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Research Article

Some Inevitable Remarks on "Tripled Fixed Point Theorems for Mixed Monotone Kannan Type Contractive Mappings"

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We advise that the proof of Theorem 12 given by Borcut et al. (2014) is not correct, and it cannot be corrected using the same technique. Furthermore, we present some similar results as an approximation to the opened question if that statement is valid.

1. Introduction

The definition of *coupled fixed point* was firstly given by Guo and Lakshmikantham in [1]. This concept, in the context of metric space, was reconsidered by Gnana Bhaskar and Lakshmikantham [2] in 2006 and by Lakshmikantham and Ciric [3] in the coincidence case. Later, Karapınar investigated this notion in the context of cone metric space. After that, Berinde and Borcut [4] presented the notion of *tripled fixed point* obtaining similar results, and the same authors extended their work to the coincidence case in [5] (see also, e.g., [6–9]).

A *coupled fixed point of* $F : X\times X \to X$ is a point $(x, y) \in X^2$ such that $F(x, y) = x$ and $F(y, x) = y$. In order to ensure existence and uniqueness of coupled fixed points, Bhaskar and Lakshmikantham introduced the concept of mapping having the mixed monotone property. Henceforth, let $\preccurlyeq$ be a partial order on X. The mapping F is said to have the *mixed monotone property (with respect to* $\preccurlyeq$*)* if $F(x, y)$ is monotone nondecreasing in x and monotone nonincreasing in y; that is, for any $x, y \in X$,

$$\begin{aligned} x_1, x_2 \in X, &\quad x_1 \preccurlyeq x_2 \Longrightarrow F(x_1, y) \preccurlyeq F(x_2, y), \\ y_1, y_2 \in X, &\quad y_1 \preccurlyeq y_2 \Longrightarrow F(x, y_1) \succcurlyeq F(x, y_2). \end{aligned} \tag{1}$$

Inspired by the previous notions, Berinde and Borcut defined the concepts of *tripled fixed point* and *mixed monotone property* as follows. A *tripled fixed point of* $F : X \times X \times X \to X$ is a point $(x, y, z) \in X^3$ such that $F(x, y, z) = x$, $F(y, x, y) = y$, and $F(z, y, z) = z$. The mapping F is said to have the *mixed monotone property (with respect to* $\preccurlyeq$*)* if $F(x, y, z)$ is monotone nondecreasing in x and z, and it is monotone nonincreasing in y; that is, for any $x, y, z \in X$,

$$\begin{aligned} x_1, x_2 \in X, &\quad x_1 \preccurlyeq x_2 \Longrightarrow F(x_1, y, z) \preccurlyeq F(x_2, y, z), \\ y_1, y_2 \in X, &\quad y_1 \preccurlyeq y_2 \Longrightarrow F(x, y_1, z) \succcurlyeq F(x, y_2, z), \\ y_1, y_2 \in X, &\quad y_1 \preccurlyeq y_2 \Longrightarrow F(x, y, z_1) \preccurlyeq F(x, y_2, z). \end{aligned} \tag{2}$$

The second equation that defines a tripled fixed point, that is, $F(y, x, y) = y$, uses the point y twice in the arguments of F. This fact is necessary to ensure the existence of tripled fixed points of a nonlinear contration because, in such a case, the mixed monotone property is applicable.

Very recently, as a continuation of their pioneering works in the tripled case, Borcut et al. announced in [10] the following result.

Theorem 1 (Borcut et al. [10], Theorem 12). *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^3 \to X$ be a mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ such that*

$$
\begin{aligned}
& d(F(x,y,z), F(u,v,w)) \\
& \quad \leq \frac{k}{8} \left[d(x, F(x,y,z)) + d(y, F(y,x,y)) \right. \\
& \qquad + d(z, F(z,y,x)) + d(u, F(u,v,w)) \\
& \qquad \left. + d(v, F(v,u,v)) + d(w, F(w,v,u)) \right]
\end{aligned} \tag{3}
$$

for all $x \succcurlyeq u$, $y \preccurlyeq v$, $z \succcurlyeq w$. Also suppose that either

