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Research Article

Asymptotic Behavior of Global Entropy Solutions for Nonstrictly Hyperbolic Systems with Linear Damping

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We investigate the large time behavior of the global weak entropy solutions to the symmetric Keyfitz-Kranzer system with linear damping. It is proved that as $t \to \infty$ the entropy solutions tend to zero in the L^p norm.

1. Introduction

In this paper, we consider the Cauchy problem to the symmetric system of Keyfitz-Kranzer type with linear damping

$$\begin{aligned} u_t + (u\phi(r))_x + au &= 0, \\ v_t + (v\phi(r))_x + bv &= 0, \end{aligned} \tag{1}$$

with initial data

$$u(x,0) = u_0(x), \qquad v(x,0) = u_0(x). \tag{2}$$

This system models the propagation models of propagation of forward longitudinal and transverse waves of elastic string which moves in a plane; see [1, 2]. General source term for the system (1) was considered in [3]. The damping term in the system (1) represents external forces proportional to velocity, and this term can produce loss of total energy of system. Consider the scalar case; for example,

$$u_t + au_x + bu = 0, \qquad u(x,0) = u_0(x). \tag{3}$$

From the integral representation of (3), it is easy to find the following solution:

$$u(x,t) = u_0(x - at)e^{-bt}. \tag{4}$$

In this case, the solution of (3) tends to zero when $t \to \infty$. In Figure 1, we show the graph of solution for the advection equation with initial data

$$u(x,0) = \begin{cases} (1 - x^2), & \text{if } x \in (-1,1), \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

For more general case, the behavior of solutions and its computation can be more complicated; for example, we consider Burger's equation with a particular initial data and linear damping; this equation models the component of the velocity in one-dimensional flow

$$u_t + \left(\frac{1}{2}u^2\right)_x + u = 0 \tag{6}$$

with intial data

$$u(x,0) = \begin{cases} A(1 - x^2), & \text{if } x \in (-1,1), \\ 0, & \text{otherwise,} \end{cases} \tag{7}$$

where A is a constant. By an application to the characteristics method, we have that solutions emanating from x_0 are given by

$$\begin{aligned} X(t,x_0) &= x_0 + A(1 - x_0^2)(1 - e^{-t}), \\ u(X,t) &= u_0(x)e^{-t}. \end{aligned} \tag{8}$$

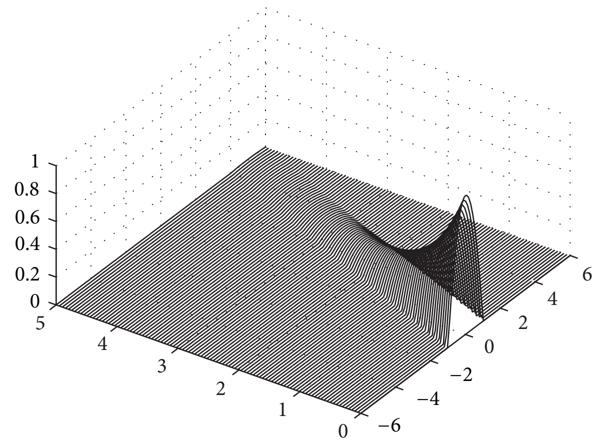

Figure 1: Graph of solution $u_t + u_x + u = 0$, $A = 1$.

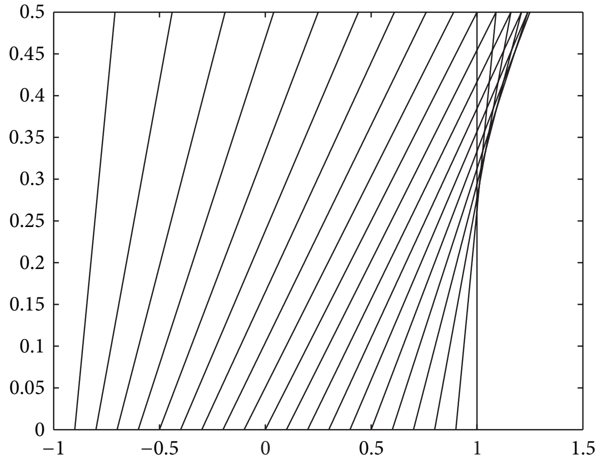

Figure 2: Characteristics for $u = 0$, $A = 1$.

