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*Research Article*

# A Note on Jordan, Adamović-Mitrinović, and Cusa Inequalities

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We improve the Jordan, Adamović-Mitrinović, and Cusa inequalities. As applications, several new Shafer-Fink type inequalities for inverse sine function and bivariate means inequalities are established, and a new estimate for sine integral is given.

## 1. Introduction

The classical Jordan inequality [1] is given by

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1, \tag{1}$$

for $x \in (0, \pi/2)$.

Some new developments on refinements, generalizations, and applications for the Jordan inequality can be found in [2] and the references therein.

In the recent past, the following two-side inequality

$$(\cos x)^{1/3} < \frac{\sin x}{x} < \frac{2+\cos x}{3} \qquad \left(0 < x < \frac{\pi}{2}\right) \tag{2}$$

has attracted the attention of many researchers (see, e.g., [2–14]). The left inequality in (2) was obtained by Mitrinović (see [1, p. 238]), while the right one is due to Huygens (see, e.g., [15]) and it is called *Cusa inequality* [3, 5, 6, 8, 14].

In [16], the following open problem was proposed: for each $p > 0$, there are greatest value $q = q(p)$ and least value $r = r(p)$ such that the double inequality

$$\frac{q\sin x}{1+p\cos x} < x < \frac{r\sin x}{1+p\cos x} \tag{3}$$

holds for all $x \in (0, \pi/2)$. This was answered by Carver in [17]. In [1, p. 238, 3.4.15], it was listed that

$$\frac{(1+p)\sin x}{1+p\cos x} < x < \frac{(\pi/2)\sin x}{1+p\cos x} \tag{4}$$

for $p \in (0, 1/2]$ and $x \in [0, \pi/2]$. Wu [18] proved that the double inequality

$$\frac{(1+p)\cos x}{1+p\cos x} < \frac{\sin x}{x} < \frac{1+q}{1+q\cos x} \tag{5}$$

holds for $x \in (0, \pi/2)$, $p \in [-1, 2]$, and $q \in [-1/4, \infty)$. In particular, he obtained that for $x \in (0, \pi/2)$,

$$\frac{3\cos x}{1+2\cos x} < \frac{\sin x}{x} < \frac{3}{4-\cos x}. \tag{6}$$

The first inequality in (6) is equivalent to the Huygen inequality:

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > 3. \tag{7}$$

Jiang [19] showed that for $x \in (0, \pi/2)$,

$$\frac{\sin x}{x} > \frac{1+2\cos x}{2+\cos x}. \tag{8}$$

Li and He [20] gave an improvement of (6) as follows:

$$\frac{7+5\cos x}{11+\cos x} < \frac{\sin x}{x} < \frac{9+6\cos x}{14+\cos x}. \tag{9}$$

The main purpose of this paper is to give sharp bounds for $(\sin x)/x$ in terms of the functions $H_1(\cos t, p)$ and $H_2(\cos t, p)$, where

$$H_1(x,p) = \frac{2p+(p+3)x}{3p+1+2x}, \quad x \in (0,1), \quad p \in (-\infty,-1] \cup [0,\infty), \tag{10}$$

$$H_2(x,p) = \frac{3p+1}{\pi p}\frac{2p+(p+3)x}{(3p+1)+2x}, \quad x \in (0,1), \quad p \in (-\infty,-1] \cup (0,\infty). \tag{11}$$

The rest of this paper is organized as follows. Several lemmas are given in Section 2. Main results and their proofs are given in Section 3, in which Theorem 7 unifies and generalizes Jordan and Cusa inequalities; Theorem 13 shows that Adamović-Mitrinović and Cusa inequalities (2) can be interpolated by $H_1(\cos x, p)$ for suitable $p$; Theorem 18 gives a hyperbolic version of Theorem 7. In Section 4, some new Shafer-Fink type inequalities for inverse sine function and several inequalities for bivariate means are presented, and a simpler but more accurate estimate for sine integral is provided.

## 2. Lemmas

**Lemma 1.** *Let $H_1$ and $H_2$ be defined by* (10) *and* (11), *respectively. Then $H_1$ and $H_2$ are, respectively, increasing and decreasing with respect to $p$ on $(-\infty,-1] \cup (0,\infty)$ with the limits*

$$\lim_{p\to-\infty} H_1(x,p) = \lim_{p\to\infty} H_1(x,p) = \frac{2+x}{3}, \qquad \lim_{p\to-\infty} H_2(x,p) = \lim_{p\to\infty} H_2(x,p) = \frac{2+x}{\pi}. \tag{12}$$

*Proof.* From (10) and (11) we have

$$\frac{\partial H_1}{\partial p} = \frac{2(x-1)^2}{(3p+1+2x)^2} > 0, \qquad \frac{\partial H_2}{\partial p} = -\frac{3x}{\pi p^2(3p+2x+1)^2}(p+1) \times ((5-2x)p+(2x+1)). \tag{13}$$

If $p \in (0,\infty)$, then we clearly see that $\partial H_2/\partial p < 0$. If $\in (-\infty,-1)$, then $(5-2x)p+(2x+1) < 4(x-1) < 0$, and then $\partial H_2/\partial p < 0$.

Simple computations give (12). □

**Lemma 2.** *Let $u_1, u_2$ be defined on $(0,1)\times(-\infty,-1] \cup [0,\infty)$ by*

$$u_1(x,p) = (2p+(3+p)x)(3p+1+2x), \tag{14}$$

$$u_2(x,p) = 2(p+3)x^3 + 8px^2 + 2p(3p+1)x + 3(p+1)^2, \tag{15}$$

*respectively. Then $u_1(x,p), u_2(x,p) > 0$.*

*Proof.* It is not difficult to see that $u_1(x,p), u_2(x,p) > 0$ for $p \in [0,\infty)$. If $p \in (-\infty,-1]$, then

$$2p+(3+p)x < 2(x-1) < 0, \qquad (3p+1+2x) < 2(x-1) < 0, \tag{16}$$

and then $u_1(x,p) > 0$. It remains to prove that $u_2(x,p) > 0$ for $p \in (-\infty,-1]$. Differentiation leads to

$$\frac{\partial u_2}{\partial p} = (12x+6)p + (2x^3+8x^2+2x+6). \tag{17}$$

Hence, $\partial u_2/\partial p < -(12x+6)+(2x^3+8x^2+2x+6) = 2x(x+5)(x-1) < 0$, which implies that $u_2$ is decreasing in $p$ on $(-\infty,-1)$, and therefore,

$$u_2(x,p) > u_2(x,-1) = 4x(x-1)^2 > 0. \tag{18}$$

This completes the proof. □

**Lemma 3.** *Let $u_3$ be defined on $(0,1)\times(-\infty,-1] \cup [0,\infty)$ by*

$$u_3(x,p) = (p+3)^2x^2 + (p+3)(7p+3)x + (-3p^3+13p^2+21p+9). \tag{19}$$

*Then*

(i) *$u_3(x,p) \geq 0$ for all $x \in (0,1)$ if and only if $p \in (-\infty, p_3]$, where $p_3 \approx 5.6630$ is the unique solution of the equation $u_3(0,p) = -3p^3+13p^2+21p+9 = 0$;*

(ii) *$u_3(x,p) \leq 0$ if and only if $p \in [9,\infty)$;*

(iii) *for every $p \in (p_3, 9)$, there exists a unique $x_1 \in (0,1)$ such that $u_3(x,p) < 0$ for $x \in (0,x_1)$ and $u_3(x,p) > 0$ for $x \in (x_1,1)$.*

*Proof.* In order to prove the desired results, we need to rewrite $u_3(x,p)$ as

$$u_3(x,p) = \left((p+3)x + \frac{7p+3}{2}\right)^2 - 3\left(p-\frac{9}{4}\right)(p+1)^2. \tag{20}$$

We clearly see that

$$u_3(0,p) = -3p^3+13p^2+21p+9, \qquad u_3(1,p) = -3(p+1)^2(p-9). \tag{21}$$

We claim that there exists unique $p_3 \in (5,6)$ such that $u_3(0,p) > 0$ for $p \in (-\infty,p_3)$ and $u_3(0,p) < 0$ for $p \in (p_3,\infty)$. Indeed, we have

$$u_3'(0,p) = -9p^2+26p+21 = -9\left(p-\frac{13-\sqrt{358}}{9}\right)\left(p-\frac{13+\sqrt{358}}{9}\right), \tag{22}$$

which implies that $u_3(0,p)$ is increasing on $((13-\sqrt{358})/9, (13+\sqrt{358})/9)$ and decreasing on $(-\infty,(13-\sqrt{358})/9)\cup((13+\sqrt{358})/9,\infty)$. Since

$$u_3\left(0,\frac{13-\sqrt{358}}{9}\right)=\frac{13952}{243}-\frac{716}{243}\sqrt{358}\approx 1.6652>0,$$
$$u_3\left(0,\frac{13+\sqrt{358}}{9}\right)=\frac{716}{243}\sqrt{358}+\frac{13952}{243}>0,$$
$$u_3(0,\infty)=-\infty, \tag{23}$$

there exists unique $p_3\in((13+\sqrt{358})/9,\infty)$ such that $u_3(0,p_3)=0$ and $u_3(0,p)>0$ for $p\in(-\infty,p_3)$ and $u_3(0,p)<0$ for $p\in(p_3,\infty)$. An easy calculation reveals that $p_3\approx 5.6630$.