(a) *F is continuous, or*

(b) *X has the following properties:*

 (i) *if a nondecreasing sequence $\{x_m\} \to x$, then $x_m \preccurlyeq x$ for all m;*

 (ii) *if a nonincreasing sequence $\{y_m\} \to y$, then $y_m \succcurlyeq y$ for all m.*

If there exist $x_0, y_0, z_0 \in X$ such that

$$
\begin{aligned}
& x_0 \preccurlyeq F(x_0, y_0, z_0), \qquad y_0 \succcurlyeq F(y_0, x_0, y_0), \\
& \qquad z_0 \preccurlyeq F(z_0, y_0, x_0),
\end{aligned} \tag{4}
$$

then F has a tripled fixed point in X; that is, there exist $x, y, z \in X$ such that

$$
F(x,y,z) = x, \qquad F(y,x,y) = y, \qquad F(z,y,x) = z. \tag{5}
$$

This note is to advise that the proof given by the authors of the previous result is not correct, and it cannot be corrected using the same technique. Furthermore, we present some similar results as an approximation to the opened question if the previous theorem is valid.

2. A Review of the Incorrect Proof

Let us review the lines of their proof. Based on $x_0, y_0, z_0 \in X$, the authors defined, recursively, for all $n \geq 0$,

$$
\begin{aligned}
& x_{n+1} = F(x_n, y_n, z_n), \qquad y_{n+1} = F(y_n, x_n, y_n), \\
& \qquad z_{n+1} = F(z_n, y_n, x_n),
\end{aligned} \tag{6}
$$

and they proved that $\{x_n\}$ and $\{z_n\}$ were monotone nondecreasing sequences and $\{y_n\}$ was a monotone nonincreasing sequence in $(X, \preccurlyeq)$. Then, they defined, for all $n \in \mathbb{N}$,

$$
\begin{aligned}
& D_{x_{n+1}} = d(x_{n+1}, x_n), \qquad D_{y_{n+1}} = d(y_{n+1}, y_n), \\
& D_{z_{n+1}} = d(z_{n+1}, z_n), \qquad D_{n+1} = D_{x_{n+1}} + D_{y_{n+1}} + D_{z_{n+1}}.
\end{aligned} \tag{7}
$$

Using the contractivity condition (3), they proved that, taking into account that $x_n \succcurlyeq x_{n-1}$, $y_n \preccurlyeq y_{n-1}$, and $z_n \succcurlyeq z_{n-1}$,

$$
\begin{aligned}
D_{x_{n+1}} &= d(x_{n+1}, x_n) = d(F(x_n, y_n, z_n), F(x_{n-1}, y_{n-1}, z_{n-1})) \\
&\leq \frac{k}{8} \left[d(x_n, F(x_n, y_n, z_n)) + d(y_n, F(y_n, x_n, y_n)) \right. \\
&\qquad + d(z_n, F(z_n, y_n, x_n)) \\
&\qquad + d(x_{n-1}, F(x_{n-1}, y_{n-1}, z_{n-1})) \\
&\qquad + d(y_{n-1}, F(y_{n-1}, x_{n-1}, y_{n-1})) \\
&\qquad \left. + d(z_{n-1}, F(z_{n-1}, y_{n-1}, x_{n-1})) \right] \\
&= \frac{k}{8} \left[d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) \right. \\
&\qquad \left. + d(x_{n-1}, x_n) + d(y_{n-1}, y_n) + d(z_{n-1}, z_n) \right] \\
&= \frac{k}{8} \left[D_{x_n} + D_{y_n} + D_{z_n} + D_{x_{n+1}} + D_{y_{n+1}} + D_{z_{n+1}} \right].
\end{aligned} \tag{8}
$$

Based on this inequality, the authors immediately announced that

$$
D_{y_{n+1}} \leq \frac{k}{8} \left[D_{x_n} + 2D_{y_n} + D_{x_{n+1}} + 2D_{y_{n+1}} \right], \tag{9}
$$

$$
D_{z_{n+1}} \leq \frac{k}{8} \left[D_{x_n} + D_{y_n} + D_{z_n} + D_{x_{n+1}} + D_{y_{n+1}} + D_{z_{n+1}} \right] \tag{10}
$$

