Y. Pan, "Singular integrals with rough kernels," *Canadian Mathematical Bulletin*, vol. 47, no. 1, pp. 3–11, 2004.

[9] J. Chen, D. Fan, and Y. Pan, "A note on a Marcinkiewicz integral operator," *Mathematische Nachrichten*, vol. 227, pp. 33–42, 2001.

[10] D. X. Chen and S. Z. Lu, "L^p boundedness of parabolic Littlewood-Paley operator with rough kernel belonging to $F(S^{n-1})$," *Acta Mathematica Scientia*, vol. 31, pp. 343–350, 2011.

[11] J. Chen and C. Zhang, "Boundedness of rough singular integral operators on the Triebel—Lizorkin spaces," *Journal of Mathematical Analysis and Applications*, vol. 337, no. 2, pp. 1048–1052, 2008.

[12] D. Fan, K. Guo, and Y. Pan, "A note of a rough singular integral operator," *Mathematical Inequalities and Applications*, vol. 2, no. 1, pp. 73–81, 1999.

[13] L. Grafakos, P. Honzik, and D. Ryabogin, "On the p-independence boundedness property of Calderón-Zygmund theory," *Journal für die Reine und Angewandte Mathematik*, vol. 602, pp. 227–234, 2007.

[14] L. C. Cheng and Y. Pan, "L^p bounds for singular integrals associated to surfaces of revolution," *Journal of Mathematical Analysis and Applications*, vol. 265, no. 1, pp. 163–169, 2002.

[15] D. Fan and S. Sato, "A note on singular integrals associated with a variable surface of revolution," *Mathematical Inequalities and Applications*, vol. 12, no. 2, pp. 441–454, 2009.

[16] Y. Pan, L. Tang, and D. Yang, "Boundedness of rough singular integrals associated with surfaces of revolution," *Advances in Mathematics*, vol. 32, no. 6, pp. 677–682, 2003 (Chinese).

[17] H. Wu, "L^p bounds for Marcinkiewicz integrals associated to surfaces of revolution," *Journal of Mathematical Analysis and Applications*, vol. 321, no. 2, pp. 811–827, 2006.

[18] Y. Zhang, J. Chen, and C. Zhang, "Boundedness of singular integrals along surfaces on Lebesgue spaces," *Applied Mathematics-A Journal of Chinese Universities*, vol. 25, no. 2, pp. 192–198, 2010.

[19] G. Hu, "$L^p(R^n)$ boundedness for the commutator of a homogeneous singular integral operator," *Studia Mathematica*, vol. 154, no. 1, pp. 13–27, 2003.

[20] Y. Chen and Y. Ding, "L^p bounds for the commutators of singular integrals and Maximal singular integrals with rough kernels," *Transactions of the American Mathematical Society*, 2014.

[21] C. Zhang and Y. Zhang, "Boundedness of oscillatory singular integral with rough kernels on Triebel-Lizorkin spaces," *Applied Mathematics*, vol. 28, no. 1, pp. 90–100, 2013.

[22] G. Hu, "$L^2(R^n)$ boundedness for the commutators of convolution operators," *Nagoya Mathematical Journal*, vol. 163, pp. 55–70, 2001.

[23] W. Kim, S. Wainger, J. Wright, and S. Ziesler, "Singular integrals and maximal functions associated to surfaces of revolution," *Bulletin of the London Mathematical Society*, vol. 28, no. 3, pp. 291–296, 1996.

[24] J. Duoandikoetxea and J. L. Rubio de Francia, "Maximal and singular integral operators via Fourier transform estimates," *Inventiones Mathematicae*, vol. 84, no. 3, pp. 541–561, 1986.

[25] J. Bony, "Calcul symbolique et propagation des singularities pour les equations aux derivees partielles non lineaires," *Annales Scientifiques de l'École Normale Supérieure*, vol. 14, no. 2, pp. 209–246, 1981.