(i) Now we prove the necessary and sufficient condition for $u_3(x,p)\ge 0$ for all $x\in(0,1)$. Since $u_3(x,-3)=144>0$, we assume that $p\neq -3$. Denote the minimum point of $u_3(x,p)$ by $x_0$. Then $x_0=-(7p+3)/(2(p+3))$. And then, due to $\partial^2u_3/\partial x^2>0$, $u_3(x,p)\ge 0$ for all $x\in(0,1)$ if and only if at least one of the following cases occur.

*Case 1.* Consider that $x_0=-(7p+3)/(2(p+3))\ge 1$, $u_3(1,p)\ge 0$. It is derived that $p\in(-3,-1]$.

*Case 2.* Consider that $x_0=-(7p+3)/(2(p+3))\le 0$, $u_3(0,p)\ge 0$. It implies that $p\in(-\infty,-3),\ \cup[-3/7,p_3]$.

*Case 3.* Consider that $x_0=-(7p+3)/(2(p+3))\in(0,1)$, $u_3(x_0,p)=-3(p-9/4)(p+1)^2\ge 0$. It yields $p\in(-1,-3/7)$.

To sum up, $u_3(x,p)\ge 0$ for all $x\in(0,1)$ if and only if $p\in(-\infty,p_3]$.

(ii) It is clear that $u_3(x,p)\le 0$ if and only if $u_3(0,p)\le 0$ and $u_3(1,p)\le 0$. Solving the inequalities for $p$ leads to $p\ge 9$.

(iii) In the case of $p\in(p_3,9)$, we clearly see that $u_3(0,p)<0$, $u_3(1,p)>0$, and $x_0=-(7p+3)/(2(p+3))<0$. This implies that there exists a unique $x_1\in(0,1)$ such that $u_3(x,p)<0$ for $x\in(0,x_1)$ and $u_3(x,p)>0$ for $x\in(x_1,1)$.

This completes the proof. □

Now let us consider the sign of function $g$ defined on $(0,\pi/2)\times(-\infty,-1]\cup[0,\infty)$ by

$$\begin{aligned}g(t,p)&=t-\Big(((2p+(p+3)\cos t)(3p+1+2\cos t))\\&\quad\times\left(2(p+3)\cos^3t+8p\cos^2t\right.\\&\qquad\left.+2p(3p+1)\cos t+3(p+1)^2\right)^{-1}\sin t\Big)\\&=t-\frac{u_1(\cos t,p)}{u_2(\cos t,p)}\sin t,\end{aligned} \tag{24}$$

where $u_1(x,p)$ and $u_2(x,p)$ are defined by (14) and (15), respectively. We have the following.

**Lemma 4.** *Let $g$ be defined on $(0,\pi/2)\times(-\infty,-1]\cup[0,\infty)$ by (24). Then*

(i) *$g(t,p)<0$ for all $t\in(0,\pi/2)$ if and only if $p\in(-\infty,-1]\cup[9,\infty)$;*

(ii) *$g(t,p)>0$ for all $t\in(0,\pi/2)$ if and only if $p\in[0,p_1]$, where*
$$p_1=\frac{2\sqrt{6\pi+1}+3\pi-2}{12-3\pi}\approx 6.3433; \tag{25}$$

(iii) *in the case of $p\in(p_1,9)$, there exists a unique $t_0\in(0,\pi/2)$ such that $g(t,p)>0$ for $t\in(0,t_0)$ and $g(t,p)<0$ for $t\in(t_0,\pi/2)$.*

*Proof.* We first give two limit relations as follows:

$$\lim_{t\to 0^+}\frac{g(t,p)}{t^5}=-\frac{1}{45}\frac{p-9}{p+1}\quad\text{if } p\neq -1,$$
$$g\left(\frac{\pi^-}{2},p\right)=-\frac{12-3\pi}{6(p+1)^2}(p-p_1)(p-p_2)\quad\text{if } p\neq -1, \tag{26}$$

where

$$p_1=\frac{2\sqrt{6\pi+1}+3\pi-2}{12-3\pi}\approx 6.3433,$$
$$p_2=-\frac{2\sqrt{6\pi+1}-3\pi+2}{12-3\pi}<0. \tag{27}$$

In fact, if $p\neq -1$, then making use of power series we get

$$g(t,p)=-\frac{1}{45}\frac{p-9}{p+1}t^5+o\left(t^5\right), \tag{28}$$

which implies the first relation. Direct computations yield the second one.

Differentiating $g(t,p)$ with respect to $t$ leads to

$$\begin{aligned}\frac{\partial g}{\partial t}&=1-\frac{u_1(\cos t,p)}{u_2(\cos t,p)}\cos t+\left(\sin^2t\right)\frac{d}{dx}\frac{u_1(x,p)}{u_2(x,p)}\Big|_{x=\cos t}\\&=\frac{4(1-x)\left(1-x^2\right)}{u_2^2(x,p)}\times h(x,p),\end{aligned} \tag{29}$$

where $u_1(x,p)$ and $u_2(x,p)$ are defined by (14) and (15), respectively, and

$$h(x,p)=\left(x+\frac{3p+1}{2}\right)\times u_3(x,p); \tag{30}$$

here $u_3(x,p)$ is defined by (19) and $x=\cos t\in(0,1)$.

(i) We now prove that $g(t,p)\le 0$ for all $t\in(0,\pi/2)$ if and only if $p\in(-\infty,-1]\cup[9,\infty)$. The necessity easily follows from the inequalities $\lim_{t\to 0^+}t^{-5}g(t,p)\le 0$ and $g(\pi/2^-,p)\le 0$ if $p\neq -1$ and $g(t,-1)=t-\tan t<0$ together with the relation (26).

Next we prove the sufficiency. If $p \in [9, \infty)$, then by Lemma 3 $u_3(x, p) \le 0$, and then $h(x, p) \le 0$. This indicates that $g$ is decreasing in $t$ on $(0, \pi/2)$, and therefore, we get $g(t, p) < g(0^+, p) = 0$. If $p \in (-\infty, -1]$, then $u_3(x, p) \ge 0$ and $x + (3p + 1)/2 < x - 1 < 0$, which yields $h(x, p) \le 0$. This also yields that $g$ is decreasing in $t$ on $(0, \pi/2)$, and so $g(t, p) < g(0^+, p) = 0$.

(ii) Similarly, we can prove that $g(t, p) > 0$ for all $t \in (0, \pi/2)$ if and only if $p \in [0, p_1]$. If $g(t, p) > 0$ for all $t \in (0, \pi/2)$, then we have $\lim_{t\to 0^+} t^{-5} g(t, p) \ge 0$ and $g(\pi/2^-, p) \ge 0$, which together with (26) and $p \in (-\infty, -1] \cup [0, \infty)$ lead to $p \in [0, p_1]$.

In order to prove the sufficiency, we distinguish two cases.

In the case of $p \in [0, p_3]$, by Lemma 3 we have $u_3(x, p) \ge 0$, which implies that $g$ is increasing in $t$ on $(0, \pi/2)$, and so, $g(t, p) > g(0^+, p) = 0$.

In the case of $p \in (p_3, p_1]$, from Lemma 3 there is a unique $x_1 \in (0, 1)$ such that $u_3(x, p) < 0$ for $x \in (0, x_1)$ and $u_3(x, p) > 0$ for $x \in (x_1, 1)$. This in conjunction with (30) and (29) shows that $g$ is decreasing in $t$ on $(\arccos x_1, \pi/2)$ and increasing on $(0, \arccos x_1)$, and consequently, we have

$$g(t, p) > g(0^+, p) = \quad \text{for } t \in (0, \arccos x_1),$$
$$g(t, p) > g\left(\frac{\pi^+}{2}, p\right) = -\frac{12 - 3\pi}{6(p+1)^2}(p - p_1)(p - p_2) \ge 0 \quad \text{for } t \in \left(\arccos x_1, \frac{\pi}{2}\right), \tag{31}$$

which proves the sufficiency.

