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Research Article

Unicity of Entire Functions concerning Shifts and Difference Operators

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We prove a unicity theorem of entire functions that share two distinct small functions with their shifts. The corollary of the theorem confirms the conjecture posed by Li and Gao (2011).

1. Introduction

Let f be a nonconstant meromorphic function in the complex plane $\mathbb{C}$. We will use the standard notations in Nevanlinna theory of meromorphic functions such as $T(r, f)$, $N(r, f)$, and $m(r, f)$ (see [1, 2]). The notation $S(r, f)$ is defined to be any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ possibly outside a set of finite linear measures. A meromorphic function a is called a small function related to f provided that $T(r, a) = S(r, f)$.

Let f and g be two nonconstant meromorphic functions, and let a be a small function related to both f and g. We say that f and g share a CM if $f - a$ and $g - a$ have the same zeros with the same multiplicities. f and g are said to share a IM if $f - a$ and $g - a$ have the same zeros ignoring multiplicities.

Let $N(r, a)$ be the counting functions of all common zeros with the same multiplicities of $f - a$ and $g - a$. If

$$N\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{g-a}\right) - 2N(r, a) = S(r, f) + S(r, g), \tag{1}$$

then we say that f and g share a CM almost.

For a nonzero complex constant $c \in \mathbb{C}$, we define difference operators as $\Delta_c f(z) = f(z+c) - f(z)$ and $\Delta_c^n f(z) = \Delta_c(\Delta_c^{n-1} f(z))$, $n \in \mathbb{N}$, $n > 2$.

In 1977, Rubel and Yang [3] proved the following result.

Theorem A. *Let f be a nonconstant entire function. If $f(z)$ and $f'(z)$ share two distinct finite values CM, then $f(z) \equiv f'(z)$.*

In fact, the conclusion still holds if the two CM values are replaced by two IM values (see Gundersen [4, 5], Mues and Steinmetz [6]).

Recently, a number of articles focused on value distribution in shifts or difference operators of meromorphic functions (see [7–11]). In particular, some papers studied the unicity of meromorphic functions sharing values with their shifts or difference operators (see [12–14]). In 2009, Heittokangas et al. [12] proved the following result concerning shifts.

Theorem B. *Let f be a nonconstant entire function of finite order, $c \in \mathbb{C}$. If $f(z)$ and $f(z+c)$ share two distinct finite values CM, then $f(z) \equiv f(z+c)$.*

In 2011, Li and Gao [14] proved the following result concerning difference operators.

Theorem C. *Let f be a nonconstant entire function of finite order, $c \in \mathbb{C}$, and let n be a positive integer. Suppose that $f(z)$ and $\Delta_c^n f(z)$ share two distinct finite values a, b CM and one of the following cases is satisfied:*

(i) *$ab = 0$;*

(ii) *$ab \neq 0$ and $\rho(f) \notin N$.*

Then $f(z) \equiv \Delta_c^n f(z)$.

In [14], Li and Gao conjectured that the restriction $\rho(f) \notin \mathbb{N}$ for the case $ab \neq 0$ can be removed. In this paper, we confirm their conjecture. In fact, we prove the following more general results.

Theorem 1. *Let f be a nonconstant entire function of finite order, let n be a positive integer, let $a(z)$, $b(z)$ be two distinct small functions related to $f(z)$, let $m_1, m_2, \ldots, m_n$ be nonzero complex numbers and $c_1, c_2, \ldots, c_n$ distinct finite values, and let*

$$F(z) = m_1 f(z + c_1) + m_2 f(z + c_2) + \cdots + m_n f(z + c_n). \tag{2}$$

If $f(z)$ and $F(z)$ share $a(z)$, $b(z)$ CM, then $f(z) \equiv F(z)$.

Corollary 2. *Let f be a nonconstant entire function of finite order, let c be a nonzero finite complex number, let n be a positive integer, and let a, b be two distinct finite values. If $f(z)$ and $\Delta_c^n f(z)$ share a, b CM, then $f(z) \equiv \Delta_c^n f(z)$.*

Remark 3. Corollary 2 confirms the conjecture of Li and Gao in [14].

Corollary 4. *Let f be a nonconstant entire function of finite order, let c be a nonzero finite complex number, and let $a(z)$, $b(z)$ be two distinct small functions related to f. If $f(z)$ and $f(z+c)$ share $a(z)$, $b(z)$ CM, then $f(z) \equiv f(z+c)$.*

2. Some Lemmas

For the proof of Theorem 1, we require the following results.