In general, in systems with source term, the characteristics are nonlinear functions and they could have asymptotic behavior; for example, the characteristics solutions for (6), (7) are asymptotic to the lines

$$\lim_{t\to\infty} X(t, x_0) = x_0 + A\left(1 - x_0^2\right). \tag{9}$$

It is easy to see that the shock occurs when $x_0 + A(1 - x_0)^2 > 1$. In Figures 2, 3, and 4, we show the characteristics solutions for several values of A.

We are looking for conditions under which the terms a, b have a dissipative effect in the solutions of (1).

Let $r(x,t) = \sqrt{u(x,t)^2 + v(x,t)^2}$, and we are going to show the following main theorem.

Theorem 1. *If the initial data $(u_0(x), v_0(x)) \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$, then the Cauchy problems (1) and (2) have a weak entropy solutions satisfying*

$$\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} < M. \tag{10}$$

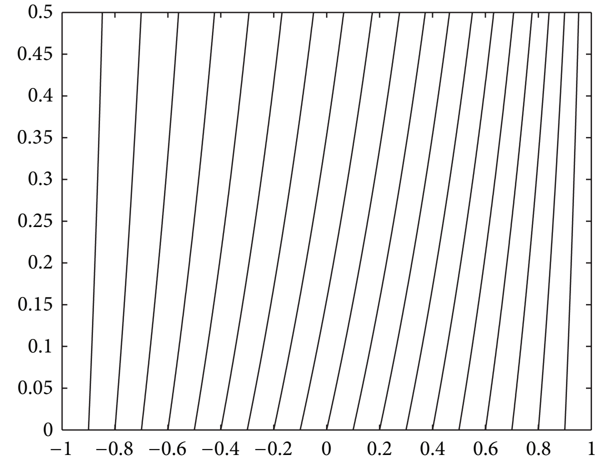

Figure 3: Linear damping $A = 1/2$.

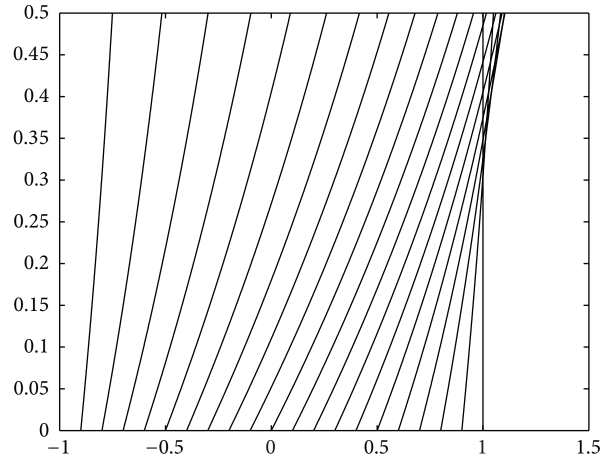

Figure 4: Linear damping $A = 2$.

Moreover, $r(u, v)$ converges to zero in L^p with exponential time decay; that is,

$$\|r(x,t)\|_{L^p(\mathbb{R})} \le K e^{-Mt} \|r(x,0)\|_{L^p(\mathbb{R})}. \tag{11}$$

2. Preliminaries

We start with some preliminaries about the general systems of conservation laws; see [4] chapter 5. Let $f : \Omega \to \mathbb{R}^n$ be a smooth vector field. Consider Cauchy problem for the system

$$\begin{aligned} u_t + f(u)_x &= g(u), \\ u(x,0) &= u_0(x). \end{aligned} \tag{12}$$

When $g(u) = 0$, the system (12) is called homogeneous system of conservation laws, if $g(u) \neq 0$, the system (12) is called inhomogeneous system or balance system of conservation laws. We will work also with the parabolic perturbation to the system (12); namely,

$$u_t + f(u)_x = \epsilon u_{xx} + g(u),$$
$$u(x,0) = u_0(x). \tag{13}$$

Denote by $A(u) = Df(u)$ the Jacobian matrix of partial derivatives of f.

Definition 2. The system (12) is strictly hyperbolic if, for every $u \in \Omega$, the matrix $A(u)$ has n real distinct eigenvalues $\lambda_1(u) < \cdots < \lambda_n(u)$.

Let $r_i(u)$ be the corresponding eigenvector to $\lambda_i(u)$. Then, one can see the following.