(see [10, page 4, inequalities (2.4) and (2.5)]). However, these last two inequalities are false. In fact, we can only prove that

$$
\begin{aligned}
D_{y_{n+1}} &= d(y_{n+1}, y_n) = d(F(y_n, x_n, y_n), F(y_{n-1}, x_{n-1}, y_{n-1})) \\
&\leq \frac{k}{8} \left[d(y_n, F(y_n, x_n, y_n)) + d(x_n, F(x_n, y_n, x_n)) \right. \\
&\qquad + d(y_n, F(y_n, x_n, y_n)) \\
&\qquad + d(y_{n-1}, F(y_{n-1}, x_{n-1}, y_{n-1})) \\
&\qquad + d(x_{n-1}, F(x_{n-1}, y_{n-1}, x_{n-1})) \\
&\qquad \left. + d(y_{n-1}, F(y_{n-1}, x_{n-1}, y_{n-1})) \right] \\
&\leq \frac{k}{8} \left[2d(y_n, y_{n+1}) + d(x_n, F(x_n, y_n, x_n)) \right. \\
&\qquad \left. + 2d(y_{n-1}, y_n) + d(x_{n-1}, F(x_{n-1}, y_{n-1}, x_{n-1})) \right] \\
&\leq \frac{k}{8} \left[2D_{y_{n+1}} + d(x_n, F(x_n, y_n, x_n)) \right. \\
&\qquad \left. + 2D_{y_n} + d(x_{n-1}, F(x_{n-1}, y_{n-1}, x_{n-1})) \right].
\end{aligned} \tag{11}
$$

However, comparing (9) with (11), we notice that $D_{x_{n+1}} = d(x_n, x_{n+1}) = d(x_n, F(x_n, y_n, z_n))$ does not necessarily coincide with $d(x_n, F(x_n, y_n, x_n))$ and, similarly, $D_{x_n} = d(x_{n-1}, x_n) = d(x_{n-1}, F(x_{n-1}, y_{n-1}, z_{n-1}))$ is not necessarily equal to

$d(x_{n-1}, F(x_{n-1}, y_{n-1}, x_{n-1}))$. Therefore, inequality (9) cannot be ensured.

Exactly in the same way, it can be possible to see that (10) cannot be proved using the contractivity condition (3). In such a case, the proof given by the authors, which decisively used inequalities (9) and (10), is false.

3. Some Berinde and Borcut's Type Tripled Fixed Point Theorems

For the moment, the question about whether Theorem 1 is valid is opened. The following results are some approximations to this problem, using contractivity conditions that are inspired in (3). The main aim of this section is to show some results in this line of research using a well-known result by Ćirić [11]. Our technique is based on some very recent works which showed that most of coupled/tripled/quadrupled fixed point results can be reduced to their corresponding unidimensional theorems in different frameworks (see, for instance, [12–18]). Before that, let us introduce some notation and basic results.

Given a binary relation $\preccurlyeq$ on X, let us define

$$(x, y, z) \sqsubseteq (u, v, w) \Longleftrightarrow [x \preccurlyeq u, y \succcurlyeq v, z \preccurlyeq w]. \tag{12}$$

If $\preccurlyeq$ is a partial order on X, then $\sqsubseteq$ is also a partial order on X^3.

Given a metric d on X, let us define $d_3^s, d_3^m : X^3 \times X^3 \to [0, \infty)$, for all $(x, y, z), (u, v, w) \in X^3$, by

$$\begin{aligned} d_3^s((x, y, z), (u, v, w)) &= d(x, u) + d(y, v) + d(z, w), \\ d_3^m((x, y, z), (u, v, w)) &= \max\{d(x, u), d(y, v), d(z, w)\}. \end{aligned} \tag{13}$$

Then d_3^s and d_3^m are metrics on X^3. In addition to this, if d is complete, then d_3^s and d_3^m are also complete.