(iii) In the case when $p \in (p_1, 9)$, we have seen that $g$ is decreasing in $t$ on $(\arccos x_1, \pi/2)$ and increasing on $(0, \arccos x_1)$ and $g(t, p) > 0$ for $t \in (0, \arccos x_1)$, but

$$g\left(\frac{\pi^-}{2}, p\right) = -\frac{12 - 3\pi}{6(p+1)^2}(p - p_1)(p - p_2) < 0. \tag{32}$$

Thus, there is a unique $t_0 \in (\arccos x_1, \pi/2)$ such that $g(t, p) > 0$ for $t \in (0, t_0)$ and $g(t, p) < 0$ for $t \in (t_0, \pi/2)$.

The whole proof is complete. □

We next observe the function $f$ defined on $(0, \pi/2) \times (-\infty, -1] \cup [0, \infty)$ by

$$f(t, p) = \ln\frac{\sin t}{t} - \ln H_1(\cos t, p) = \ln\frac{\sin t}{t} - \ln\frac{2p + (p+3)\cos t}{(3p+1) + 2\cos t}. \tag{33}$$

Differentiation yields that

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{\cos t}{\sin t} - \frac{1}{t} - \frac{2\sin t}{3p + 1 + 2\cos t} + \frac{(p+3)\sin t}{2p + (p+3)\cos t} \\ &= \Big(\big(2(p+3)\cos^3 t + (3p^2 + 14p + 3)\cos^2 t + 3(p+1)^2(\sin^2 t) + 2p(3p+1)\cos t\big) \\ &\quad \times \big((\sin t)(2p + 3\cos t + p\cos t)(3p + 2\cos t + 1)\big)^{-1}\Big) - \frac{1}{t} \\ &= \frac{1}{\sin t}\frac{u_2(\cos t, p)}{u_1(\cos t, p)} - \frac{1}{t} = \frac{1}{t\sin t}\frac{u_2(\text{cos}t, p)}{u_1(\cos t, p)} \times g(t, p),\end{aligned} \tag{34}$$

where $u_1(x, p)$, $u_2(x, p)$, and $g(t, p)$ are defined by (14), (15), and (24), respectively. From Lemmas 2 and 4 the following assertion is immediate.

**Lemma 5.** *Let $f$ be the function defined on $(0, \pi/2) \times (-\infty, -1] \cup [0, \infty)$ by (33). Then*

(i) *$f$ is decreasing in $t$ on $(0, \pi/2)$ if and only if $p \in (-\infty, -1] \cup [9, \infty)$;*

(ii) *$f$ is increasing in $t$ on $(0, \pi/2)$ if and only if $p \in [0, p_1]$, where $p_1 \approx 6.3433$ is given by (25);*

(iii) *in the case when $p \in (p_1, 9)$, there is a unique $t_0 \in (0, \pi/2)$ such that $f$ is increasing in $t$ on $(0, t_0)$ and decreasing on $(t_0, \pi/2)$.*

Lastly, for later use, we also give the following.

**Lemma 6.** *Let $H_1$ be defined on $(0, 1) \times (-\infty, -1] \cup [0, \infty)$ by (10). Then $H_1(x^3, p) \ge x$ if and only if $p \in (-\infty, -1] \cup [1, \infty)$, and $H_1(x^3, p) \le x$ if and only if $p = 0$.*

*Proof.* For $p \in (-\infty, \infty)$, we define

$$u_4(x, p) = 2x^2 + (1 - p)x - 2p = 2\left(x - \frac{p-1}{4}\right)^2 - \frac{1}{7}(14p + p^2 + 1). \tag{35}$$

Then $u_4(x, p) \ge 0$ holds for all $x \in (0, 1)$ if and only if $p \in (-\infty, 0]$.

In fact, $u_4(x, p) \ge 0$ if and only if at least one case of the following occurs.

*Case 1.* Consider that $(p - 1)/4 \ge 1$, $u_4(1, p) = 3 - 3p \ge 0$. It is impossible.

*Case 2.* Consider that $(p - 1)/4 \le 0$, $u_4(0, p) = -2p \ge 0$. It indicates $p \in (-\infty, 0]$.

*Case 3.* Consider that $0 < (p - 1)/4 < 1$, $u_4((p - 1)/4, p) \ge 0$. It is impossible.

In the same way, we can prove that $u_4(x, p) \le 0$ holds for all $x \in (0, 1)$ if and only if $p \in [1, \infty)$.

We now prove that $H_1(x^3, p) \geq x$ if and only if $p \in (-\infty, -1] \cup [1, \infty)$. Factoring yields

$$H_1\left(x^3, p\right) - x = -2(x-1)^2 \frac{2x^2 + (1-p)x - 2p}{3p + 2x^3 + 1} = -(x-1)^2 \frac{u_4(x,p)}{3p + 2x^3 + 1}. \tag{36}$$

If $p \in (-\infty, -1]$, then $3p + 2x^3 + 1 < 0$, and then, $H_1(x^3, p) \geq x$ if and only if $u_4(x, p) \geq 0$, which is equivalent to $p \in (-\infty, -1] \cap (-\infty, 0] = (-\infty, -1]$. If $p \in [0, \infty)$, then $3p + 2x^3 + 1 > 0$, and then, $H_1(x^3, p) \geq x$ if and only if $u_4(x, p) \leq 0$, which is equivalent to $p \in [0, \infty) \cap [1, \infty) = [1, \infty)$. Consequently, $H_1(x^3, p) \geq x$ if and only if $p \in (-\infty, -1] \cup [1, \infty)$.

Next we show that $H_1(x^3, p) \leq x$ if and only if $p = 0$. In fact, if $p \in (-\infty, -1]$, then $H_1(x^3, p) \leq x$ if and only if $u_4(x, p) \leq 0$, which yields $p \in [1, \infty)$. It is clearly a contradiction. If $p \in [0, \infty)$, then the statement in question if and only if $u_4(x, p) \geq 0$, which leads to $p \in [0, \infty) \cap (-\infty, 0] = \{0\}$. Thus the proof is complete. □

## 3. Main Results

**Theorem 7.** *Let $p \in (-\infty, -1] \cup [0, \infty)$. Then for $t \in (0, \pi/2)$,*

$$\frac{\sin t}{t} < \frac{2p + (p+3)\cos t}{(3p+1) + 2\cos t} \tag{37}$$

*holds if and only if $p \in (-\infty, -1] \cup [9, \infty)$. Moreover, we have*

$$H_2(\cos t, p) = \lambda_p \frac{2p + (p+3)\cos t}{(3p+1) + 2\cos t} < \frac{\sin t}{t} < \frac{2p + (p+3)\cos t}{(3p+1) + 2\cos t} = H_1(\cos t, p) \tag{38}$$

*for $p \in (-\infty, -1] \cup [9, \infty)$, where $\lambda_p = (3p+1)/(\pi p)$ is the best possible. And the lower and upper bounds in (38) are decreasing and increasing in $p$ on $(-\infty, -1] \cup (0, \infty)$, respectively.*

*Proof.* Clearly, the desired result is equivalent to $f(t, p) < 0$ if and only if $p \in (-\infty, -1] \cup [9, \infty)$, where $f(t, p)$ is defined by (33). To this end, we give two limit relations. The first one follows by expanding $f(t, p)$ in power series for $t$. We have

$$f(t, p) = -\frac{1}{180} \frac{p-9}{p+1} t^4 + o\left(t^4\right) \quad \text{if } p \neq -1, \tag{39}$$

which yields

$$\lim_{t \to 0^+} \frac{f(t, p)}{t^4} = -\frac{1}{180} \frac{p-9}{p+1} \quad \text{if } p \neq -1. \tag{40}$$

The second one is derived by a simple computation; that is,

$$f\left(\frac{\pi^-}{2}, p\right) = \ln \frac{3p+1}{\pi p}. \tag{41}$$

Now we prove that $f(t, p) < 0$ for all $t \in (0, \pi/2)$ if and only if $p \in (-\infty, -1] \cup [9, \infty)$.

