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Research Article

An Automatic Quadrature Schemes and Error Estimates for Semibounded Weighted Hadamard Type Hypersingular Integrals

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The approximate solutions for the semibounded Hadamard type hypersingular integrals (HSIs) for smooth density function are investigated. The automatic quadrature schemes (AQSs) are constructed by approximating the density function using the third and fourth kinds of Chebyshev polynomials. Error estimates for the semibounded solutions are obtained in the class of $h(t) \in C^{N,\alpha}[-1,1]$. Numerical results for the obtained quadrature schemes revealed that the proposed methods are highly accurate when the density function $h(t)$ is any polynomial or rational functions. The results are in line with the theoretical findings.

1. Introduction

In an attempt to solve the Cauchy's problem for hyperbolic partial differential equations, Hadamard [1] introduced the concept of hypersingular integrals. He defined the hypersingular integrals to be the finite part of a divergent integral in which singularities are arranged at the endpoints of the interval for one-dimensional integrals and on the domain boundary for multidimensional integrals. These forms of integrals are later identified as Hadamards finite part integrals or simply Hadamards integrals. Numerous applications of the approximate solutions of Hadamard integrals have been found in mechanics, electrodynamics, aerodynamics, and acoustics with various numerical approaches [2–5]. Some authors applied various regularization methods [2, 4, 5] for the transformation of the hypersingular integrals into singular or weakly singular integrals, while others applied the direct numerical computation of the finite part integrals by using various quadrature or cubature formulae [4, 6]. Hypersingular integral represents a natural extension of singular integrals in the Cauchy principal value:

$$H(h,x) = \oint_{-1}^{1} \frac{w(t)\,h(t)}{(t-x)^2}\,dt = \frac{d}{dx}C(h,x), \tag{1}$$

where

$$C(h,x) = \int_{-1}^{1} \frac{w(t)\,h(t)}{t-x}\,dt. \tag{2}$$

Hui and Shia [7] developed Gaussian quadrature formula for hypersingular integrals with second-order singularities which have been extensively used for the solution of elasticity problems. Chen [8] used the hypersingular integral equation approach to study the plane elastic problem for crack problem. Nik Long and Eshkuvatov [9] formulated the multiple curved crack problems into hypersingular integral equations via the complex variable function method. Obayis et al. [10] developed an automatic quadrature scheme (AQS) based on Clenshaw-Curtis Chebyshev quadrature methods for the evaluation of HSIs of the form

$$Q_i(f,x) = \oint_{-1}^{1} \frac{w_i(t) f(t)}{(t-x)^2} dt, \quad x \in (-1,1), \ i = \{0,1,2\}, \tag{3}$$

where $w_0(t) = 1$, $w_1(t) = \sqrt{1-t^2}$, and $w_2(t) = 1/\sqrt{1-t^2}$ are the weight functions and $f(t)$ is a smooth or continuous. Error estimation for the developed AQS for $i = 0$ was obtained in the class of functions $C^{N+2,\alpha}[-1,1]$.

In this note, we have developed the automatic quadrature scheme based on Chebyshev Gauss quadrature formula for the semibounded solutions of the hypersingular integrals of the form

$$H_i(h,x) = \frac{w_i(x)}{\pi} \oint_{-1}^{1} \frac{h(t)}{w_i(t)(t-x)^2} dt, \quad x \in (-1,1), \tag{4}$$

where $w_1(t) = \sqrt{(1-t)/(1+t)}$ and $w_2(t) = \sqrt{(1+t)/(1-t)}$ are the weights and $h(t)$ is a smooth function. The proposed automatic quadrature schemes constructed cover the different weights functions that have not been considered by Obayis et al. [10]. Unlike the work of Obayis et al. [10] which takes advantage of Clenshaw-Curtis Chebyshev quadrature methods, the construction of our AQSs is based on Chebyshev Gauss quadrature methods. The method proposed by Hui and Shia [7] works only for even number N of knots points whereas AQS for our problems works well for all N and converges very fast to the exact solution.

Present in Section 2 are the mathematical concepts of the third and fourth kinds of Chebyshev polynomials and singular integrals. In Section 3, the construction of automatic quadrature schemes for semibounded solutions of hypersingular integrals is given. Rate of convergence of the suggested method is discussed in Section 4. Numerical examples and results are provided in Section 5.

2. Mathematical Concepts of Third and Fourth Kinds of Chebyshev Polynomials and Singular Integrals

The Chebyshev polynomials $V_m(t)$ and $W_m(t)$ are defined as follows.

Definition 1 (Mason and Handscomb [11]). The chebyshev polynomial $V_m(t)$ of the third kind on $[-1,1]$ is a polynomial of degree m in t defined by

$$V_m(t) = \frac{\cos(m+1/2)\theta}{\cos(1/2)\theta}, \quad t = \cos\theta, \ 0 \le \theta \le \pi. \tag{5}$$

Definition 2 (Mason and Handscomb [11]). The chebyshev polynomial $W_m(t)$ of the fourth kind on $[-1,1]$ is a polynomial of degree m in t defined by

$$W_m(t) = \frac{\sin(m+1/2)\theta}{\sin(1/2)\theta}, \quad t = \cos\theta, \ 0 \le \theta \le \pi. \tag{6}$$

Both polynomials

$$P_{im}(t) = \begin{cases} V_m(t), & i = 1, \\ W_m(t), & i = 2, \end{cases} \tag{7}$$

satisfy the recurrence relation

$$P_{im}(t) = 2tP_{i(m-1)}(t) - P_{i(m-2)}(t), \quad m = 2,3,\ldots, \quad i = \{1,2\} \tag{8}$$

with the same starting value

$$P_{i0}(t) = \begin{cases} V_0(t) = 1, & i = 1, \\ W_0(t) = 1, & i = 2, \end{cases} \qquad P_{i1}(t) = \begin{cases} V_1(t) = 2t-1, & i = 1, \\ W_1(t) = 2t+1, & i = 2. \end{cases} \tag{9}$$

The third and fourth kinds of Chebyshev polynomials are orthogonal on $[-1,1]$ with the weights $w_1(t)$ and $w_2(t)$, respectively. They also have continuous orthogonality given in [11], respectively, as

$$\langle V_m, V_j \rangle = \int_{-1}^{1} \sqrt{\frac{1+t}{1-t}} V_m(t) V_j(t)\, dt = \begin{cases} 0, & m \ne j \\ \pi, & m = j, \end{cases}$$
$$\langle W_m, W_j \rangle = \int_{-1}^{1} \sqrt{\frac{1-t}{1+t}} W_m(t) W_j(t)\, dt = \begin{cases} 0, & m \ne j; \ (m,j \le N), \\ \pi, & m = j. \end{cases} \tag{10}$$

The third kind and fourth kind of Chebyshev polynomials have discrete orthogonality, respectively, as

$$\langle V_m, V_j \rangle = \sum_{k=0}^{N} (1+t_k) V_m(t_k) V_j(t_k) = \begin{cases} 0, & m \ne j, \ (m,j \le N), \\ N+1, & 0 \le j = m \le N, \end{cases}$$
$$\langle W_m, W_j \rangle = \sum_{k=0}^{N} (1-t_k) W_m(t_k) W_j(t_k) = \begin{cases} 0, & m \ne j, \ (m,j \le N), \\ N+1, & j = m, \ (0 \le m \le N). \end{cases} \tag{11}$$

The four kinds of Chebyshev polynomials are connected via the relations

$$(1+t) V_k(t) = T_{k+1}(t) + T_k(t), \tag{12}$$

$$W_k(t) = U_k(t) + U_{k-1}(t), \tag{13}$$

$$(1-t) W_k(t) = T_{k+1}(t) - T_k(t), \tag{14}$$

$$V_k(t) = U_k(t) - U_{k-1}(t), \tag{15}$$

and the differentiation formulae

$$\frac{d}{dx}T_k(t) = kU_{k-1}(t),$$
$$\frac{d}{dx}U_k(t) = \frac{(k+1)T_{k+1}(t) - tU_k}{t^2 - 1}. \tag{16}$$

The Hilbert type integral transform is as follows.