Given a mapping $F : X^3 \to X$, let us denote by $T_F^3 : X^3 \to X^3$ the mapping

$$T_F^3(x, y, z) = (F(x, y, z), F(y, x, y), F(z, y, x)) \quad \forall (x, y, z) \in X^3. \tag{14}$$

Notice that a tripled fixed point of F is nothing but a fixed point of T_F^3. If F is d-continuous, then T_F^3 is d_3^s-continuous. Furthermore, if F has the mixed monotone property with respect to $\preccurlyeq$, then T_F^3 is nondecreasing with respect to $\sqsubseteq$ (see [12]). We also recall the following result.

Definition 2. Let d be a metric on X and let $\preccurlyeq$ be a partial order on X. We will say that $(X, d, \preccurlyeq)$ is *regular* if it verifies the following two properties:

(i) if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \preccurlyeq x$ for all n;

(ii) if a nonincreasing sequence $\{y_n\} \to y$, then $y_n \succcurlyeq y$ for all n.

Lemma 3. *If $(X, d, \preccurlyeq)$ is regular, then $(X^3, d_3^s, \sqsubseteq)$ is also regular.*

The first version of the following theorem was given by Ćirić in 1972 (see [11]) in the case of metric spaces that were not necessarily partially ordered. A partially ordered version can be found, for example, in [19]. Our main results will be consequences of the next result.

Theorem 4 (see e.g., [19]). *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T : X \to X$ be a nondecreasing mapping and let $k \in [0, 1)$ be such that*

$$d(Tx, Ty) \leq k \max\left(d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right) \tag{15}$$

for all $x, y \in X$ such that $x \preccurlyeq y$. Also assume that F is continuous or $(X, d, \preccurlyeq)$ is regular. If there exist $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$, then F has a fixed point.

In the following result, we found some terms that play an important role in the contractivity condition (3).

Theorem 5. *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^3 \to X$ be a mapping having the mixed monotone property on X. Suppose that there exists $k \in [0, 1)$ such that*

$$\begin{aligned} &d(F(x, y, z), F(u, v, w)) \\ &\quad \leq \frac{k}{4} \max\Big(d(x, u) + d(y, v) + d(z, w), \\ &\qquad d(x, F(x, y, z)) + d(z, F(z, y, x)), \\ &\qquad d(u, F(u, v, w)) + d(v, F(w, v, u)), \\ &\qquad (d(u, F(x, y, z)) + d(w, F(z, y, x)) \\ &\qquad + d(x, F(u, v, w)) + d(z, F(w, v, u))) \\ &\qquad \times (2)^{-1}\Big) \end{aligned} \tag{16}$$

for all $x, y, z, u, v, w \in X$ such that $x \preccurlyeq u$, $y \succcurlyeq v$, and $z \preccurlyeq w$. Also assume that F is continuous or $(X, d, \preccurlyeq)$ is regular. If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \preccurlyeq F(x_0, y_0, z_0), \quad y_0 \succcurlyeq F(y_0, x_0, y_0), \quad z_0 \preccurlyeq F(z_0, y_0, x_0), \tag{17}$$

then F has a tripled fixed point in X; that is, there exist $x, y, z \in X$ such that

$$F(x, y, z) = x, \quad F(y, x, y) = y, \quad F(z, y, x) = z. \tag{18}$$

Proof. As F has the mixed monotone property on X with respect to $\preccurlyeq$, it follows that T_F^3 is $\sqsubseteq$-nondecreasing. Let us define, for all $x, y, z, u, v, w \in X$,