The necessity easily follows by solving the simultaneous inequalities:

$$\lim_{t \to 0^+} \frac{f(t, p)}{t^4} = -\frac{1}{180} \frac{p-9}{p+1} \leq \quad \text{if } p \neq -1,$$
$$f(t, -1) = \ln \frac{\sin t}{t} < 0, \tag{42}$$
$$f\left(\frac{\pi^-}{2}, p\right) = \ln \frac{3p+1}{\pi p} \leq 0,$$

which implies $p \in (-\infty, -1] \cup [9, \infty)$.

The sufficiency is due to Lemma 5. In fact, If $p \in (-\infty, -1] \cup [9, \infty)$, then by Lemma 5 we see that $f$ is decreasing in $t$ on $(0, \pi/2)$. Hence, $f(t, p) < f(0^+, p) = 0$.

Utilizing the monotonicity of $f$ in $t$ on $(0, \pi/2)$ gives (38). And from Lemma 1 it is seen that the lower and upper bounds in (38) are decreasing and increasing in $p$ on $(-\infty, -1] \cup (0, \infty)$, respectively.

Thus the proof is finished. □

By Theorem 7 and Lemma 1, we have the following interesting chain of inequalities.

**Corollary 8.** *For $t \in (0, \pi/2)$, one has*

$$\begin{aligned} \frac{2}{\pi} &= H_2(\cos t, -1) < \cdots < H_2(\cos t, -\infty) \\ &= \frac{2 + \cos t}{\pi} = H_2(\cos t, \infty) < \cdots < H_2(\cos t, 9) < \frac{\sin t}{t} \\ &< H_1(\cos t, 9) < \cdots < H_1(\cos t, \infty) = \frac{2 + \cos t}{3} \\ &= H_1(\cos t, -\infty) \cdots < H_1(\cos t, -1) = -1. \end{aligned} \tag{43}$$

*Remark 9.* It is clear that our results unify and refine Jordan and Cusa's inequalities and show that the first one in (9) is sharp. Also, Theorem 7 contains other known results, for example, taking $p = -3$ in (38) we get

$$\frac{8}{\pi} \frac{1}{4 - \cos t} < \frac{\sin t}{t} < 3 \frac{1}{4 - \cos t}, \tag{44}$$

which contain (6). After a simple transformation, (44) can be written as

$$\frac{8}{\pi} \frac{t}{\sin t} + \cos t < 4 < 3 \frac{t}{\sin t} + \cos t, \tag{45}$$

where the second inequality in (45) is due to Neuman and Sándor [6, (2.12)].

**Theorem 10.** *Let $p \in (-\infty, -1] \cup [0, \infty)$. Then for $t \in (0, \pi/2)$*

$$\frac{2p + (p+3)\cos t}{(3p+1) + 2\cos t} < \frac{\sin t}{t} \tag{46}$$

*holds if and only if $p \in [0, p_0]$, where $p_0 = (\pi - 3)^{-1} \approx 7.0625$.*

*Moreover, for $p \in (0, p_1]$, one has*

$$H_1(\cos t, p) = \frac{2p + (p+3)\cos t}{(3p+1) + 2\cos t} < \frac{\sin t}{t} < \lambda_p \frac{2p + (p+3)\cos t}{(3p+1) + 2\cos t} = H_2(\cos t, p), \tag{47}$$

*where $p_1 \approx 6.3433$, $\lambda_p = (3p+1)/(\pi p)$ is the best possible. And $H_1(\cos t, p)$, $H_2(\cos t, p)$ are decreasing and increasing in $p$ on $(-\infty, -1] \cup (0, \infty)$, respectively.*

*For $p \in (p_1, p_0]$ one has*

$$H_1(\cos t, p) = \frac{2p + (p+3)\cos t}{(3p+1) + 2\cos t} < \frac{\sin t}{t} < \delta_p \frac{2p + (p+3)\cos t}{(3p+1) + 2\cos t} = \delta_p H_1(\cos t, p), \tag{48}$$

*where $\delta_p = (\sin t_0/t_0)(((3p+1)+2\cos t_0)/(2p+(p+3)\cos t_0))$ is the best possible and $t_0$ is the unique root of the equation*

$$\frac{(2p + (3+p)\cos t)(3p + 1 + 2\cos t)}{2(p+3)\cos^3 t + 8p\cos^2 t + 2p(3p+1)\cos t + 3(p+1)^2} \times \sin t = t \tag{49}$$

*on $(0, \pi/2)$.*

*Proof.* Since the inequality (46) is equivalent to $f(t,p) > 0$, it suffices to prove that $f(t,p) > 0$ holds for $t \in (0, \pi/2)$ if and only if $p \in [0, p_0]$.

Similarly, solving the simultaneous inequalities $\lim_{t\to 0} t^{-4} f(t,p) \geq 0$ and $f(\pi/2^-, p) \geq 0$ with $p \in (-\infty, -1] \cup (0, \infty)$ yields $p \in [0, p_0]$, which proves the necessity.

Conversely, the condition $p \in [0, p_0]$ is also sufficient for $f(t,p) > 0$ to be valid. For this end, we divide the proof into two cases.

*Case 1.* Consider that $p \in [0, p_1]$. By Lemma 5 it is seen that $f$ is increasing in $t$ on $(0, \pi/2)$, which indicates that $f(t,p) > f(0^+, p) = 0$.

*Case 2.* Consider that $p \in (p_1, p_0]$. By Lemma 5 we see that there is a unique $t_0 \in (0, \pi/2)$ such that $f$ is increasing in $t$ on $(0, t_0)$ and decreasing on $(t_0, \pi/2)$. It is acquired that

$$\begin{aligned} &f(t_0, p) > f(t,p) > f(0^+, p) = 0 \quad \text{for } t \in (0, t_0), \\ &f(t_0, p) > f(t,p) > f(\pi/2^-, p) = \ln\frac{3p+1}{\pi p} \geq 0 \quad \text{for } t \in \left(t_0, \frac{\pi}{2}\right); \end{aligned} \tag{50}$$

that is,

$$f(t_0, p) \geq f(t,p) > \quad \text{for } t \in (0, \pi/2), \tag{51}$$

which proves the sufficiency.

In the first case, application of the monotonicity of $f$ in $t$ on $(0, \pi/2)$ leads to (47), and $\lambda_p = (3p+1)/(\pi p)$. In the second case, (51) also yields (47), and

$$\delta_p = \exp f(t_0, p) = \frac{\sin t_0}{t_0} \frac{(3p+1) + 2\cos t_0}{2p + (p+3)\cos t_0}. \tag{52}$$

Thus we complete the proof. □

*Remark 11.* Taking $p = 7$ in (46), we get the first inequality in (9).

Letting $p = p_0 = (\pi - 3)^{-1}$ and solving (49) by mathematical computation software, we find that $t_0 \approx 1.3055$ and $\delta_{p_0} \approx 1.0015$. Letting $p = p_1$ be defined by (25) yields $\lambda_{p_1} = (3p_1 + 1)/(\pi p_1) \approx 1.0051$. By Theorem 10 we get the following.

**Corollary 12.** *For $t \in (0, \pi/2)$, one has*

$$\begin{aligned} &\frac{2p_0 + (p_0+3)\cos t}{(3p_0+1) + 2\cos t} < \frac{\sin t}{t} < \delta_{p_0} \frac{2p_0 + (p_0+3)\cos t}{(3p_0+1) + 2\cos t}, \\ &\frac{2p_1 + (p_1+3)\cos t}{(3p_1+1) + 2\cos t} < \frac{\sin t}{t} < \lambda_{p_1} \frac{2p_1 + (p_1+3)\cos t}{(3p_1+1) + 2\cos t}, \end{aligned} \tag{53}$$

*where $\delta_{p_0} \approx 1.0015$ and $\lambda_{p_1} \approx 1.0051$ are the best possible constants.*

Letting $x = \cos^{1/3} t$ in Lemma 6 and using Theorems 7 and 10, we obtain a chain of inequalities that interpolates Adamović-Mitrinović and Cusa's inequalities (2) by $H_1(\cos x, p)$.