Lemma 3 (Mason and Handscomb [11]). *Consider the following*

$$\int_0^{\pi} \frac{\cos m\theta}{\cos\theta - \cos\alpha} d\theta = \pi \frac{\sin m\alpha}{\sin\alpha}, \tag{17}$$

for any $\alpha \in [0, \pi]$, $m = 1, 2, 3, \ldots$.

Lemma 4 (Mason and Handscomb [11]). *Consider the following*

$$\int_0^{\pi} \frac{\sin m\theta \sin\theta}{\cos\theta - \cos\alpha} d\theta = -\pi \cos m\alpha, \tag{18}$$

for any $\alpha \in [0, \pi]$, $m = 1, 2, 3, \ldots$.

The estimations for some integrals are established.

Lemma 5 (Israilov [12]). *Let* $0 < \sigma \le 1$ *and* $1 - x \ge \delta_N$, $0 < \delta_N < 1/2$, *and for all* $i = \{1, 2\}$

$$\gamma_i(x) = w_i(x) \int_{-1}^{x-\delta_N} \frac{dt}{w_i(t)(t-x)},$$
$$\gamma_i^*(x) = w_i(x) \int_{x+\delta_N}^{1} \frac{dt}{w_i(t)(t-x)}, \tag{19}$$
$$T_i(x, \delta_N, \sigma) = w_i(x) \int_{x-\delta_N}^{x+\delta_N} \frac{|t-x|^{\sigma-1}\, dt}{w_i(t)},$$

where $w_1(x) = \sqrt{(1-x)/(1+x)}$ *and* $w_2(x) = \sqrt{(1+x)/(1-x)}$.

The following estimations are true:

(i) $|\gamma_i(x)| \le \ln(2/\delta_N) + \pi/\sqrt{2}$,

(ii) $\gamma_1(-x) = -\gamma_2^*(x)$, $\gamma_2(-x) = -\gamma_1^*(x)$,

(iii) $T_i(x, \delta_N, \sigma) \le \phi_i(\sigma)\delta_N^{\sigma}$, $\max\{\phi_1(\sigma), \phi_2(\sigma)\} \le \sqrt{10} \times 2^{-\sigma} + (1 + \sqrt{10} \times 2^{-1-\sigma})\sigma^{-1}$.

3. Automatic Quadrature Schemes for Semibounded Hypersingular Integrals

To construct AQSs for the semibounded HSIs (4), the density function $h(t)$ is approximated with the truncated series of third and forth kinds of Chebyshev polynomials:

$$h(t) \approx S_{iN}(t) = \begin{cases} \sum_{m=0}^{N} b_{1m} V_m(t), & i = 1, \\ \sum_{m=0}^{N} b_{2m} W_m(t), & i = 2. \end{cases} \tag{20}$$

Substituting (20) into (4), we have

$$\begin{aligned} H_i(h, x) &\approx H_{iN}(h, x) \\ &= \frac{w_i(t)}{\pi} \int_{-1}^{1} \frac{S_{iN}(t)}{w_i(x)(t-x)^2} dt \\ &= \begin{cases} \sqrt{\dfrac{1-x}{1+x}} \sum_{m=0}^{N} b_{1m} \dfrac{d}{dx}\left[\dfrac{1}{\pi}\int_{-1}^{1} \dfrac{(1+t)V_m(t)}{\sqrt{1-t^2}(t-x)} dt\right], & i = 1, \\ \sqrt{\dfrac{1+x}{1-x}} \sum_{m=0}^{N} b_{2m} \dfrac{d}{dx}\left[\dfrac{1}{\pi}\int_{-1}^{1} \dfrac{(1-t)W_m(t)}{\sqrt{1-t^2}(t-x)} dt\right], & i = 2, \end{cases} \\ &= \begin{cases} \sqrt{\dfrac{1-x}{1+x}} q_1(x), & i = 1, \\ \sqrt{\dfrac{1+x}{1-x}} q_2(x), & i = 2, \end{cases} \end{aligned} \tag{21}$$

where

$$q_i(x) = \begin{cases} \dfrac{d}{dx} \sum_{m=0}^{N} b_{1m} W_m(x), & i = 1, \\ \dfrac{d}{dx} \sum_{m=0}^{N} b_{2m} V_m(x), & i = 2. \end{cases} \tag{22}$$

By applying the orthogonality relations (10), we obtain the coefficients b_{im}, $i = \{1, 2\}$ as

$$b_{im} = \begin{cases} \dfrac{1}{\pi}\int_{-1}^{1} \sqrt{\dfrac{1+t}{1-t}} h(t) V_m(t)\, dt, & i = 1, \\ \dfrac{1}{\pi}\int_{-1}^{1} \sqrt{\dfrac{1-t}{1+t}} h(t) W_m(t)\, dt, & i = 2. \end{cases} \tag{23}$$

For the discrete coefficients b_{im}, we choose the collocation points, $t = t_k$, at the zeros of T_{N+1}; namely,

$$t = t_k = \cos\frac{(2k+1)\pi}{2N+2}, \quad k = 0, 1, 2, \ldots, N, \tag{24}$$

and, using the orthogonality conditions (11) and relations (12) and (14), respectively, as well as applying composite trapezium rule, we arrive at

$$b_{im} \approx \begin{cases} \dfrac{1}{N+3/2} \sum_{k=0}^{N} (1+t_k) V_m(t_k) h(t_k), & i = 1, \\ \dfrac{1}{N+3/2} \sum_{k=0}^{N} (t_k - 1) W_m(t_k) h(t_k), & i = 2. \end{cases} \tag{25}$$

4. Error Estimation for the Semibounded Hypersingular Integrals

Let us introduce the classes of functions.

(i) $H^{\alpha}([-1, 1], L)$ is a class of function satisfying Hölder condition on the interval with the index α and constant L.

(ii) $C^{N,\alpha}[-1,1] = \{h(t) : h^{(N)}(t) \in H^{\alpha}([-1,1], L_N)\}$.

(iii) $C[-1,1]$ is a class of continuous functions on the interval $[-1,1]$ with the norm

$$\|h\| = \|h(x)\|_{C[-1,1]} = \max_{x\in[-1,1]} |h(x)|. \tag{26}$$

(iv) $C^r[-1,1] = \{h(t) : h^r(t) \in C([-1,1])\}$.