$$\begin{aligned}
M&((x,y,z),(u,v,w))\\
&= \max\Big(d_3^s((x,y,z),(u,v,w)),\\
&\qquad d_3^s\left((x,y,z),T_F^3(x,y,z)\right),\\
&\qquad d_3^s\left((u,v,w),T_F^3(u,v,w)\right),\\
&\qquad \Big(d_3^s\left((x,y,z),T_F^3(u,v,w)\right)\\
&\qquad\quad + d_3^s\left((u,v,w),T_F^3(x,y,z)\right)\Big)\\
&\qquad \times (2)^{-1}\Big)\\
&= \max\big(d(x,u)+d(y,v)+d(z,w), d(x,F(x,y,z))\\
&\qquad + d(y,F(y,x,y)) + d(z,F(z,y,x)),\\
&\qquad d(u,F(u,v,w)) + d(v,F(v,u,v))\\
&\qquad + d(w,F(w,v,u)),\\
&\qquad (d(x,F(u,v,w)) + d(y,F(v,u,v))\\
&\qquad + d(z,F(w,v,u)) + d(u,F(x,y,z))\\
&\qquad + d(v,F(y,x,y)) + d(w,F(z,y,x)))\\
&\qquad \times (2)^{-1}\big).
\end{aligned} \tag{19}$$

Assume that $x \preccurlyeq u$, $y \succcurlyeq v$, and $z \preccurlyeq w$; that is, $(x,y,z) \sqsubseteq (u,v,w)$. In this case, the contractivity condition (16) guarantees that

$$\begin{aligned}
d&(F(x,y,z),F(u,v,w))\\
&\le \frac{k}{4}\max\big(d(x,u)+d(y,v)+d(z,w),\\
&\qquad d(x,F(x,y,z)) + d(z,F(z,y,x)),\\
&\qquad d(u,F(u,v,w)) + d(v,F(w,v,u)),\\
&\qquad (d(u,F(x,y,z)) + d(w,F(z,y,x))\\
&\qquad + d(x,F(u,v,w)) + d(z,F(w,v,u)))\\
&\qquad \times (2)^{-1}\big)\\
&\le \frac{k}{4}M((x,y,z),(u,v,w)).
\end{aligned} \tag{20}$$

Taking into account that $z \preccurlyeq w$, $y \succcurlyeq v$, and $x \preccurlyeq u$, we also find the same upper bound:

$$\begin{aligned}
d&(F(z,y,x),F(w,v,u))\\
&\le \frac{k}{4}\max\big(d(z,w)+d(y,v)+d(x,u),\\
&\qquad d(z,F(z,y,x)) + d(x,F(x,y,z)),\\
&\qquad d(v,F(w,v,u)) + d(u,F(u,v,w)),\\
&\qquad (d(w,F(z,y,x)) + d(u,F(x,y,z))\\
&\qquad + d(z,F(w,v,u)) + d(x,F(u,v,w)))\\
&\qquad \times (2)^{-1}\big)\\
&\le \frac{k}{4}M((x,y,z),(u,v,w)).
\end{aligned} \tag{21}$$

Furthermore, as $v \preccurlyeq y$, $u \succcurlyeq x$, and $v \preccurlyeq y$,

$$\begin{aligned}
d&(F(y,x,y),F(v,u,v))\\
&\le \frac{k}{4}\max\big(d(y,v)+d(x,u)+d(y,v),\\
&\qquad d(y,F(y,x,y)) + d(y,F(y,x,y)),\\
&\qquad d(v,F(v,u,v)) + d(v,F(v,u,v)),\\
&\qquad (d(v,F(y,x,y)) + d(v,F(y,x,y))\\
&\qquad + d(y,F(v,u,v)) + d(y,F(v,u,v)))\\
&\qquad \times (2)^{-1}\big)\\
&= \frac{k}{4}\max\big(d(x,u)+2d(y,v),\\
&\qquad 2d(y,F(y,x,y)), 2d(v,F(v,u,v)),\\
&\qquad d(v,F(y,x,y)) + d(y,F(v,u,v))\big)\\
&\le \frac{k}{4}2M((x,y,z),(u,v,w)) = \frac{k}{2}M((x,y,z),(u,v,w)).
\end{aligned} \tag{22}$$