**Theorem 13.** *For $t \in (0, \pi/2)$, the inequalities*

$$\begin{aligned} \frac{2p + (p+3)\cos t}{(3p+1) + 2\cos t} &< \cos^{1/3} t < \frac{2q + (q+3)\cos t}{(3q+1) + 2\cos t} \\ &< \frac{\sin t}{t} < \frac{2r + (r+3)\cos t}{(3r+1) + 2\cos t} < \frac{2 + \cos t}{3} \\ &< \frac{2s + (s+3)\cos t}{(3s+1) + 2\cos t} \end{aligned} \tag{54}$$

*hold if and only if $p = 0$, $q \in [0, p_0]$, $r \in [9, \infty)$, and $s \in (-\infty, -1]$, where $p_0 = (\pi - 3)^{-1}$.*

Using the monotonicity of $f(t,p)$ in $t$ on $(0, \pi/4)$ given by parts one and two of Lemma 5, we see that

$$\begin{aligned} &\ln\left(\frac{4}{\pi} \frac{3p + \sqrt{2} + 1}{(2\sqrt{2}+1)p + 3}\right) \\ &= f\left(\frac{\pi}{4}, p\right) \leq f\left(\frac{t}{2}, p\right) = \ln\frac{2\sin(t/2)}{t} \\ &- \ln H_1\left(\cos\frac{t}{2}, p\right) \leq f(0, p) = 0 \end{aligned} \tag{55}$$

hold for $p \in (-\infty,-1] \cup [9,\infty)$. And then we have

$$\frac{4}{\pi}\frac{3p+\sqrt{2}+1}{\left(2\sqrt{2}+1\right)p+3}H_1\left(\cos\frac{t}{2},p\right)\cos\frac{t}{2} < \frac{\sin t}{t} = H_1\left(\cos\frac{t}{2},p\right)\cos\frac{t}{2}. \tag{56}$$

It is clear that the right-hand in (56) is increasing in $p$ on $(-\infty,-1] \cup [0,\infty)$, but the monotonicity of left-hand is to be checked. We define

$$H_3(x,p) = \frac{4}{\pi}\frac{3p+\sqrt{2}+1}{\left(2\sqrt{2}+1\right)p+3}H_1(x,p), \tag{57}$$

where $x = \cos(t/2) \in [1/\sqrt{2},1]$. Logarithmic differentiation leads to

$$\begin{aligned}\frac{\partial \ln H_3}{\partial p} &= \frac{3}{\left(3p+\sqrt{2}+1\right)} - \frac{2\sqrt{2}+1}{\left(p\left(2\sqrt{2}+1\right)+3\right)} - \frac{3}{(3p+2x+1)} + \frac{x+2}{2p+x(p+3)} \\ &= -\left(\left(6\left(2\sqrt{2}+1\right)\left(x-\frac{\sqrt{2}}{2}\right)\left(\frac{22-9\sqrt{2}}{7}-x\right)\right)\right. \\ &\quad\times\left(\left(3p+\sqrt{2}+1\right)\left(p\left(2\sqrt{2}+1\right)+3\right)(3p+2x+1)\right. \\ &\quad\left.\left.\times(2p+x(p+3))\right)^{-1}\right)(p+1)(p-u_5(x)),\end{aligned} \tag{58}$$

where

$$u_5(x) = \frac{\left(5-2\sqrt{2}\right)x-\left(\sqrt{2}+2\right)}{\left(5\sqrt{2}-2\right)-\left(2\sqrt{2}+1\right)x}. \tag{59}$$

Since

$$u_5'(x) = -\frac{12\left(3-2\sqrt{2}\right)}{\left(5\sqrt{2}-2-\left(2\sqrt{2}+1\right)x\right)^2} < 0, \tag{60}$$

we have $-1 = u_5(1) < u_5(x) < u_5(1/\sqrt{2}) = -(24\sqrt{2}+5)/49 \approx -0.7947$. Consequently, $\partial(\ln H_3)/\partial p < 0$ for $p \in (-\infty,-1] \cup [0,\infty)$.

The result can be stated as a theorem.

**Theorem 14.** *Let $p \in (-\infty,-1] \cup [0,\infty)$. Then for $t \in (0,\pi/2)$ the inequalities*

$$\sigma_p\frac{2p\cos(t/2)+(p+3)\cos^2(t/2)}{(3p+1)+2\cos(t/2)} < \frac{\sin t}{t} < \frac{2p\cos(t/2)+(p+3)\cos^2(t/2)}{(3p+1)+2\cos(t/2)}. \tag{61}$$

*hold if and only if $p \in (-\infty,-1] \cup [9,\infty)$, where $\sigma_p = (4/\pi)((3p+\sqrt{2}+1)/((2\sqrt{2}+1)p+3))$ is the best constant. And the right-hand and left-hand in (61) are increasing and decreasing in p, respectively. Inequality (61) is reversed if and only if $p \in [0,p_1]$, where $p_1 \approx 6.3433$ is defined by (25).*

Putting $p = 9,\infty,0,1$ in Theorem 14 we have the following.

**Corollary 15.** *For $t \in (0,\pi/2)$ the following inequalities hold:*

$$\frac{2\left(41\sqrt{2}-25\right)}{7\pi}\frac{2\cos^2(t/2)+3\cos(t/2)}{\cos(t/2)+14} < \frac{\sin t}{t} < 3\frac{2\cos^2(t/2)+3\cos(t/2)}{\cos(t/2)+14}, \tag{62}$$

$$\frac{4\left(2\sqrt{2}-1\right)}{7}\frac{\cos^2(t/2)+2\cos(t/2)}{\pi} < \frac{\sin t}{t} < \frac{\cos^2(t/2)+2\cos(t/2)}{3}, \tag{63}$$

$$3\frac{\cos^2(t/2)}{2\cos(t/2)+1} < \frac{\sin t}{t} < \frac{4\left(\sqrt{2}+1\right)}{\pi}\frac{\cos^2(t/2)}{2\cos(t/2)+1}, \tag{64}$$

$$\frac{2\cos^2(t/2)+1}{\cos(t/2)+2} < \frac{\sin t}{t} < \frac{2\left(3-\sqrt{2}\right)}{\pi}\frac{2\cos^2(t/2)+1}{\cos(t/2)+2}. \tag{65}$$

Further, let $H_4$ be defined on $[1/\sqrt{2},1]\times(-\infty,-1]\cup[0,\infty)$ by

$$H_4(x,p) = \frac{H_1\left(2x^2-1,p\right)}{xH_1(x,p)}, \tag{66}$$

where $H_1$ is defined by (10). We can show that the monotonicity of $H_4$ in $x$ for certain fixed $p$. Differentiation again yields

$$\frac{\partial \ln H_4(x,p)}{\partial x} = \frac{2}{1+3p+2x} - \frac{1}{x} - \frac{p+3}{2p+(p+3)x} + \frac{4(p+3)x}{(p-3)+2(p+3)x^2} - \frac{8x}{4x^2+3p-1}. \tag{67}$$

It is easy to verify that

$$\begin{aligned}\frac{\partial \ln H_4(x,9)}{\partial x} &= -2\frac{(x-1)^2\left(594x^2+240x^3+8x^4+910x+273\right)}{x(2x+3)(x+14)\left(2x^2+13\right)\left(4x^2+1\right)} < 0, \\ \frac{\partial \ln H_4(x,\infty)}{\partial x} &= -2\frac{(1-x)(2x+1)}{x(x+2)\left(2x^2+1\right)} < 0, \\ \frac{\partial \ln H_4(x,1)}{\partial x} &= 2\frac{(1-x)\left(2x^3+8x^2+x+1\right)}{x(2x-1)(x+2)\left(2x^2+1\right)} > 0.\end{aligned} \tag{68}$$

Consequently, we have

$$1=\frac{H_1(1,p)}{H_1(1,p)}<\frac{H_1(2x^2-1,p)}{xH_1(x,p)}<\frac{H_1(0,p)}{(1/\sqrt{2})H_1(1/\sqrt{2},p)}=\frac{4p}{3p+1}\frac{3p+1+\sqrt{2}}{(2\sqrt{2}+1)p+3}\quad \text{for } p=9,\infty. \tag{69}$$

It is reversed for $p = 1$. From these we can obtain the following.

**Theorem 16.** *For $t\in(0,\pi/2)$ the following inequalities hold:*

$$\begin{aligned}
\frac{28}{9\pi}\frac{6\cos t+9}{\cos t+14}&<\frac{41(2\sqrt{2}-25)}{7\pi}\frac{2\cos^2(t/2)+3\cos(t/2)}{\cos(t/2)+14}\\
&<\frac{\sin t}{t}<\frac{6\cos^2(t/2)+9\cos(t/2)}{\cos(t/2)+14}\\
&<\frac{6\cos t+9}{\cos t+14},\\
\frac{2+\cos t}{\pi}&<\frac{12(2\sqrt{2}-1)}{7\pi}\frac{\cos^2(t/2)+2\cos(t/2)}{3}\\
&<\frac{\sin t}{t}<\frac{\cos^2(t/2)+2\cos(t/2)}{3}\\
&<\frac{2+\cos t}{3},\\
\frac{2\cos t+1}{\cos t+2}&<\frac{2\cos^2(t/2)+1}{\cos(t/2)+2}<\frac{\sin t}{t}\\
&<\frac{2(3-\sqrt{2})}{\pi}\frac{2\cos^2(t/2)+1}{\cos(t/2)+2}\\
&<\frac{4}{\pi}\frac{2\cos t+1}{\cos t+2}.
\end{aligned} \tag{70}$$

Additionally, Lemma 4 implies an optimal two-side inequality.