Let $e_{iN}(t) = h(t) - S_{iN}(t)$ and let the maximum norm of the error be defined as

$$\|e_{iN}\| = \max_{t\in[-1,1]} |e_{iN}(t)|. \tag{27}$$

It is known in [11] that for any $x \in (-1,1)$

$$e_{1N}(x) = \frac{h^{(N)}(\eta_x) W_N(x)}{2^N N!}, \quad \eta_x \in [-1,1],$$
$$e_{2N}(x) = \frac{h^{(N)}(\eta_x) V_N(x)}{2^N N!}, \quad \eta_x \in [-1,1], \tag{28}$$

$$e'_{1N}(x) = \frac{1}{2^N N!}\left[W_N(x)\frac{d}{dx}h^{(N)}(\eta_x) + h^{(N)}(\eta_x) W'_N(x)\right], \quad \eta_x \in [-1,1],$$
$$e'_{2N}(x) = \frac{1}{2^N N!}\left[V_N(x)\frac{d}{dx}h^{(N)}(\eta_x) + h^{(N)}(\eta_x) V'_N(x)\right], \quad \eta_x \in [-1,1]. \tag{29}$$

As a consequence of (29), we arrive at the following estimations:

$$\|e_{1N}\| = \|e_{2N}\| \le \frac{(2N+1) M_1}{2^N \cdot N!}, \tag{30}$$

$$\|e'_{1N}\| = \|e'_{2N}\| \le \frac{M}{12\cdot 2^{N-3}(N-3)!}\left[1 + \frac{L_{1N}}{N} + \frac{7N-4}{2(N-1)(N-2)}\right], \tag{31}$$

where

$$M = \max\{M_1, M_2\}, \qquad M_1 = \max_{\eta_x\in[-1,1]} \left|h^{(N)}(\eta_x)\right|,$$
$$M_2 = \max_{\eta_x\in[-1,1]} \left|\frac{d}{dx}h^{(N)}(\eta_x)\right| < \infty, \tag{32}$$

$$L_{1N} = 1 + \frac{3}{N} + \frac{7N-4}{2(N-1)(N-2)}\left(1 + \frac{3}{N}\right). \tag{33}$$

Lemma 6. *The Lipschitz constants L_{jiN}, $j = \{3,4\}$, $i = \{1,2\}$ of $V_N(x)$ and $W_N(x)$ are*

$$\left|V_N(x) - V_N(y)\right| \le L_{31N}|x-y|,$$
$$\left|V'_N(x) - V'_N(y)\right| \le L_{32N}|x-y|,$$
$$\left|W_N(x) - W_N(y)\right| \le L_{41N}|x-y|,$$
$$\left|W'_N(x) - W'_N(y)\right| \le L_{42N}|x-y|, \tag{34}$$

where

$$L_{31N} = L_{41N} = \frac{N}{3}(N+1)(2N+1),$$
$$L_{32N} = L_{42N} = \frac{N}{15}(N+1)(N+2)(2N+1)(N-1). \tag{35}$$

Proof. Let L_{31N} be the Lipschitz constant of the third kind Chebyshev polynomial $V_N(t)$; that is,

$$\left|V_N(t) - V_N(x)\right| \le L_{31N}|t-x|, \tag{36}$$

where

$$L_{31N} = \max_{\zeta_t\in[-1,1]} \left|V'_N(\zeta_t)\right|, \tag{37}$$

which gives sequence of numbers:

$$0, 2, 10, 28, 60, 110, 182, 280, 408, 570 \ldots. \tag{38}$$

These sequences can be generated by

$$L_{31(N)} = L_{31(N-1)} + 2N^2, \quad L_{31(0)} = 0, \tag{39}$$

which leads to

$$L_{31N} = \frac{N}{3}(N+1)(2N+1). \tag{40}$$

In a similar way we can get

$$L_{41N} = \frac{N}{3}(N+1)(2N+1). \tag{41}$$

Let $L_{32(N)}$ be the Lipschitz constant of the derivative of third kind Chebyshev polynomial $V'_N(t)$; that is,

$$\left|V'_N(t) - V'_N(x)\right| \le L_{32N}|t-x|, \tag{42}$$

where

$$L_{32N} = \max_{\zeta_t\in[-1,1]} \left|V''_N(\zeta_t)\right|. \tag{43}$$

This Lipschitz constants generate the following sequence of numbers:

$$0, 0, 8, 56, 216, 616, 1456, 3024, 5712, 10032, \ldots, \tag{44}$$

and its recurrence relation is

$$L_{32(N+2)} = L_{32(N+1)} + M_N, \tag{45}$$

where

$$L_{32(0)} = 0,$$
$$M_{(N)} = 8D_{(N+1)},$$
$$D_{(N+1)} = D_{(N)} + R_{(N+1)}, \quad D_{(1)} = 1,$$
$$R_{(N+1)} = R_{(N)} + (N+1)^2 \quad R_{(1)} = 1. \tag{46}$$

These recurrence relations lead to

$$L_{32(N)} = \frac{N}{5}(N+1)(N+2)(2N+1)(N-1). \quad (47)$$

In a similar way we obtain

$$L_{42(N)} = \frac{N}{5}(N+1)(N+2)(2N+1)(N-1). \quad (48)$$

□

Main results are given in the following Theorem 7.

Theorem 7. *Let $h(t) \in C^{N,\alpha}[-1,1]$ for $0 < \alpha \le 1$ and let the series of Chebyshev polynomials $S_{iN}(t)$ be defined by (20); then the AQS (21) has an error bound*

$$\|E_i(h)\|_C \le \frac{0.11M}{2^{N-4}(N-4)!}\left[1+\frac{2.38}{N}L_{3N}+\frac{0.5(7N-4)}{(N-1)(N-2)}\right], \quad i=\{1,2\}, \quad (49)$$

where M is defined by (32) and

$$L_{3N} = 1 + \frac{2.53}{\ln N}\left[L_{1N} + 0.07L_{2N} + \frac{0.42(7N-4)}{(N-1)(N-2)}\left(1+\frac{1.11}{\ln N}\right) + \frac{0.03\left(23N^3-69N^2+100N-48\right)}{(N-1)(N-2)(N-3)(N-4)}\right],$$
$$L_{1N} = 1 + \frac{3}{N} + \frac{7N-4}{2(N-1)(N-2)} + \frac{3(7N-4)}{2N(N-1)(N-2)},$$
$$L_{2N} = 1 + \frac{3}{N} + \frac{23N^3-69N^2+100N-48}{(N-1)(N-2)(N-3)(N-4)}\left(1+\frac{3}{N}\right). \quad (50)$$

Proof of Theorem 7. In view of error norms (27), we can compute the error bound of AQS (21) for $i=\{1,2\}$ as follows:

$$\begin{aligned}E_i(h,x) &= H_i(h,x) - H_{iN}(h,x)\\ &= \frac{w_i(x)}{\pi}\int_{-1}^{1}\frac{e_{iN}(t)\,dt}{w_i(t)(t-x)^2}\\ &= \frac{w_i(x)}{\pi}\int_{-1}^{1}\frac{\left[e_{iN}(t)-e_{iN}(x)-e'_{iN}(x)(t-x)\right]dt}{w_i(t)(t-x)^2}\\ &\quad+\frac{w_i(x)}{\pi}\int_{-1}^{1}\frac{\left[e_{iN}(x)+e'_{iN}(x)(t-x)\right]dt}{w_i(t)(t-x)^2}.\end{aligned} \quad (51)$$

Taylor series expansion of $e_{iN}(t)$ around x is given by

$$e_{iN}(t) = e_{iN}(x) + e'_{iN}(\zeta_t)(t-x), \quad (52)$$

where

$$\zeta_t \in \begin{cases}(x,t), & \text{if } t\in(x,1),\\ (t,x), & \text{if } t\in(-1,x).\end{cases} \quad (53)$$