Joining the last three inequalities, we deduce that, for all $(x,y,z),(u,v,w) \in X^3$ such that $(x,y,z) \sqsubseteq (u,v,w)$,

$$\begin{aligned}
d_3^s&\left(T_F^3(x,y,z),T_F^3(u,v,w)\right)\\
&= d_3^s((F(x,y,z),F(y,x,y),F(z,y,x)),
\end{aligned}$$

$$
\begin{aligned}
&(F(u,v,w), F(v,u,v), F(w,v,u))) \\
&= d(F(x,y,z), F(u,v,w)) + d(F(y,x,y), F(v,u,v)) \\
&\quad + d(F(z,y,x), F(w,v,u)) \\
&= \left(\frac{k}{4} + \frac{k}{4} + \frac{k}{2}\right) M((x,y,z),(u,v,w)) \\
&= kM((x,y,z),(u,v,w)).
\end{aligned} \tag{23}
$$

As $(x_0, y_0, z_0) \sqsubseteq T_F^3(x_0, y_0, z_0)$, Theorem 4 guarantees that T_F^3 has a fixed point; that is, F has a tripled fixed point. □

Theorem 6. *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^3 \to X$ be a mapping having the mixed monotone property on X. Suppose that there exists $k \in [0, 1)$ such that*

$$
\begin{aligned}
&d(F(x,y,z), F(u,v,w)) \\
&\quad \le k \max\{d(x, F(x,y,z)), d(z, F(z,y,x)), \\
&\qquad d(u, F(u,v,w)), d(w, F(w,v,u))\}
\end{aligned} \tag{24}
$$

for all $x, y, z, u, v, w \in X$ such that $x \preccurlyeq u$, $y \succcurlyeq v$, and $z \preccurlyeq w$. Also assume that F is continuous or $(X, d, \preccurlyeq)$ is regular. If there exist $x_0, y_0, z_0 \in X$ such that

$$
\begin{gathered}
x_0 \preccurlyeq F(x_0, y_0, z_0), \qquad y_0 \succcurlyeq F(y_0, x_0, y_0), \\
z_0 \preccurlyeq F(z_0, y_0, x_0),
\end{gathered} \tag{25}
$$

then F has a tripled fixed point in X; that is, there exist $x, y, z \in X$ such that

$$
F(x,y,z) = x, \qquad F(y,x,y) = y, \qquad F(z,y,x) = z. \tag{26}
$$

Proof. Following the lines of the previous proof, consider X^3 provided with the metric d_3^m. Assume that $x \preccurlyeq u$, $y \succcurlyeq v$, and $z \preccurlyeq w$; that is, $(x,y,z) \sqsubseteq (u,v,w)$. In this case, the contractivity condition (24) guarantees that

$$
\begin{aligned}
&d(F(x,y,z), F(u,v,w)) \\
&\quad \le k \max\{d(x, F(x,y,z)), d(z, F(z,y,x)), \\
&\qquad d(u, F(u,v,w)), d(w, F(w,v,u))\},
\end{aligned} \tag{27}
$$

and the same upper bound is valid for $d(F(z,y,x), F(w,v,u))$. Moreover, as $v \preccurlyeq y$, $u \succcurlyeq x$, and $v \preccurlyeq y$,

$$
\begin{aligned}
&d(F(v,u,v), F(y,x,y)) \\
&\quad \le k \max\{d(v, F(v,u,v)), d(y, F(y,x,y))\}.
\end{aligned} \tag{28}
$$

Therefore

$$
\begin{aligned}
&d_3^m\left(T_F^3(x,y,z), T_F^3(u,v,w)\right) \\
&\quad = d_3^m((F(x,y,z), F(y,x,y), F(z,y,x)), \\
&\qquad (F(u,v,w), F(v,u,v), F(w,v,u))) \\
&\quad = \max\{d(F(x,y,z), F(u,v,w)), \\
&\qquad d(F(y,x,y), F(v,u,v)), \\
&\qquad d(F(z,y,x), F(w,v,u))\} \\
&\quad \le k \max\{d(x, F(x,y,z)), d(y, F(y,x,y)), \\
&\qquad d(z, F(z,y,x)), d(u, F(u,v,w)), \\
&\qquad d(v, F(v,u,v)), d(w, F(w,v,u))\} \\
&\quad = k \max\left\{d_3^m\left((x,y,z), T_F^3(x,y,z)\right),\right. \\
&\qquad \left. d\left((u,v,w), T_F^3(u,v,w)\right)\right\}.
\end{aligned} \tag{29}
$$

Theorem 4 guarantees that T_F^3 has a fixed point; that is, F has a tripled fixed point. □

The following particularization is also inspired by some Berinde and Borcut's results.