**Theorem 17.** *Let $p\in(-\infty,-1]\cup[0,\infty)$ and let $u_1(x,p)$ and $u_2(x,p)$ be defined by* (14) *and* (15)*, respectively. Then for $t\in(0,\pi/2)$ the two-side inequality*

$$\frac{u_2(\cos t,p)}{u_1(\cos t,p)}<\frac{\sin t}{t}<\frac{u_2(\cos t,q)}{u_1(\cos t,q)} \tag{71}$$

*holds if and only if $p\in(-\infty,-1]\cup[9,\infty)$ and $q\in[0,p_1]$, where $p_1\approx 6.3433$. And, for $x\in(0,1)$, the function $p\mapsto u_2(x,p)/u_1(x,p)$ is decreasing on $(-\infty,-1]\cup[0,\infty)$.*

*Proof.* Since $u_1(x,p),u_2(x,p)>0$ for $p\in(-\infty,-1]\cup[0,\infty)$ and $x\in(0,1)$ by Lemma 2 and $g(t,p)$ defined by (24) can be written as

$$g(t,p)=-t\frac{u_1(\cos t,p)}{u_2(\cos t,p)}\left(\frac{\sin t}{t}-\frac{u_2(\cos t,p)}{u_1(\cos t,p)}\right), \tag{72}$$

it follows from Lemma 4 that (71) k holds if and only if $p\in(-\infty,-1]\cup[9,\infty)$ and $q\in[0,p_1]$. It remains to check the monotonicity of $u_2(\cos t,p)/u_1(\cos t,p)$ in $p$. Differentiation yields

$$\frac{d}{dp}\frac{u_2(x,p)}{u_1(x,p)}=-6(x+1)(x-1)^2\times\frac{(p+1)((5+x)p+5x+1)}{(2p+3x+px)^2(3p+2x+1)^2}, \tag{73}$$

where $x\in(0,1)$. If $p\in[0,\infty)$, then the numerator of the fraction in right-hand above is clearly positive. Consider that $(p+1)((5+x)p+5x+1)>0$. If $p\in(-\infty,-1]$, then $(p+1)\le 0$ and $((5+x)p+5x+1)\le 5(x-1)<0$, which yields that the numerator is nonnegative.

This proves the assertion. □

Similarly, we can obtain a hyperbolic version of Theorems 7 and 10

**Theorem 18.** *Let $p\in(-\infty,-1]\cup[0,\infty)$. Then for $t\in(0,\infty)$*

$$\frac{2+(1+3p)\cosh t}{3+p+2p\cosh t}<\frac{\sinh t}{t} \tag{74}$$

*holds if and only if $p\in(-\infty,-1]\cup[1/9,\infty)$. It is reversed if and only if $p=0$.*

*Proof.* Let $F$ be the function defined on $(0,\infty)\times(-\infty,-1]\cup[0,\infty)$ by

$$F(t,p)=\frac{3+p+2p\cosh t}{2+(1+3p)\cosh t}\sinh t-t. \tag{75}$$

Then the inequalities (74) are equivalent to $F(t,p)>0$. Expanding in power series yields

$$F(t,p)=\frac{t^5}{180}\frac{9p-1}{p+1}+o(t^5), \tag{76}$$

which implies

$$\begin{aligned}\lim_{t\to 0}\frac{F(t,p)}{t^5}&=\frac{1}{20}\frac{p-1/9}{p+1}\quad \text{if } p\neq-1,\\ F(t,-1)&=\sinh t-t>0.\end{aligned} \tag{77}$$

On the other hand, we have

$$\lim_{t\to\infty}\frac{F(t,p)}{\sinh t}=\frac{2p}{1+3p}. \tag{78}$$

Now we prove desired results.

(i) We first prove that $F(t,p)>0$ holds if and only if $p\in(-\infty,-1]\cup[1/9,\infty)$.

If $F(t,p)>0$ for all $t>0$, then we have

$$\begin{aligned}\lim_{t\to 0}\frac{F(t,p)}{t^5}&=\frac{1}{20}\frac{p-1/9}{p+1}\ge 0,\\ F(t,-1)&=\sinh t-t>0,\\ \lim_{t\to\infty}\frac{F(t,p)}{\sinh t}&=\frac{2p}{1+3p}\ge 0.\end{aligned} \tag{79}$$

Solving the inequalities yields $p\in(-\infty,-1]\cup[1/9,\infty)$.

We prove the condition $p \in (-\infty, -1] \cup [1/9, \infty)$ is sufficient for $F(t, p) > 0$ to hold for $t \in (0, \infty)$. Differentiation gives

$$\begin{aligned}\frac{\partial F}{\partial t} &= \frac{(3p+1+2\cosh t)}{2p+(p+3)\cosh t}\cosh t \\ &\quad - \frac{3(p+1)^2\sinh^2 t}{(2p+(p+3)\cosh t)^2} - 1 \\ &= (x-1)^2 \frac{2p(3p+1)x + (3p^2+6p-1)}{(x+3px+2)^2},\end{aligned} \tag{80}$$

where $x = \cosh t \in (1, \infty)$.

Due to $p \in (-\infty, -1] \cup [1/9, \infty)$, we see that $2p(3p+1) > 0$, which yields

$$\begin{aligned}&2p(3p+1)x + (3p^2+6p-1) \\ &\quad > 2p(3p+1) + (3p^2+6p-1) \\ &\quad = (p+1)(9p-1) \geq 0.\end{aligned} \tag{81}$$

Then $\partial F/\partial t > 0$; that is, $F$ is increasing in $t$ on $(0, \infty)$. It is obtained that $F(t, p) > F(0, p) = 0$, which proves the sufficiency.

(ii) Next we prove that the reverse inequality of (74) holds if and only if $p = 0$. The necessity follows from

$$\begin{aligned}\lim_{t\to 0}\frac{F(t,p)}{t^5} &= \frac{1}{20}\frac{p-1/9}{p+1} \leq 0, \\ \lim_{t\to\infty}\frac{F(t,p)}{\sinh t} &= \frac{2p}{1+3p} \leq 0,\end{aligned} \tag{82}$$

and the assumption $p \in (-\infty, -1] \cup [0, \infty)$. We get $p = 0$.

Now we prove $F(t, p) < 0$ when $p = 0$. We have

$$\frac{\partial F}{\partial t} = -\frac{(x-1)^2}{(x+3px+2)^2} < 0, \tag{83}$$

where $x = \cosh t \in (1, \infty)$, then $F(t, 0) < F(0, 0) = 0$.

Thus the proof of Theorem 18 is complete. □

Denote

$$H_5(x, p) = \frac{2+(1+3p)x}{3+p+2px}. \tag{84}$$

It is easy to verify that $H_5(x, p) = H_1(x, p^{-1})$ for $p \neq 0$. By Lemma 1, we see that $H_5$ is decreasing in $p$ on $(-\infty, -1] \cup [0, \infty)$. Thus, as a consequence of Theorem 14, we have the following.

**Corollary 19.** *One has*

$$\begin{aligned}\frac{2+\cosh t}{3} &> \frac{\sinh t}{t} > H_5\left(\cosh t, \frac{1}{9}\right) \\ &> \cdots > H_5(\cosh t, \infty) = \frac{3\cosh t}{2\cosh t+1} \\ &= H_5(\cosh t, -\infty) > \cdots H_5(\cosh t, -1) = 1.\end{aligned} \tag{85}$$

Furthermore, note that $H_5(x, p) = H_1^{-1}(x^{-1}, p)$ and by Lemma 6 we have the following.

**Corollary 20.** *One has*

$$\begin{aligned}\frac{2+\cosh t}{3} &> \frac{\sinh t}{t} > \cosh^{1/3} t > \frac{1+2\cosh t}{2+\cosh t} \\ &> H_5(\cosh t, p),\end{aligned} \tag{86}$$

*where* $p \in (-\infty, -1] \cup (1, \infty)$.