Due to (51)-(52), we have

$$\left|E_i(h,x)\right| \le E_{i1}(h,x) + E_{i2}(h,x) + E_{i3}(h,x), \quad (54)$$

where

$$\begin{aligned}E_{i1}(h,x) &= \left|\frac{w_i(x)}{\pi}\int_{-1}^{1}\frac{\left[e'_{iN}(\zeta_t)-e'_{iN}(x)\right]dt}{w_i(t)(t-x)}\right|,\\ E_{i2}(h,x) &= \left|\frac{w_i(x)}{\pi}e_{iN}(x)\int_{-1}^{1}\frac{dt}{w_i(t)(t-x)^2}\right|,\\ E_{i3}(h,x) &= \left|\frac{w_i(x)}{\pi}e'_{iN}(x)\int_{-1}^{1}\frac{dt}{w_i(t)(t-x)}\right|.\end{aligned} \quad (55)$$

It is easy to check that

$$\begin{aligned}&\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\frac{dt}{(t-x)^2} = \int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\frac{dt}{(t-x)^2} = 0,\\ &\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\frac{dt}{t-x} = -\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\frac{dt}{t-x} = \pi.\end{aligned} \quad (56)$$

For the estimation of $E_{i1}(h,x)$, we divide the interval $(-1,1)$ into three subintervals $(-1,x-\delta_N)$, $(x+\delta_N,1)$, and $[x-\delta_N, x+\delta_N]$:

$$E_{i1}(h,x) \le \frac{1}{\pi}\left[F_{i1}(x)+F_{i2}(x)+F_{i3}(x)\right], \quad (57)$$

where

$$\begin{aligned}F_{i1}(x) &= \left|w_i(x)\int_{-1}^{x-\delta_N}\frac{\left[e'_{iN}(\zeta_t)-e'_{iN}(x)\right]dt}{w_i(t)(t-x)}\right|,\\ F_{i2}(x) &= \left|w_i(x)\int_{x+\delta_N}^{1}\frac{\left[e'_{iN}(\zeta_t)-e'_{iN}(x)\right]dt}{w_i(t)(t-x)}\right|,\\ F_{i3}(x) &= \left|w_i(x)\int_{x-\delta_N}^{x+\delta_N}\frac{\left[e'_{iN}(\zeta_t)-e'_{iN}(x)\right]dt}{w_i(t)(t-x)}\right|.\end{aligned} \quad (58)$$

For $F_{i1}(x)$, according to Lemma 5, we obtain

$$\begin{aligned}F_{i1}(x) &\le 2\left\|e'_{iN}\right\|\left|w_i(x)\int_{-1}^{x-\delta_N}\frac{dt}{w_i(t)(t-x)}\right|\\ &\le 2\left\|e'_{iN}\right\|\left[\ln\frac{2}{\delta_N}+\frac{\pi}{\sqrt{2}}\right].\end{aligned} \quad (59)$$

In the similar way, for $F_{i2}(x)$, we have

$$\begin{aligned}F_{i2}(x) &\le 2\left\|e'_{iN}\right\|\left|w_i(x)\int_{x+\delta_N}^{1}\frac{dt}{w_i(t)(t-x)}\right|\\ &\le 2\left\|e'_{iN}\right\|\left[\ln\frac{2}{\delta_N}+\frac{\pi}{\sqrt{2}}\right].\end{aligned} \quad (60)$$

Utilizing (30) and (31) into (59) and (60), we arrive at

$$F_{i1}(x) = F_{i2}(x) \le \frac{(1/6)M}{2^{N-3}(N-3)!}\left[1+\frac{L_{1N}}{N}+\frac{7N-4}{(N-1)(N-2)}\right] \times \left[\ln\frac{2}{\delta_N}+\frac{\pi}{\sqrt{2}}\right]. \quad (61)$$

For the error of $F_{i3}(x)$,

$$F_{i3}(x) \le w_i(x)\int_{x-\sigma_N}^{x+\sigma_N}\frac{\left|e'_{iN}(\zeta_t)-e'_{iN}(x)\right|}{w_i(t)\left|\zeta_t-x\right|}dt. \quad (62)$$

It follows from (29) and Lemma 6 that

$$\begin{aligned}
&\left|e'_{iN}(\zeta_t)-e'_{iN}(x)\right| \\
&\le \frac{1}{2^N N!} \times \left\{\left|P_{iN}(\zeta_t)-P_{iN}(x)\right|\left|\frac{d}{dx}h^{(N)}(\zeta_t)\right| + \left|h^{(N)}(\zeta_t)\right|\left|P'_{iN}(\zeta_t)-P'_{iN}(x)\right|\right\} \\
&\le \frac{1}{2^N N!} \times \left\{L_{jiN}\left|\zeta_t-x\right|\left|\frac{d}{dx}h^{(N)}(\zeta_t)\right| + L_{jiN}\left|h^{(N)}(\zeta_t)\right|\left|\zeta_t-x\right|\right\} \\
&\le \frac{M}{480\cdot 2^{N-5}(N-5)!}\,\frac{N^2(N+1)(2N+1)}{(N-1)(N-2)(N-3)(N-4)} \times \left[1+\frac{1}{N}+\frac{3}{N^2}\right]\left|\zeta_t-x\right| \\
&\le \frac{M}{240\cdot 2^{N-5}(N-5)!} \times \left[1+\frac{23N^3-69N^2+100N-48}{2(N-1)(N-2)(N-3)(N-4)}\right] \times \left[1+\frac{1}{N}+\frac{3}{N^2}\right]\left|\zeta_t-x\right| \\
&\le \frac{M}{240\cdot 2^{N-5}(N-5)!} \times \left[1+\frac{L_{2N}}{N}+\frac{23N^3-69N^2+100N-48}{2(N-1)(N-2)(N-3)(N-4)}\right] \times \left|\zeta_t-x\right|,
\end{aligned} \quad (63)$$

where

$$L_{2N} = 1+\frac{3}{N}+\frac{23N^3-69N^2+100N-48}{(N-1)(N-2)(N-3)(N-4)}\left(1+\frac{3}{N}\right). \quad (64)$$

Substituting (63) into (62) yields

$$F_{i3}(x) \le \frac{M}{240\cdot 2^{N-5}(N-5)!} \times \left[1+\frac{L_{2N}}{N}+\frac{23N^3-69N^2+100N-48}{2(N-1)(N-2)(N-3)(N-4)}\right] \times \left|w_i(x)\int_{x-\delta_N}^{x+\delta_N}\frac{dt}{w_i(t)}\right|. \quad (65)$$

Using Lemma 5 gives

$$F_{i3}(x) \le \frac{3.37M}{240\cdot 2^{N-5}(N-5)!} \times \left[1+\frac{L_{2N}}{N}+\frac{23N^3-69N^2+100N-48}{2(N-1)(N-2)(N-3)(N-4)}\right]\delta_N. \quad (66)$$

By choosing $\delta_N = 2/N^2$ for $N \ge 5$, we obtain

$$F_{i3}(x) \le \frac{0.11M}{2^{N-3}(N-3)!} \times \left[1+\frac{L_{2N}}{N}+\frac{23N^3-69N^2+100N-48}{2(N-1)(N-2)(N-3)(N-4)}\right]. \quad (67)$$

By substituting (61) and (67) into (57), we arrive at

$$\begin{aligned}
E_{i1}(h,x) \le \frac{0.64M\ln_N}{\pi\cdot 2^{N-3}(N-3)!} &\times \left[1+\frac{1.27}{\ln N}+\frac{2.11}{\ln N}\left(L_{1N}+0.08L_{2N}\right)\right. \\
&+ \frac{2.54(7N-4)}{(N-1)(N-2)}\left(1+\frac{1.11}{\ln N}\right) \\
&\left.+ \frac{0.17\left(23N^3-69N^2+100N-48\right)}{(N-1)(N-2)(N-3)(N-4)}\right].
\end{aligned} \quad (68)$$

$$\begin{aligned}
E_{i1}(h,x) \le \frac{2.46M\ln_N}{2^{N-3}(N-3)!} &\times \left[1+\frac{0.10}{\ln N}+\frac{L_{1N}}{N}+\frac{0.10L_{1N}}{N\ln N}\right. \\
&+ \frac{23.16(7N-4)}{(N-1)(N-2)}+\frac{0.05(7N-4)}{(N-1)(N-2)} \\
&\left.+ \frac{10.59\left(23N^3-69N^2+100N-48\right)}{2(N-1)(N-2)(N-3)(N-4)}\right].
\end{aligned} \quad (69)$$

Table 1: Results of HSIs for $w_1(t)$.