Corollary 7. *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^3 \to X$ be a mapping having the mixed monotone property on X. Suppose that there exists $a, b, c, e \in [0, 1)$ such that $a + b + c + e < 1$ and*

$$
\begin{aligned}
&d(F(x,y,z), F(u,v,w)) \\
&\quad \le ad(x, F(x,y,z)) + bd(z, F(z,y,x)) \\
&\qquad + cd(u, F(u,v,w)) + ed(w, F(w,v,u))
\end{aligned} \tag{30}
$$

for all $x, y, z, u, v, w \in X$ such that $x \preccurlyeq u$, $y \succcurlyeq v$, and $z \preccurlyeq w$. Also assume that F is continuous or $(X, d, \preccurlyeq)$ is regular. If there exist $x_0, y_0, z_0 \in X$ such that

$$
\begin{gathered}
x_0 \preccurlyeq F(x_0, y_0, z_0), \qquad y_0 \succcurlyeq F(y_0, x_0, y_0), \\
z_0 \preccurlyeq F(z_0, y_0, x_0),
\end{gathered} \tag{31}
$$

then F has a tripled fixed point in X; that is, there exist $x, y, z \in X$ such that

$$
F(x,y,z) = x, \qquad F(y,x,y) = y, \qquad F(z,y,x) = z. \tag{32}
$$

Proof. Let us define $k = a + b + c + e \in [0,1)$. If $(x,y,z),(u,v,w) \in X^3$ are such that $(x,y,z) \sqsubseteq (u,v,w)$, then

$$\begin{aligned}
&d(F(x,y,z),F(u,v,w)) \\
&\leq ad(x,F(x,y,z)) + bd(z,F(z,y,x)) \\
&\quad + cd(u,F(u,v,w)) + ed(w,F(w,v,u)) \\
&\leq (a+b+c+e)\max\{d(x,F(x,y,z)),d(z,F(z,y,x)), \\
&\qquad d(u,F(u,v,w)),d(w,F(w,v,u))\},
\end{aligned} \tag{33}$$

so the previous theorem is applicable. □

The following result presents a contractivity condition more similar to (3) than (16). It follows from the previous result using $a = b = c = e = k/4$.

Corollary 8. *Let $(X,\preccurlyeq)$ be a partially ordered set and suppose that there is a metric d on X such that (X,d) is a complete metric space. Let $F : X^3 \to X$ be a mapping having the mixed monotone property on X. Suppose that there exists $k \in [0,1)$ such that*

$$\begin{aligned}
&d(F(x,y,z),F(u,v,w)) \\
&\leq \frac{k}{4}[d(x,F(x,y,z)) + d(z,F(z,y,x)) \\
&\quad + d(u,F(u,v,w)) + d(w,F(w,v,u))]
\end{aligned} \tag{34}$$

for all $x,y,z,u,v,w \in X$ such that $x \preccurlyeq u$, $y \succcurlyeq v$, and $z \preccurlyeq w$. Also assume that F is continuous or $(X,d,\preccurlyeq)$ is regular. If there exist $x_0, y_0, z_0 \in X$ such that

$$\begin{gathered}
x_0 \preccurlyeq F(x_0,y_0,z_0), \qquad y_0 \succcurlyeq F(y_0,x_0,y_0), \\
z_0 \preccurlyeq F(z_0,y_0,x_0),
\end{gathered} \tag{35}$$

then F has a tripled fixed point in X; that is, there exist $x,y,z \in X$ such that

$$F(x,y,z) = x, \qquad F(y,x,y) = y, \qquad F(z,y,x) = z. \tag{36}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

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