Lemma 5 (see [15]). *Let f and g be two nonconstant meromorphic functions satisfying*

$$\begin{aligned} N\left(r, \frac{1}{f}\right) + N(r, f) &= S(r, f), \\ N\left(r, \frac{1}{g}\right) + N(r, g) &= S(r, g). \end{aligned} \tag{3}$$

If $f(z)$ and $g(z)$ share 1 CM almost, then either $f(z) \equiv g(z)$ or $f(z)g(z) \equiv 1$.

Lemma 6 (see [15]). *Let f and g be two nonconstant meromorphic functions satisfying*

$$N(r, f) = S(r, f), \qquad N(r, g) = S(r, g). \tag{4}$$

If $f(z)$ and $g(z)$ share 0 and 1 CM almost, and

$$\overline{\lim_{\substack{r \to \infty \\ r \in I}}} \frac{N(r, 0) + N(r, 1)}{T(r, f) + T(r, g)} < \frac{2}{3}, \tag{5}$$

where $I \subset [0, \infty)$ is a set of infinitely linear measure, then

$$f(z) = \frac{ag(z) + b}{cg(z) + d}, \tag{6}$$

where a, b, c, and d are constants satisfying $ad - bc \neq 0$.

Lemma 7 (see [10]). *Let f be a nonconstant meromorphic function of finite order, $c \in \mathbb{C}$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r, f), \tag{7}$$

for all r outside a possible exceptional set E with finite logarithmic measure $\int_E dr/r < \infty$.

In the following, $S(r, f)$ denotes any function satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside a set with finite logarithmic measure.

3. Proof of Theorem 1

We prove Theorem 1 by contradiction. Suppose that $f(z) \not\equiv F(z)$. Then it follows from $f(z)$ and $F(z)$ being two distinct entire functions that $f(z)$ and $F(z)$ share $a(z)$, $b(z)$, and ∞ CM. By the Nevanlinna second fundamental theorem for three small functions, we have

$$\begin{aligned} T(r, f) &\leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f - a}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{f - b}\right) + S(r, f) \\ &\leq \overline{N}\left(r, \frac{1}{F - a}\right) + \overline{N}\left(r, \frac{1}{F - b}\right) + S(r, f) \\ &\leq 2T(r, F) + S(r, f). \end{aligned} \tag{8}$$

Similarly, we have $T(r, F) \leq 2T(r, f) + S(r, F)$. Therefore, $S(r, f) = S(r, F)$.

Set

$$\begin{aligned} f_1(z) &= \frac{f(z) - a(z)}{b(z) - a(z)}, \\ F_1(z) &= \frac{F(z) - a(z)}{b(z) - a(z)}. \end{aligned} \tag{9}$$

Thus $f_1(z)$, $F_1(z)$ share 0, 1, and ∞ CM almost.

Obviously, we have

$$\begin{aligned} T(r, f_1) &= T(r, f) + S(r, f), \\ T(r, F_1) &= T(r, F) + S(r, f), \\ S(r, F) = S(r, F_1) &= S(r, f_1) = S(r, f). \end{aligned} \tag{10}$$

By Nevanlinna's second fundamental theorem, we have

$$\begin{aligned} T(r, f_1) &\leq N\left(r, \frac{1}{f_1}\right) + N\left(r, \frac{1}{f_1 - 1}\right) + N(r, f_1) + S(r, f_1) \\ &\leq N(r, 0) + N(r, 1) + S(r, f) \end{aligned}$$

$$\begin{aligned}&\le N\left(r,\frac{1}{F_1-f_1}\right)+S(r,f)\\&\le T(r,F_1-f_1)+S(r,f)\\&\le T(r,F-f)+S(r,f)\\&\le m(r,F-f)+S(r,f).\end{aligned}\tag{11}$$

Since $F-f=m_1f(z+c_1)+m_2f(z+c_2)+\cdots+m_nf(z+c_n)-f(z)=f(z)[m_1(f(z+c_1)/f(z))+m_2(f(z+c_2)/f(z))+\cdots+m_n(f(z+c_n)/f(z))-1]$, thus

$$\begin{aligned}&m(r,F-f)\\&\quad\le m(r,f)\\&\qquad+m\left(r,m_1\frac{f(z+c_1)}{f(z)}+\cdots+m_n\frac{f(z+c_n)}{f(z)}-1\right)\\&\quad\le m(r,f)+S(r,f).\end{aligned}\tag{12}$$