Definition 3. One says that the ith characteristic field is genuinely nonlinear if

$$\nabla\lambda_i(u)\cdot r_i(0) \neq 0. \tag{14}$$

If instead

$$\nabla\lambda_i(u)\cdot r_i(0) = 0, \tag{15}$$

we say that the ith characteristic field is linearly degenerate.

For the following definitions, see [5, 6].

Definition 4. A k-Riemann invariant is a smooth function $w_k : \mathbb{R}^n \to \mathbb{R}$, such that

$$\nabla w_k(u)\cdot r_k(u) = 0. \tag{16}$$

Definition 5. A pair of function $\eta, q : \mathbb{R}^n \to \mathbb{R}$ is called a entropy-entropy flux pair if it satisfies

$$\nabla\eta(u)A(u) = \nabla q(u), \tag{17}$$

if $\eta(u)$ is a convex function, then the pair (η, q) is called convex entropy-entropy flux pair.

Definition 6. A bounded measurable function $u(x,t)$ is an entropy (or admissible) solution for the Cauchy problem (12) if it satisfies the following inequality:

$$\eta(u)_t + q(u)_x + \nabla\eta(u)g(u) \leq 0, \tag{18}$$

in the distributional sense, where (η, q) is any convex entropy-entropy flux pair.

We consider the general system of Keyfitz-Kranzer type

$$u_t + (u\phi(u,v))_x = 0,$$
$$v_t + (v\phi(u,u))_x = 0, \tag{19}$$

to get some general observations about this type of system. Let $F(u,v) = (u\phi(u,v), v\phi(u,v))$ in (19), and we have that the eigenvalues and eigenvector of the Jacobian matrix Df are given by

$$\lambda_1(u,v) = \phi(u,v), \qquad r_1 = \left(1, -\frac{\phi_u}{\phi_v}\right), \tag{20}$$

$$\lambda_2(u,v) = \phi(u,v) + (u,v)\cdot\nabla\phi(u,v), \qquad r_2 = \left(1, \frac{v}{u}\right). \tag{21}$$

From (20) and (21), we have that $\nabla\phi\cdot r_1 = 0$ and $\nabla Z(u,v)\cdot r_2 = 0$, where $Z(u,v) = u/v$, and then the Riemann invariants are given by

$$W(u,v) = \phi(r),$$
$$Z(u,v) = \frac{u}{v}. \tag{22}$$

Lemma 7. *The system (1) is always linear degenerate in the first characteristic field. If*

$$(u,v)\nabla\phi(u,v) \neq 0, \tag{23}$$

then the system (1) is strictly hyperbolic and nonlinear degenerate in the second characteristic field. Moreover,

$$\nabla\lambda_2(u,v)\cdot r_2 = \frac{2(u,v)\nabla\phi(u,v) + (u,v)H(\phi)(u,v)^T}{u}, \tag{24}$$

where H represents the Hessian matrix.

Lemma 8. *Let $\eta(u,v) \in \mathbf{C}^1(\mathbb{R}_+)$ be a Lipschitz function in a neighborhood of the origin, and let $q(u,v) = \psi(u,v) + \eta(u,v)\phi(u,v)$ be a function, such that ψ satisfies*

$$\nabla\psi(u,v) = ((u,v)\cdot\nabla\eta(u,v) - \eta(u,v))\nabla\phi(u,v). \tag{25}$$

Then, the pair

$$(n(u,v), q(u,v)) \tag{26}$$

is a entropy-entropy flux pair for the system (1). Moreover, if $\eta(u,v)$ is a convex function, then the pair (26) is a convex entropy-entropy flux pair.

3. Global Existence of Weak Entropy Solutions and Asymptotic Behavior

We consider the parabolic regularization of the system (1). Namely,

$$u_t + (u\phi(r))_x + au = \epsilon u_{xx},$$
$$v_t + (v\phi(r))_x + bv = \epsilon v_{xx}, \tag{27}$$

with initial data

$$u^\epsilon(x,0) = u_0^\epsilon * j_\epsilon, \qquad v^\epsilon(x,0) = v_0^\epsilon * j_\epsilon, \tag{28}$$

where j_ϵ is a mollifier. In this case, $\phi(u,v) = \phi(r)$, with $r = \sqrt{u^2+v^2}$. By (20), the eigenvectors and eigenvalues are given by

$$\lambda_1(u,v) = \phi(r), \qquad r_1 = \left(1, -\frac{u}{v}\right),$$
$$\lambda_2(u,v) = \phi(r) + r\phi'(r), \qquad r_2 = \left(1, \frac{v}{u}\right). \tag{29}$$

The following conditions will be necessary in our next discussion:

(C_1) $r\phi(r) \to 0$, as $r \to 0$, $r\phi'(r) \neq 0$;

(C_2) $a > b$.