## 4. Applications

In this section, we give some applications of our results.

*4.1. Shafer-Fink Type Inequalities.* In [1, p. 247, 3.4.31], it was listed that the inequality

$$\arcsin x > \frac{6(\sqrt{x+1}-\sqrt{1-x})}{4+\sqrt{x+1}+\sqrt{1-x}} > \frac{3x}{2+\sqrt{1-x^2}} \tag{87}$$

holds for $x \in (0, 1)$, which is due to Shafer [21]. Fink [22] proved that the double inequality

$$\frac{3x}{2+\sqrt{1-x^2}} \leq \arcsin x \leq \frac{\pi x}{2+\sqrt{1-x^2}} \tag{88}$$

is true for $x \in [0, 1]$. There has been some improvements and generalizations of Shafer-Fink inequality (see [23]). Letting $\sin t = x$ in Theorems 7, 10, 13, 14, 16 and 17 we can obtain corresponding Shafer-Fink type inequalities, which clearly contain many known results. For example, Theorems 7 and 10 can be changed into the following.

**Proposition 21.** *For* $x \in (0, 1)$, *the two-side inequality*

$$\begin{aligned}\frac{x}{H_1(\sqrt{1-x^2}, p)} &= x\frac{(3p+1)+2\sqrt{1-x^2}}{2p+(p+3)\sqrt{1-x^2}} < \arcsin x \\ &< \frac{\pi p}{3p+1}x\frac{(3p+1)+2\sqrt{1-x^2}}{2p+(p+3)\sqrt{1-x^2}} \\ &= \frac{x}{H_2(\sqrt{1-x^2}, p)}\end{aligned} \tag{89}$$

*holds if and only if* $p \in (-\infty, -1] \cup [9, \infty)$, *where* $\pi p/(3p+1)$ *is the best possible. And, the lower and upper bounds in* (89) *are decreasing and increasing in* $p$ *on* $(-\infty, -1] \cup (0, \infty)$, *respectively.*

*Inequality* (89) *is reversed if* $p \in [0, p_1]$, *where* $p_1 \approx 6.3433$ *is defined by* (25).

Letting $\sin t = x$, then $\cos(t/2) = (1/2)(\sqrt{1+x} + \sqrt{1-x})$. Theorem 14 can be restated as follows.

**Proposition 22.** *For $x \in (0,1)$, the two-side inequality*

$$2\frac{(3p+1)\left(\sqrt{1+x}-\sqrt{1-x}\right)+2x}{4p+(p+3)\left(\sqrt{1+x}+\sqrt{1-x}\right)} < \arcsin x < \frac{2}{\sigma_p}\frac{(3p+1)\left(\sqrt{1+x}-\sqrt{1-x}\right)+2x}{4p+(p+3)\left(\sqrt{1+x}+\sqrt{1-x}\right)} \quad (90)$$

*holds if and only if $p \in (-\infty,-1] \cup [9,\infty)$, where $\sigma_p = (4/\pi)((3p+\sqrt{2}+1)/((2\sqrt{2}+1)p+3))$ is the best constant. And, the lower and upper bounds in (90) are decreasing and increasing in $p$ on $(-\infty,-1] \cup (0,\infty)$, respectively.*

*Inequality (90) is reversed if $p \in [0,p_1]$, where $p_1 \approx 6.3433$ is defined by (25).*

As another example, Theorem 16 can be rewritten as follows.

**Proposition 23.** *For $x \in (0,1)$, all the following chains of inequalities hold:*

$$\frac{x}{3}\frac{\sqrt{1-x^2}+14}{2\sqrt{1-x^2}+3} < \frac{1}{3}\frac{x+14\left(\sqrt{x+1}-\sqrt{1-x}\right)}{3+\sqrt{x+1}+\sqrt{1-x}} < \arcsin x < \frac{\left(41\sqrt{2}+25\right)\pi}{782}\frac{x+14\left(\sqrt{x+1}-\sqrt{1-x}\right)}{3+\sqrt{x+1}+\sqrt{1-x}} < \frac{3\pi x}{28}\frac{\sqrt{1-x^2}+14}{2\sqrt{1-x^2}+3}, \quad (91)$$

$$\frac{3x}{2+\sqrt{1-x^2}} < \frac{6\left(\sqrt{x+1}-\sqrt{1-x}\right)}{4+\sqrt{x+1}+\sqrt{1-x}} < \arcsin x < \frac{\left(1+2\sqrt{2}\right)\pi}{12}\frac{6\left(\sqrt{x+1}-\sqrt{1-x}\right)}{4+\sqrt{x+1}+\sqrt{1-x}} < \frac{\pi x}{2+\sqrt{1-x^2}}, \quad (92)$$

$$\frac{\pi x}{4}\frac{\sqrt{1-x^2}+2}{2\sqrt{1-x^2}+1} < \frac{\left(\sqrt{2}+3\right)\pi}{14}\frac{x+2\left(\sqrt{x+1}-\sqrt{1-x}\right)}{1+\sqrt{x+1}+\sqrt{1-x}} < \arcsin x < \frac{x+2\left(\sqrt{x+1}-\sqrt{1-x}\right)}{1+\sqrt{x+1}+\sqrt{1-x}} < x\frac{\sqrt{1-x^2}+2}{2\sqrt{1-x^2}+1}. \quad (93)$$

*Remark 24.* Inequalities (92) are due to Zhu [23].

*4.2. Inequalities for Certain Means.* For $a, b > 0$ with $a \neq b$, the first and second Seiffert means [24, 25]; Nueman-Sándor means [26] are defined by

$$P = P(a,b) = \frac{a-b}{2\arcsin((a-b)/(a+b))},$$
$$T = T(a,b) = \frac{a-b}{2\arctan((a-b)/(a+b))}, \quad (94)$$
$$\mathrm{NS} = \mathrm{NS}(a,b) = \frac{a-b}{2\operatorname{arcsinh}((a-b)/(a+b))},$$

respectively. More new means can be found in [27]. We also denote the logarithmic mean, arithmetic mean, geometric mean, and quadratic mean of $a$ and $b$ by $L$, $A$, $G$, and $Q$. There has been some inequalities for these means; we quote [7, 26–36]. Now we establish some new ones involving these means.

Let $x = \arcsin((b-a)/(a+b)), \arctan((b-a)/(a+b))$. Then $(\sin x)/x = P/A$, $\cos x = G/A$; $(\sin x)/x = T/Q$, $\cos x = A/Q$. And then Theorems 7, 10, 13, 14, 16 and 17 can be stated as equivalent ones involving means $P$, $A$, $G$, and $T$, $Q$. For example, from Theorems 7 and 17 we have the following.

**Proposition 25.** *For $a,b > 0$ with $a \neq b$, both the two-side inequalities*

$$\frac{2pA+(p+3)G}{(3p+1)A+2G}A < P < A\frac{2qA+(q+3)G}{(3q+1)A+2G},$$
$$\frac{2pQ+(p+3)A}{(3p+1)Q+2A}Q < T < Q\frac{2qQ+(q+3)A}{(3q+1)Q+2A} \quad (95)$$

*hold if and only if $p \in [0,p_0]$ and $q \in (-\infty,-1] \cup [9,\infty)$, where $p_0 = (\pi-3)^{-1} \approx 7.0625$.*

Making changes of variables $x = \operatorname{arctanh}((b-a)/(a+b))$, $\operatorname{arcsinh}((b-a)/(a+b))$ yield $(\sinh x)/x = L/G$, $\cosh x = A/G$; $(\sinh x)/x = \mathrm{NS}/A$, $\cosh x = Q/A$, respectively. And then, Theorem 18 can be equivalently written as follows.

**Proposition 26.** *For $a, b > 0$ with $a \neq b$, both the inequalities*

$$\frac{2G+(1+3p)A}{(3+p)G+2pA}G < L,$$
$$\frac{2A+(1+3p)Q}{(3+p)A+2pQ}A < \mathrm{NS} \quad (96)$$

*hold if and only if $p \in (-\infty,-1] \cup [1/9,\infty)$. They are reversed if and only if $p = 0$.*

*4.3. The Estimate for the Sine Integral.* For the estimations for the sine integral defined by

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}dt, \quad (97)$$

there has been some results (see [37–39]). By our results we can obtain many estimates for $\mathrm{Si}(x)$. Here we give a simpler but more accurate one.