By (11), we have

$$\begin{aligned}T(r,f_1)&\le N(r,0)+N(r,1)+S(r,f)\\&\le m(r,f)+S(r,f)\le T(r,f)+S(r,f)\\&=T(r,f_1)+S(r,f).\end{aligned}\tag{13}$$

It follows that

$$N(r,0)+N(r,1)=T(r,f_1)+S(r,f).\tag{14}$$

On the other hand, by Nevanlinna first fundamental theorem, we have

$$\begin{aligned}2T(r,f_1)&=T\left(r,\frac{1}{f_1}\right)+T\left(r,\frac{1}{f_1-1}\right)+S(r,f)\\&\le N(r,0)+N(r,1)+m\left(r,\frac{1}{f_1}\right)\\&\quad+m\left(r,\frac{1}{f_1-1}\right)+S(r,f)\\&\le T(r,f_1)+m\left(r,\frac{1}{f_1}\right)\\&\quad+m\left(r,\frac{1}{f_1-1}\right)+S(r,f).\end{aligned}\tag{15}$$

So we get

$$\begin{aligned}T(r,f_1)&\le m\left(r,\frac{1}{f_1}\right)+m\left(r,\frac{1}{f_1-1}\right)+S(r,f)\\&\le m\left(r,\frac{1}{f-a}\right)+m\left(r,\frac{1}{f-b}\right)+S(r,f).\end{aligned}\tag{16}$$

Set

$$\begin{aligned}a_1(z)&=m_1a(z+c_1)+m_2a(z+c_2)+\cdots+m_na(z+c_n),\\b_1(z)&=m_1b(z+c_1)+m_2b(z+c_2)+\cdots+m_nb(z+c_n).\end{aligned}\tag{17}$$

If $a_1(z)\equiv b_1(z)$, we can deduce by (16) that

$$\begin{aligned}T(r,f_1)&\le m\left(r,\frac{1}{f-a}+\frac{1}{f-b}\right)+S(r,f)\\&\le m\left(r,\frac{F-a_1}{f-a}+\frac{F-b_1}{f-b}\right)\\&\quad+m\left(r,\frac{1}{F-a_1}\right)+S(r,f)\\&\le T(r,F)+S(r,f)\\&\le T(r,m_1f(z+c_1)+m_2f(z+c_2)\\&\qquad+\cdots+m_nf(z+c_n))+S(r,f)\\&=m(r,m_1f(z+c_1)+m_2f(z+c_2)\\&\qquad+\cdots+m_nf(z+c_n))+S(r,f)\\&\le m(r,f)+S(r,f)\le T(r,f)+S(r,f)\\&=T(r,f_1)+S(r,f).\end{aligned}\tag{18}$$

If $a_1(z)\not\equiv b_1(z)$, set

$$L(F)=\begin{vmatrix}F & a_1 & b_1\\ F' & a_1' & b_1'\\ F'' & a_1'' & b_1''\end{vmatrix}.\tag{19}$$

Then we have

$$\begin{aligned}m\left(r,\frac{F-a_1}{f-a}\right)&=m\left(r,\frac{F-b_1}{f-b}\right)=S(r,f),\\m\left(r,\frac{L(F)}{F-a_1}\right)&=m\left(r,\frac{L(F)}{F-b_1}\right)=S(r,f).\end{aligned}\tag{20}$$

It followed from (16) that

$$\begin{aligned}T(r,f_1)&\le m\left(r,\frac{F-a_1}{f-a}\right)+m\left(r,\frac{1}{F-a_1}\right)\\&\quad+m\left(r,\frac{F-b_1}{f-b}\right)+m\left(r,\frac{1}{F-b_1}\right)+S(r,f)\\&\le m\left(r,\frac{1}{F-a_1}\right)+m\left(r,\frac{1}{F-b_1}\right)+S(r,f)\\&\le m\left(r,\frac{1}{F-a_1}+\frac{1}{F-b_1}\right)+S(r,f)\\&\le m\left(r,\frac{1}{L(F)}\right)+S(r,f)\\&\le T(r,L(F))+S(r,f)\\&\le T(r,F)+S(r,f)\end{aligned}$$

$$\begin{aligned}
&\leq T\left(r, m_1 f\left(z+c_1\right)+m_2 f\left(z+c_2\right)\right.\\
&\qquad \left.+\cdots+m_n f\left(z+c_n\right)\right)+S(r, f)\\
&= m\left(r, m_1 f\left(z+c_1\right)+m_2\, f\left(z+c_2\right)\right.\\
&\qquad \left.+\cdots+m_n f\left(z+c_n\right)\right)+S(r, f)\\
&\leq m(r, f)+S(r, f)\\
&\leq T(r, f)+S(r, f)=T\left(r, f_1\right)+S(r, f).
\end{aligned} \tag{21}$$