The condition (C_1) guarantees the strict hyperbolicity to the system (27) according to Lemma 7, while condition (C_2) ensures the existence of a positive invariant region. Now, we consider the following subset of $\mathbb{R}$:

$$\Sigma = \left\{(u,v) : \phi(r) \le C_0,\ 0 < C_1 \le \frac{u}{v} \le C_2\right\}. \tag{30}$$

We affirm that Σ is an invariant region. Let $h(u,v) = (au, bv)$, if $(\overline{u}, \overline{v}) \in \gamma_1$, where γ_1 is the level curve of $W = \phi(r)$. We have that

$$(\nabla W \cdot h)(\overline{u}, \overline{v}) = (a+b)\, r\phi'(r) > 0, \tag{31}$$

and if $\overline{u}, \overline{v} \in \gamma_2$, where γ_2 is the level curve of $Z = u/v$, we have that

$$(\nabla Z \cdot h)(\overline{u}, \overline{v}) = (a-b)\left(\frac{u}{v}\right) > 0. \tag{32}$$

By Theorem 14.7 of [5], Σ is an invariant region for the system (27). It is easy to verify that (au, bv) satisfies the condition $H_1 \cdots H_5$ in [3]; thus, we have the following Lemma.

Proposition 9. *If $(u_0, v_0) \in \Sigma$ and the C condition holds, then the Cauchy problems (27), (28) have a global weak entropy solution.*

Now, for the global behavior of solutions, using ideas of the author in [7], we construct the following entropy-entropy flux pairs as follows:

$$n(r) = r^m, \quad m \ge 2. \tag{33}$$

From (25), we have

$$q(r) = (m-1)\int_0^r s^m \phi'(s)\, ds + r^m \phi(r). \tag{34}$$

Integrating by parts, we have that

$$q(r) = m\phi(r) - m(m-1)\int_0^r s^{m-1}\phi(s)\, ds. \tag{35}$$

Let $M = \sup_{(u,v)\in[0,\|u\|_{L^\infty}]\times[0,\|v\|_{L^\infty}]}\{\phi(r)\}$. Then, we have that

$$|q(r)| \le 2mMr^m. \tag{36}$$

Multiplying (1) by $\nabla\eta$, we have that

$$\eta(r)_t + q(r)_x \le -3mMr^m. \tag{37}$$

Now, we choose $h(x) \in C^2(\mathbb{R})$ as a function, such as $|h'(x)| \le 1$, $|h''(x)| \le 1$, and $h(x) = |x|$ for $|x| \ge 1$ and set $k(x) = e^{-h(x)}$. Then, $k'(x) \le k(x)$. Multiplying by $k(x)$ in (37) and integrating over x, we have

$$\frac{d}{dt}\int_{-\infty}^{\infty} \eta(r)\, g(x) \le \int_{-\infty}^{\infty} q(r)\, k'(x) + -3mM\int_{-\infty}^{\infty} r^m dx. \tag{38}$$

By the inequality (36), we have

$$\frac{d}{dt}\int_{-\infty}^{\infty} \eta(r)\, k(x)\, dx \le -mM\int_{-\infty}^{\infty} r^m k(x)\, dx. \tag{39}$$

If $\psi(t) = \int_{-\infty}^{\infty} \eta(r)k(x)dx$, we have

$$\frac{d}{dt}\psi(t) \le -mM\psi(t). \tag{40}$$

By Gronwall's inequality, we have

$$\psi(t) \le e^{-mMt}\psi(0). \tag{41}$$

Thus, we have

$$\left(\int_{-\infty}^{\infty} r^m(t)\, k(x)\, dx\right)^{1/m} \le e^{-Mt}\left(\int_{-\infty}^{\infty} r^m(0)\, k(x)\, dx\right)^{1/m}. \tag{42}$$

Passing to limit $m \to \infty$, in (42), we have the inequality (10).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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