**Proposition 27.** *For $x \in (0, \pi/2]$, we have*

$$\frac{4\sqrt{2}-2}{7\pi}\left(x+\sin x+8\sin\frac{x}{2}\right) < \mathrm{Si}(x) < \frac{1}{6}\left(x+\sin x+8\sin\frac{x}{2}\right). \quad (98)$$

*Proof.* By (63) we see that the inequalities

$$\frac{4\left(2\sqrt{2}-1\right)}{7}\frac{\cos^2(t/2)+2\cos(t/2)}{\pi} < \frac{\sin t}{t} < \frac{\cos^2(t/2)+2\cos(t/2)}{3} \quad (99)$$

hold for $t \in [0, \pi/2]$. Integrating both sides over $[0, x]$ and simple calculation yield (98). □

*Remark 28.* By (98) we have

$$1.3682 \approx \frac{2\sqrt{2}-1}{7\pi}\left(\pi+8\sqrt{2}+2\right) < \int_0^{\pi/2}\frac{\sin t}{t}dt < \frac{1}{12}\left(\pi+8\sqrt{2}+2\right) \approx 1.3713. \quad (100)$$

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## References

[1] D. S. Mitrinović, *Analytic Inequalities*, vol. 165, Springer, New York, NY, USA, 1970.

[2] F. Qi, D.-W. Niu, and B.-N. Guo, "Refinements, generalizations, and applications of Jordan's inequality and related problems," *Journal of Inequalities and Applications*, vol. 2009, Article ID 271923, 52 pages, 2009.

[3] C.-P. Chen and W.-S. Cheung, "Sharp Cusa and Becker-Stark inequalities," *Journal of Inequalities and Applications*, vol. 2011, article 136, 6 pages, 2011.

[4] K. S. K. Iyengar, B. S. M. Rao, and T. S. Nanjundiah, "Some trigonometrical inequalities," *Half-Yearly Journal of Mysore University B*, vol. 6, pp. 1–12, 1945.

[5] C. Mortici, "The natural approach of Wilker-Cusa-Huygens inequalities," *Mathematical Inequalities & Applications*, vol. 14, no. 3, pp. 535–541, 2011.

[6] E. Neuman and J. Sándor, "On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker, and Huygens inequalities," *Mathematical Inequalities & Applications*, vol. 13, no. 4, pp. 715–723, 2010.

[7] E. Neuman, "Inequalities for the Schwab-Borchardt mean and their applications," *Journal of Mathematical Inequalities*, vol. 5, no. 4, pp. 601–609, 2011.

[8] J. Sándor and M. Bencze, "On Huygens' trigonometric inequality," *RGMIA Research Report Collection*, vol. 8, no. 3, article 14, 2005.

[9] S.-H. Wu and Á. Baricz, "Generalizations of Mitrinović, Adamović and Lazarević's inequalities and their applications," *Publicationes Mathematicae Debrecen*, vol. 75, no. 3-4, pp. 447–458, 2009.

[10] Z.-H. Yang, "Refinements of Mitrinović-Cusa inequality," http://arxiv.org/abs/1206.4911.

[11] Z.-H. Yang, "Renements of a two-sided inequality for trigonometric functions," *Journal of Mathematical Inequalities*, vol. 7, no. 4, pp. 601–615, 2013.

[12] Z.-H. Yang, "New sharp Jordan type inequalities and their applications," *Gulf Journal of Mathematics*, vol. 2, no. 1, pp. 1–10, 2014.

[13] L. Zhu, "Some new wilker-type inequalities for circular and hyperbolic functions," *Abstract and Applied Analysis*, vol. 2009, Article ID 485842, 9 pages, 2009.

[14] L. Zhu, "A source of inequalities for circular functions," *Computers & Mathematics with Applications*, vol. 58, no. 10, pp. 1998–2004, 2009.

[15] C. Huygens, *Oeuvres Completes 1888–1940*, Sociéte Hollondaise des Science, Haga, Sweden.

[16] A. Oppenheim, "Problems for solutions E1277," *The American Mathematical Monthly*, vol. 64, no. 7, pp. 504–505, 1957.

[17] W. B. Carver, "Extreme parameters in an inequality," *The American Mathematical Monthly*, vol. 65, no. 3, pp. 206–209, 1958.

[18] Y.-F. Wu, "Remarks on generalizations of Jordan and Kober's inequalities," *Studies in College Mathematics*, vol. 11, no. 4, pp. 38–39, 42, 2008 (Chinese).

[19] W. D. Jiang, "On Shafer-Fink inequality and Carlson inequality," *The College Mathematics Journal*, vol. 23, no. 4, pp. 152–154, 2007 (Chinese).

[20] M. Li and D. He, "Better estimations for upper and lower bounds of sin $x/x$ and its application," *Journal of Beijing Union University*, vol. 24, no. 2, pp. 47–48, 2010 (Chinese).

[21] R. E. Shafer, "An inequality for the inverse tangent E1867," *The American Mathematical Monthly*, vol. 74, no. 6, pp. 726–727, 1967.

[22] A. M. Fink, "Two inequalities," *Univerzitet u Beogradu. Publikacije Elektrotehnickog Fakulteta. Serija Matematika*, vol. 6, pp. 48–49, 1995.

[23] L. Zhu, "On Shafer-Fink inequalities," *Mathematical Inequalities & Applications*, vol. 8, no. 4, pp. 571–574, 2005.

[24] H.-J. Seiffert, "Werte zwischen dem geometrischen und dem arithmetischen Mittel zweier Zahlen," *Elemente der Mathematik*, vol. 42, no. 4, pp. 105–107, 1987.

[25] H. -J. Seiffert, "Aufgabe $\beta$ 16," *Die Wurzel*, vol. 29, pp. 221–222, 1995.

[26] E. Neuman and J. Sándor, "On the Schwab-Borchardt mean," *Mathematica Pannonica*, vol. 14, no. 2, pp. 253–266, 2003.

[27] Z.-H. Yang, "Three families of two-parameter means constructed by trigonometric functions," *Journal of Inequalities and Applications*, vol. 2013, article 541, 27 pages, 2013.

[28] Y.-M. Chu, Y.-F. Qiu, M.-K. Wang, and G.-D. Wang, "The optimal convex combination bounds of arithmetic and harmonic means for the Seiffert's mean," *Journal of Inequalities and Applications*, vol. 2010, Article ID 436457, 7 pages, 2010.

[29] Y.-M. Chu, M.-K. Wang, and Y.-F. Qiu, "An optimal double inequality between power-type Heron and Seiffert means," *Journal of Inequalities and Applications*, vol. 2010, Article ID 146945, 11 pages, 2010.

[30] Y.-M. Chu, M.-K. Wang, and W.-M. Gong, "Two sharp double inequalities for Seiffert mean," *Journal of Inequalities and Applications*, vol. 2011, article 44, 7 pages, 2011.

[31] Y.-M. Chu, B.-Y. Long, W.-M. Gong, and Y.-Q. Song, "Sharp bounds for Seiffert and Neuman-Sándor means in terms of generalized logarithmic means," *Journal of Inequalities and Applications*, vol. 2013, article 10, 13 pages, 2013.

[32] I. Costin and G. Toader, "A nice separation of some Seiffert-type means by power means," *International Journal of Mathematics and Mathematical Sciences*, vol. 2012, Article ID 430692, 6 pages, 2012.

[33] A. Witkowski, "Interpolations of Schwab-Borchardt mean," *Mathematical Inequalities & Applications*, vol. 16, no. 1, pp. 193–206, 2013.

[34] Z.-H. Yang, "Sharp bounds for the second Seiffert mean in terms of powermeans," http://arxiv.org/abs/1206.5494.

[35] Z.-H. Yang, "The monotonicity results and sharp inequalities for some power-type means of two arguments," http://arxiv.org/abs/1210.6478v1.

[36] Z.-H. Yang, "Estimates for Neuman-Sándor mean by power means and their relative errors," *Journal of Mathematical Inequalities*, vol. 7, no. 4, pp. 711–726, 2013.

[37] F. Qi, "Extensions and sharpenings of Jordan's and Kober's inequality," *Journal of Mathematics for Technology*, vol. 12, no. 4, pp. 98–102, 1996 (Chinese).

[38] S. Wu and L. Debnath, "A new generalized and sharp version of Jordan's inequality and its applications to the improvement of the Yang Le inequality," *Applied Mathematics Letters*, vol. 19, no. 12, pp. 1378–1384, 2006.

[39] S.-H. Wu, "Sharpness and generalization of Jordan's inequality and its application," *Taiwanese Journal of Mathematics*, vol. 12, no. 2, pp. 325–336, 2008.