By (18) and (21), we can deduce that

$$T\left(r, f_1\right)=T(r, F)+S(r, f)=T\left(r, F_1\right)+S(r, f). \tag{22}$$

It follows from (14) and (22) that

$$\overline{\lim_{\substack{r \rightarrow \infty \\ r \in I}}} \frac{N(r, 0)+N(r, 1)}{T\left(r, f_1\right)+T\left(r, F_1\right)}=\frac{1}{2}<\frac{2}{3}. \tag{23}$$

By Lemma 6, we have

$$f_1(z)=\frac{A F_1(z)+B}{C F_1(z)+D}, \tag{24}$$

where A, B, C, and D are complex numbers satisfying $AD-BC \neq 0$.

Now, we consider three cases.

Case 1. Consider $N(r,0)=S(r,f_1)$. Thus

$$N\left(r, \frac{1}{f_1}\right)+N\left(r, f_1\right)=S\left(r, f_1\right)=S(r, f). \tag{25}$$

Similarly, we have

$$N\left(r, \frac{1}{F_1}\right)+N\left(r, F_1\right)=S\left(r, F_1\right)=S(r, f). \tag{26}$$

By Lemma 5, we get that either $f_1 \equiv F_1$ or $f_1 F_1 \equiv 1$.

If $f_1 \equiv F_1$, we can easily deduce that $f \equiv F$, which is a contradiction with our assumption.

If $f_1 F_1 \equiv 1$, that is

$$(f(z)-a)(F(z)-a) \equiv(b-a)^2, \tag{27}$$

then we have

$$(f-a)^2=\frac{(b-a)^2}{(F-a)/(f-a)}. \tag{28}$$

From (28), we have

$$\begin{aligned}
2T(r, f) &\leq T\left(r,(f-a)^2\right)+S(r, f)\\
&= T\left(r, \frac{1}{(b-a)^2 /((F-a) /(f-a))}\right)+S(r, f)\\
&\leq T\left(r, \frac{F-a}{f-a}\right)+S(r, f)\\
&= N\left(r, \frac{F-a}{f-a}\right)+m\left(r, \frac{F-a}{f-a}\right)+S(r, f)\\
&\leq m\left(r, \frac{\left(F-a_1\right)+\left(a_1-a\right)}{f-a}\right)+S(r, f)\\
&\leq m\left(r, \frac{a_1-a}{f-a}\right)+S(r, f) \leq T(r, f)+S(r, f).
\end{aligned} \tag{29}$$

It follows that $T(r,f) \leq S(r,f)$, a contradiction.

Case 2. Consider $N(r,1)=S(r,f_1)$. Using the same argument as used in Case 1, we deduce that $T(r,f) \leq S(r,f)$, a contradiction.

Case 3. Consider $N(r,0) \neq S(r,f_1)$, $N(r,1) \neq S(r,f_1)$. Since f_1 and F_1 share 0, 1 CM almost, we deduce from (24) that

$$f_1(z)=\frac{(C+D) F_1(z)}{C F_1(z)+D}. \tag{30}$$

If $C=0$, then $f_1 \equiv F_1$; that is, $f \equiv F$, a contradiction. Hence $C \neq 0$. Thus we have

$$N\left(r, \frac{1}{F_1+(D/C)}\right)=N\left(r, f_1\right)=S\left(r, f_1\right)=S(r, f). \tag{31}$$

Obviously, $D/C \neq 0$, $D/C \neq -1$. Thus by Nevanlinna second fundamental theorem and (14), we get

$$\begin{aligned}
2T\left(r, f_1\right) &= 2T\left(r, F_1\right)+S\left(r, f_1\right)\\
&\leq \overline{N}\left(r, \frac{1}{F_1}\right)+\overline{N}\left(r, \frac{1}{F_1-1}\right)\\
&\quad+\overline{N}\left(r, \frac{1}{F_1+(D/C)}\right)+S(r, f)\\
&\leq N(r, 0)+N(r, 1)+S(r, f)\\
&\leq T\left(r, f_1\right)+S(r, f).
\end{aligned} \tag{32}$$

It follows that $T(r,f_1) \leq S(r,f_1)$, a contradiction. Thus we prove that $f(z) \equiv F(z)$. This completes the proof of Theorem 1.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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