N	x	Exact solution	Approximate solution	Absolute error
	−0.99999	−2236.0444988237596342	−2236.0444988237596345	3.13×10^{-16}
	−0.91111	−21.535241487484869307	−21.535241487484869310	3.08×10^{-18}
	−0.71111	−9.3563357403104129848	−9.3563357403104129863	1.42×10^{-18}
	−0.31111	−3.0963683860810919235	−3.0963683860810919240	5.86×10^{-19}
2	0.00000	−1.0000000000000000000	−1.0000000000000000003	3.00×10^{-19}
	0.31111	0.17718528456311131724	0.17718528456311131709	1.27×10^{-19}
	0.71111	0.75786494952828070264	0.75786494952828070264	6.39×10^{-21}
	0.91111	0.57031832778553277005	0.57031832778553277007	1.40×10^{-20}
	0.99999	0.00670813126012938238	0.00670813126012938238	2.24×10^{-22}

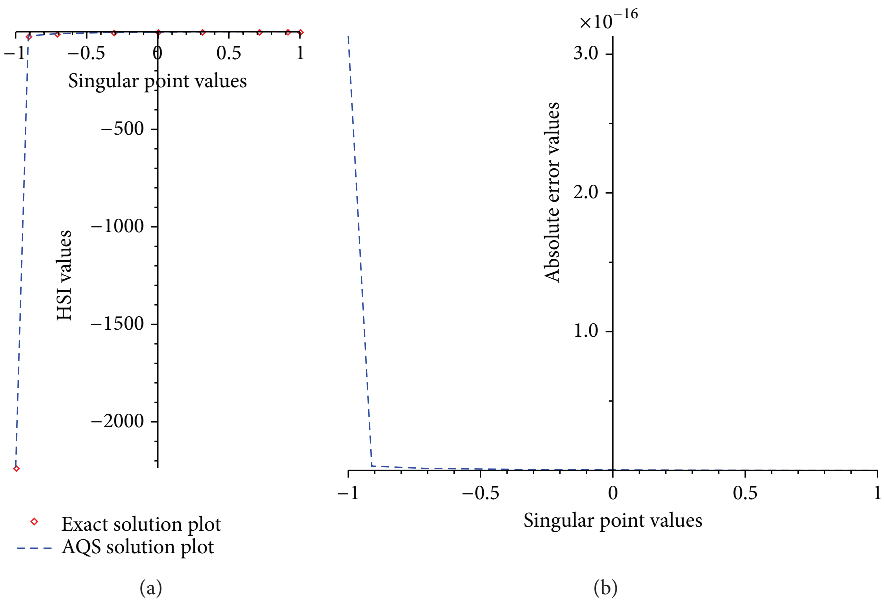


Figure 1: (a) Exact and AQS solutions when $N = 2$ and $h(t) = 2t^2 - 3t + 5$ and (b) absolute error between exact and AQS solution when $N = 2$ and $h(t) = 2t^2 - 3t + 5$.

Due to (31) and (56),

$$E_{i2}(h, x) = 0,$$

$$E_{i3}(h, x) \le \frac{2}{\sqrt{1-x^2}} \frac{0.08M}{2^{N-3}(N-3)!} \times \left[1 + \frac{L_{1N}}{N} + \frac{7N-4}{2(N-1)(N-2)}\right]. \tag{70}$$

Since, from the hypothesis in Lemma 5,

$$1 - x^2 \ge 1 - (1-\delta_N)^2,$$

$$\frac{1}{\sqrt{1-x^2}} \le \frac{1}{\sqrt{1-(1-\delta_N)^2}} \le \frac{\sqrt{2}}{\sqrt{3\delta_N}} = \frac{N}{\sqrt{3}}, \tag{71}$$

we have

$$E_{i3}(h, x) \le \frac{0.09M}{2^{N-3}(N-3)!} \times N\left[1 + \frac{L_{1N}}{N} + \frac{7N-4}{2(N-1)(N-2)}\right]. \tag{72}$$

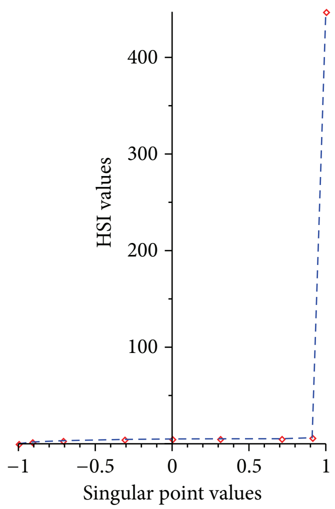
Substituting (69)–(72) into (54) yields

$$\begin{aligned}\left|E_i(h, x)\right| \le{}& \frac{0.68M \ln N}{\pi \cdot 2^{N-3}(N-3)!} \times \left[1 + \frac{1.27}{\ln N} + \frac{2.11}{\ln N}(L_{1N} + 0.08L_{2N})\right.\\ &+ \frac{2.54(7N-4)}{(N-1)(N-2)}\left(1 + \frac{1.11}{\ln N}\right)\\ &+ \frac{0.17\left(23N^3 - 69N^2 + 100N - 48\right)}{(N-1)(N-2)(N-3)(N-4)} + \frac{0.42N}{\ln N}\\ &\left.+ \frac{0.42}{\ln N}L_{1N} + \frac{0.42N(7N-4)}{2\ln N(N-1)(N-2)}\right].\end{aligned} \tag{73}$$

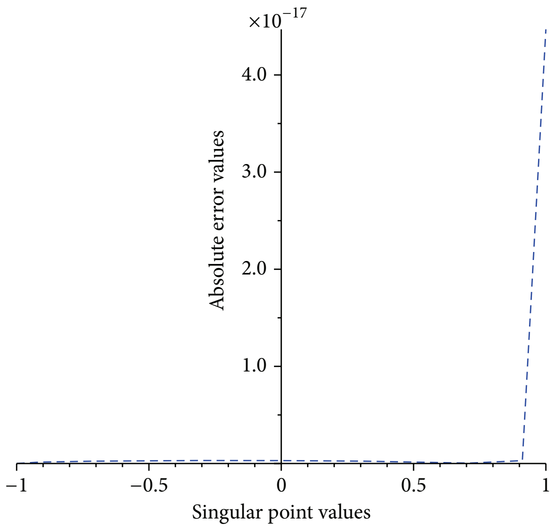
Choosing $N \ge 5$ the statement of Theorem 7 follows. □

TABLE 2: Results of HSIs for $w_2(t)$.

N	x	Exact solution	Approximate solution	Absolute error
	−0.99999	0.02012457266627356241	0.0201245726662735624	1.57×10^{-21}
	−0.91111	1.8643200698228626472	1.8643200698228626470	1.43×10^{-19}
	−0.71111	3.2232147018485970132	3.2232147018485970129	2.40×10^{-19}
	−0.31111	4.5263577087926478228	4.5263577087926478225	3.08×10^{-19}
2	0.00000	5.0000000000000000000	4.9999999999999999997	3.00×10^{-19}
	0.31111	5.1810684429214884711	5.1810684429214884708	2.42×10^{-19}
	0.71111	5.2460548398163357508	5.2460548398163357508	3.79×10^{-20}
	0.91111	6.2854320328769430625	6.2854320328769430630	2.99×10^{-19}
	0.99999	447.23036596367022628	447.23036596367022632	4.47×10^{-17}



(a)



(b)

FIGURE 2: (a) Exact and AQS solutions when $N = 2$ and $h(t) = 2t^2 - 3t + 5$ and (b) absolute error between exact and AQS solution when $N = 2$ and $h(t) = 2t^2 - 3t + 5$.

5. Numerical Results and Discussion

In this session, we evaluate the hypersingular integrals (4) for different choices of $h(t)$, and compare with the exact solution.

Example 1. Consider the HSIs of the form

$$H_1(h,x) = \frac{1}{\pi}\sqrt{\frac{1-x}{1+x}} \;{=}\!\!\!\!\!\!\int_{-1}^{1} \sqrt{\frac{1+t}{1-t}} \frac{h(t)}{(t-x)^2} dt, \quad (74)$$

where $h(t) = 2t^2 - 3t + 5$. The exact solution is $H_1(h,x) = 2\sqrt{(1-x)/(1+x)}(-1+4x)$.

The numerical results in Table 1 show that the AQS gives a very good result for the quadratic function $h(t) = 2t^2 - 3t + 5$ for the case $i = 1$ and $N = 2$. This is further illustrated in Figure 1(a) where the graphs of the exact and AQS solutions are superimposed for the same choices of singular point values. Details of the accuracy of the proposed AQS is observed from the absolute error in Figure 1(b).

Example 2. Consider the HSIs of the form

$$H_2(h,x) = \frac{1}{\pi}\sqrt{\frac{1+x}{1-x}} \;{=}\!\!\!\!\!\!\int_{-1}^{1} \sqrt{\frac{1-t}{1+t}} \frac{h(t)}{(t-x)^2} dt, \quad (75)$$

where $h(t) = 2t^2 - 3t + 5$. The exact solution is $H_2(h,x) = \sqrt{(1+x)/(1-x)}(5-4x)$.

The numerical results in Table 2 show that the AQS (21) gives a very good result for the quadratic function $h(t) = 2t^2 - 3t + 5$ for the case $i = 2$ and $N = 2$. Figure 2(a) indicates that the exact and AQS solutions are nearly the same. Further details of the accuracy of the absolute error of the proposed AQS are demonstrated in Figure 2(b).

Example 3. Consider the HSIs of the form

$$H_1(h,x) = \frac{1}{\pi}\sqrt{\frac{1-x}{1+x}} \;{=}\!\!\!\!\!\!\int_{-1}^{1} \sqrt{\frac{1+t}{1-t}} \frac{h(t)}{(t-x)^2} dt, \quad (76)$$

Table 3: Results of HSIs for $w_1(t)$.

N	x	Exact solution	Approximate solution	Absolute error
10	−0.99999	−258.19308036170334212	−258.17083250027430060	2.23×10^{-2}
	−0.91111	−2.2578118246552894121	−2.2578564435672151572	4.46×10^{-5}
	−0.77111	−1.0634637975870125656	−1.0634476360338990858	1.62×10^{-5}
	−0.33111	−0.2924243743021256602	−0.2924196578457308097	4.72×10^{-6}
	0.00000	−0.1443375672974064411	−0.1443397190188599320	2.15×10^{-6}
	0.33111	−0.0753154744680519215	−0.0753153016422631300	1.73×10^{-7}
	0.77111	−0.0270285335549129334	−0.0270280566695517436	4.77×10^{-7}
	0.91111	−0.0146928400579184367	−0.0146941562302624467	1.32×10^{-6}
	0.99999	−0.0001434451425475821	−0.0001431650832316927	2.80×10^{-7}
20	−0.99999	−258.19308036170334212	−258.19308021411085378	1.48×10^{-7}
	−0.91111	−2.2578118246552894121	−2.2578118246107028644	4.46×10^{-11}
	−0.77111	−1.0634637975870125656	−1.0634637975433417458	4.37×10^{-11}
	−0.33111	−0.2924243743021256602	−0.2924243742855352783	1.66×10^{-11}
	0.00000	−0.1443375672974064411	−0.1443375672897319217	7.68×10^{-12}
	0.33111	−0.0753154744680519215	−0.0753154744695265617	1.48×10^{-12}
	0.77111	−0.0270285335549129334	−0.0270285335485592653	6.35×10^{-12}
	0.91111	−0.0146928400579184367	−0.0146928400450019102	1.29×10^{-11}
	0.99999	−0.0001434451425475821	−0.0001434451390347442	3.51×10^{-12}
40	−0.99999	−258.19308036170334212	−258.19308036170338346	4.04×10^{-14}
	−0.91111	−2.2578118246552894121	−2.2578118246552894116	4.50×10^{-19}
	−0.77111	−1.0634637975870125656	−1.0634637975870125637	1.85×10^{-18}
	−0.33111	−0.2924243743021256602	−0.2924243743021256612	9.67×10^{-19}
	0.00000	−0.1443375672974064411	−0.1443375672974064412	1.13×10^{-19}
	0.33111	−0.0753154744680519215	−0.0753154744680519211	4.00×10^{-19}
	0.77111	−0.0270285335549129334	−0.0270285335549129336	1.72×10^{-19}
	0.91111	−0.0146928400579184367	−0.0146928400579184373	5.73×10^{-19}
	0.99999	−0.0001434451425475821	−0.0001434451425475826	4.86×10^{-19}

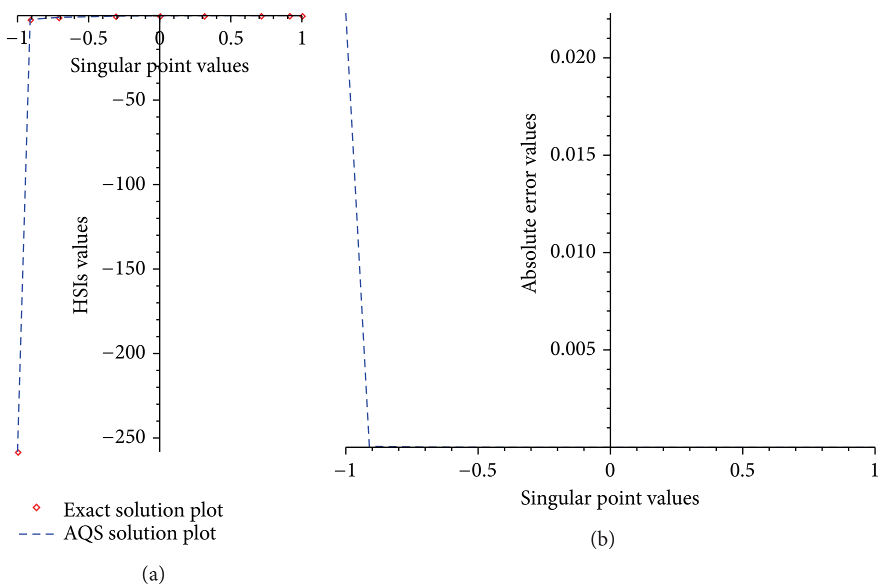


Figure 3: (a) Exact and AQS solutions when $N = 10$ and $h(t) = 1/(2+t)$ and (b) absolute error between exact and AQS solution when $N = 10$ and $h(t) = 1/(2+t)$.

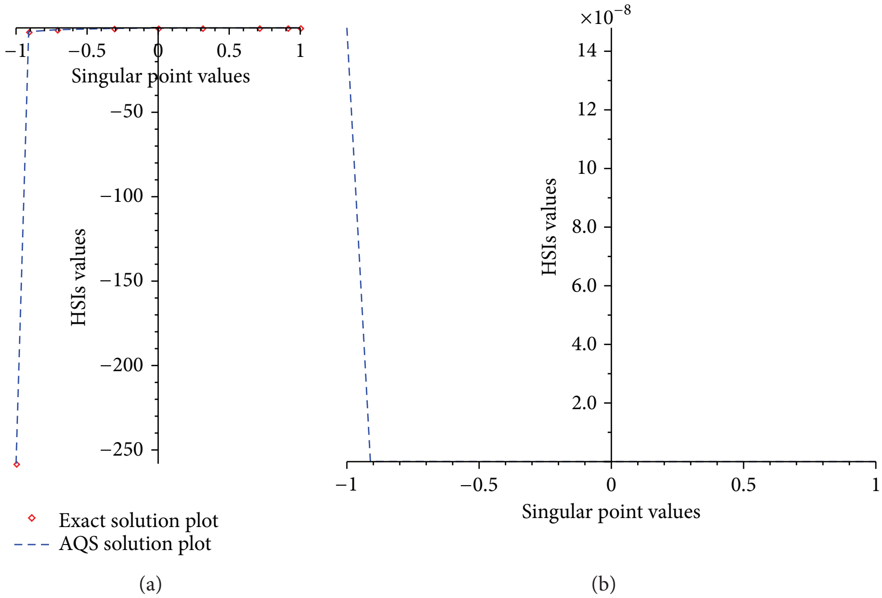


FIGURE 4: (a) Exact and AQS solutions when $N = 20$ and $h(t) = 1/(2+t)$ and (b) absolute error between exact and AQS solution when $N = 20$ and $h(t) = 1/(2+t)$.

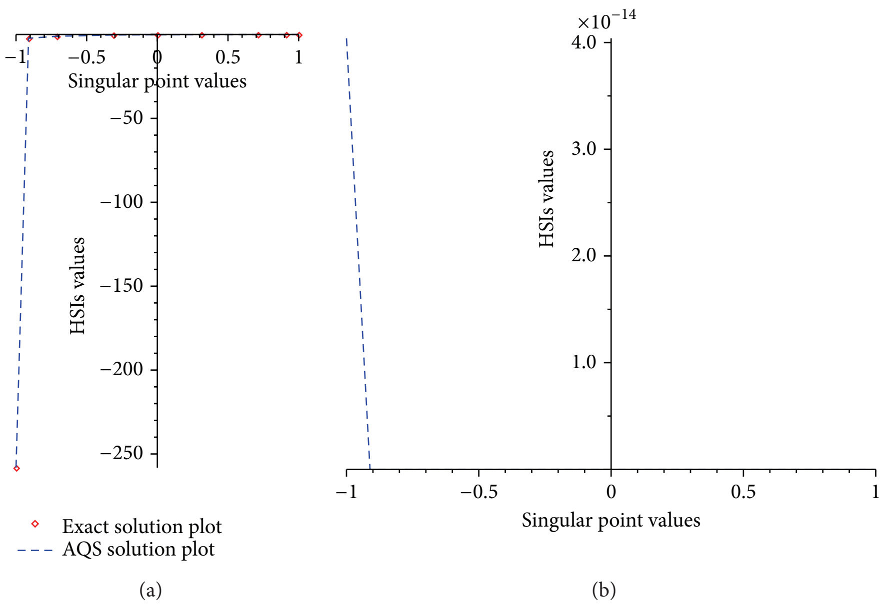


FIGURE 5: (a) Exact and AQS solutions when $N = 40$ and $h(t) = 1/(2+t)$ and (b) absolute error between exact and AQS solution when $N = 40$ and $h(t) = 1/(2+t)$.

where $h(t) = 1/(2+t)$. The exact solution is $H_1(h,x) = -\sqrt{(1-x)/(1+x)}(\sqrt{3}/3(2+x)^2)$.

The numerical results in Table 3 show that the AQS (21) is highly accurate for the rational function $h(t) = 1/(2+t)$ for the case $i = 1$ and $N = 40$. Figure 3(a) reveals that there js good agreement between the exact solution and AQS solution which can be seen from absolute error in Figure 3(b) when $N = 10$. The absolute error is improved in Figure 4(b) when $N = 20$ with the corresponding exact and AQS solution in Figure 4(a). Best result is obtained in Figures 5(a) and 5(b) when $N = 40$.

Example 4. Consider the HSIs of the form

$$H_2(h,x) = \frac{1}{\pi}\sqrt{\frac{1+x}{1-x}} \int_{-1}^{1} \sqrt{\frac{1-t}{1+t}} \frac{h(t)}{(t-x)^2} dt, \quad (77)$$

Table 4: Results of HSIs for $w_2(t)$.

N	x	Exact solution	Approximate solution	Absolute error
	−0.99999	0.00387291557000340015	0.00387141593292902562	1.50×10^{-6}
	−0.91111	0.31504763162812503076	0.31505448317559772604	6.85×10^{-6}
	−0.77111	0.41231131092313516294	0.41230879229200872013	2.52×10^{-6}
	−0.33111	0.44083450592426358350	0.44083359884530457966	9.07×10^{-7}
10	0.00000	0.43301270189221932338	0.43301675286046974386	4.05×10^{-6}
	0.33111	0.44963976686376800332	0.44963389568653105901	5.87×10^{-6}
	0.77111	0.62742591722366873321	0.62741343576104182143	1.25×10^{-5}
	0.91111	0.94767578680690190767	0.94770492350476572039	2.91×10^{-5}
	0.99999	86.066655193121619385	86.053938870581623578	01.27×10^{-2}
	−0.99999	0.00387291557000340015	0.00387291555159212453	1.84×10^{-11}
	−0.91111	0.31504763162812503076	0.31504763156789346359	6.02×10^{-11}
	−0.77111	0.41231131092313516294	0.41231131089736085386	2.58×10^{-11}
	−0.33111	0.44083450592426358350	0.44083450592691374976	2.65×10^{-12}
20	0.00000	0.43301270189221932338	0.43301270187830840205	1.39×10^{-11}
	0.33111	0.44963976686376800332	0.44963976684337128944	2.04×10^{-11}
	0.77111	0.62742591722366873321	0.62742591718912403402	3.46×10^{-11}
	0.91111	0.94767578680690190767	0.94767578677558019222	3.13×10^{-11}
	0.99999	86.066655193121619385	86.066655108152556362	8.50×10^{-8}
	−0.99999	0.00387291557000340015	0.00387291557000339911	1.04×10^{-18}
	−0.91111	0.31504763162812503076	0.31504763162812503215	1.40×10^{-18}
	−0.77111	0.41231131092313516294	0.41231131092313516515	2.28×10^{-18}
	−0.33111	0.44083450592426358350	0.44083450592426358289	6.14×10^{-19}
40	0.00000	0.43301270189221932338	0.43301270189221932269	6.75×10^{-19}
	0.33111	0.44963976686376800332	0.44963976686376800195	1.31×10^{-18}
	0.77111	0.62742591722366873321	0.62742591722366873124	1.91×10^{-18}
	0.91111	0.94767578680690190767	0.94767578680690190858	1.31×10^{-18}
	0.99999	86.066655193121619385	86.066655193121618680	7.05×10^{-16}

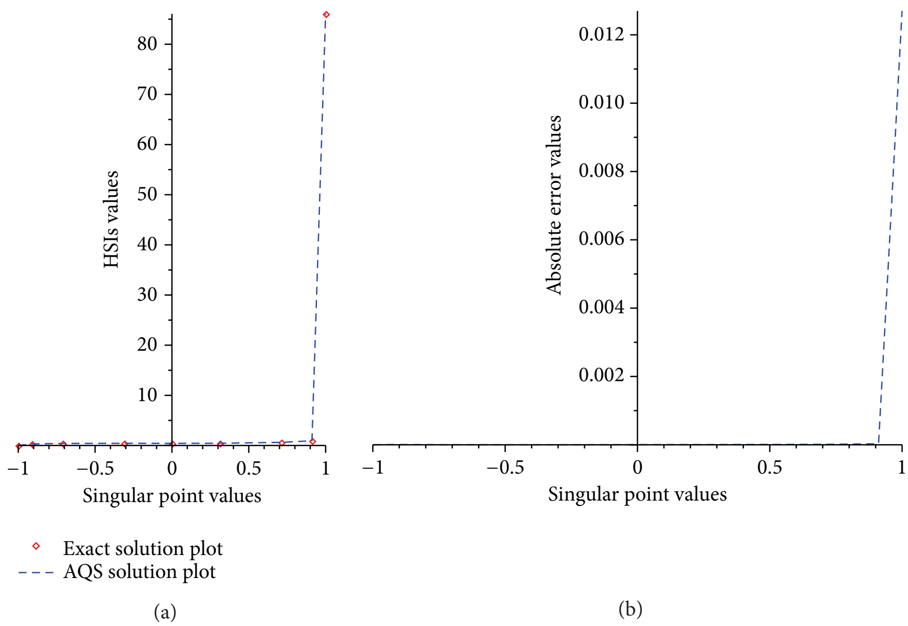


Figure 6: (a) Exact and AQS solutions when $N = 10$ and $h(t) = 1/(2+t)$ and (b) absolute error between exact and AQS solution when $N = 10$ and $h(t) = 1/(2+t)$.

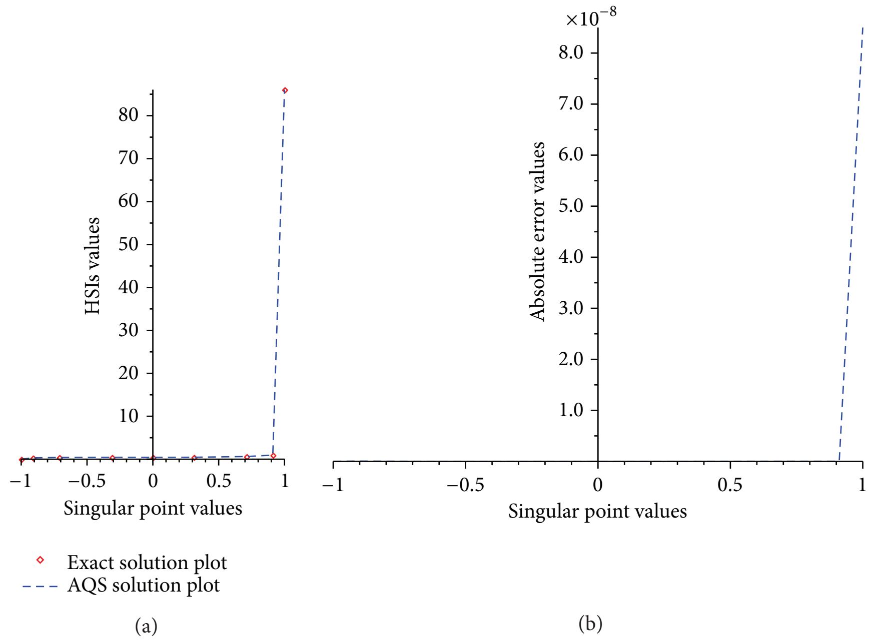


FIGURE 7: (a) Exact and AQS solutions when $N = 20$ and $h(t) = 1/(2+t)$ and (b) absolute error between exact and AQS solution when $N = 20$ and $h(t) = 1/(2+t)$.

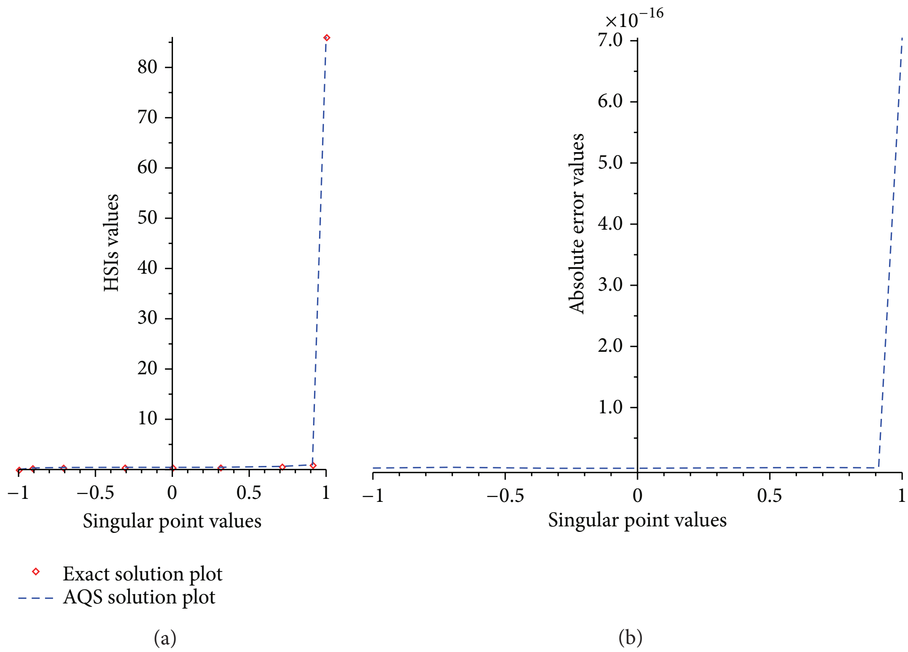


FIGURE 8: (a) Exact and AQS solutions when $N = 40$ and $h(t) = 1/(2+t)$ and (b) absolute error between exact and AQS solution when $N = 40$ and $h(t) = 1/(2+t)$.

where $h(t) = 1/(2+t)$. The exact solution is $H_2(h,x) = \sqrt{(1+x)/(1-x)}(\sqrt{3}/(2+x)^2)$.

The numerical results in Table 4 show that the AQS (21) is highly accurate for the rational function $h(t) = 1/(2+t)$ for the case $i = 2$ and $N = 40$. Figure 6(a) shows that there is good agreement between the exact solution and AQS solution which can be seen from absolute error in Figure 6(b) when $N = 10$. The absolute error is reduced in Figure 7(b) when $N = 20$ with the corresponding exact and AQS solution in Figure 7(a). The best result is obtained and illustrated in Figures 8(a) and 8(b) when $N = 40$.

6. Conclusion

In this paper, we have constructed the automatic quadrature schemes for the semibounded weighted Hadamard type hypersingular integrals using the truncated series of the third and fourth kinds of Chebyshev polynomials. The unknown coefficients are found by applying the discrete orthogonality condition and the choice of collocation points on $(-1, 1)$. The error estimations for the semibounded solutions of the hypersingular integrals are obtained. Numerical examples have shown the accuracy of the developed methods and agreed with theoretical findings.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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