Research Article

Generalized Common Fixed Point Results with Applications

Marwan Amin Kutbi,[1] Muhammad Arshad,[2] Jamshaid Ahmad,[3] and Akbar Azam[3]

[1] *Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia*
[2] *Department of Mathematics & Statistics, International Islamic University H-10, Islamabad, Pakistan*
[3] *Department of Mathematics, COMSATS Institute of Information Technology, Chak Shahzad, Islamabad 44000, Pakistan*

Correspondence should be addressed to Muhammad Arshad; marshad_zia@yahoo.com

We obtained some generalized common fixed point results in the context of complex valued metric spaces. Moreover, we proved an existence theorem for the common solution for two Urysohn integral equations. Examples are presented to support our results.

1. Introduction and Preliminaries

Since the appearance of the Banach contraction mapping principle, a number of papers were dedicated to the improvement and generalization of that result. Most of these deal with the generalizations of the contractive condition in metric spaces.

Gähler [1] generalized the idea of metric space and introduced a 2-metric space which was followed by a number of papers dealing with this generalized space. Plenty of material is also available in other generalized metric spaces, such as, rectangular metric spaces, semimetric spaces, pseudometric spaces, probabilistic metric spaces, fuzzy metric spaces, quasimetric spaces, quasisemi metric spaces, D-metric spaces, G-metric space, partial metric space, and cone metric spaces (see [2–14]). Azam et al. [15] improved the Banach contraction principle by generalizing it in complex valued metric space involving rational inequity which could not be handled in cone metric spaces [3, 5, 11, 15] due to limitations regarding product and quotient. Rouzkard and Imdad [16] extended the work of Azam et al. [15]. Sintunavarat and Kumam [17] obtained common fixed point results by replacing constant of contractive condition to control functions. Recently, Klin-eam and Suanoom [12] extend the concept of complex valued metric spaces and generalized the results of Azam et al. [15] and Rouzkard and Imdad [16]. In this paper we continue the study of complex valued metric spaces and established some fixed point results for mappings satisfying a rational inequality. The idea of complex valued metric spaces can be exploited to define complex valued normed spaces and complex valued Hilbert spaces and then it will bring wonderful research activities in nonlinear analysis.

In this paper we continue our investigations initiated by Azam et al. [15] and prove a common fixed point result for two mappings and applied it to get the coincidence and common fixed points of three and four mappings.

We begin with listing some notations, definitions, and basic facts on these topics that we will need to convey our theorems. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$\text{iff } z_1 \precsim z_2, \quad \text{Re}(z_1) \leqslant \text{Re}(z_2), \quad \text{Im}(z_1) \leqslant \text{Im}(z_2). \quad (1)$$

It follows that

$$z_1 \precsim z_2 \qquad (2)$$

if one of the following conditions is satisfied:

$$\begin{aligned}
&\text{(i) Re}(z_1) = \text{Re}(z_2), \qquad \text{Im}(z_1) < \text{Im}(z_2), \\
&\text{(ii) Re}(z_1) < \text{Re}(z_2), \qquad \text{Im}(z_1) = \text{Im}(z_2), \\
&\text{(iii) Re}(z_1) < \text{Re}(z_2), \qquad \text{Im}(z_1) < \text{Im}(z_2), \\
&\text{(iv) Re}(z_1) = \text{Re}(z_2), \qquad \text{Im}(z_1) = \text{Im}(z_2).
\end{aligned} \quad (3)$$

In particular, we will write $z_1 \precsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied. Note that

$$0 \preceq z_1 \precsim z_2 \implies |z_1| < |z_2|,$$

$$z_1 \preceq z_2, \qquad z_2 \prec z_3 \implies z_1 \prec z_3. \tag{4}$$

Definition 1. Let X be a nonempty set. Suppose that the self-mapping $d : X \times X \to \mathbb{C}$ satisfies:

(1) $0 \preceq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, y) \preceq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X, and (X, d) is called a complex valued metric space. A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that

$$B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A. \tag{5}$$

A point $x \in X$ is called a limit point of A whenever for every $0 \prec r \in \mathbb{C}$,

$$B(x, r) \cap (A \setminus \{x\}) \neq \phi. \tag{6}$$

A is called open whenever each element of A is an interior point of A. Moreover, a subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B. The family

$$F = \{B(x, r) : x \in X, 0 \prec r\} \tag{7}$$

is a subbasis for a Hausdorff topology τ on X.

Let x_n be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x, and x is the limit point of $\{x_n\}$. We denote this by $\lim_{n \to \infty} x_n = x$, or $x_n \to x$, as $n \to \infty$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$, then $\{x_n\}$ is called a Cauchy sequence in (X, d). If every Cauchy sequence is convergent in (X, d), then (X, d) is called a complete complex valued metric space. Let X be a nonempty set and $T, f : X \to X$. The mappings T, f are said to be weakly compatible if they commute at their coincidence point (i.e., $Tfx = fTx$ whenever $Tx = fx$). A point $y \in X$ is called point of coincidence of T and f if there exists a point $x \in X$ such that $y = Tx = fx$. We require the following lemmas.

Lemma 2 (see [15]). *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 3 (see [15]). *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.*

Definition 4 (see [18]). Two families of self-mappings $\{T_i\}_1^m$ and $\{S_i\}_1^n$ are said to be pairwise commuting if:

(1) $T_i T_j = T_j T_i$, $i, j \in \{1, 2, \ldots, m\}$;

(2) $S_k S_l = S_l S_k$, $k, l \in \{1, 2, \ldots, n\}$;

(3) $T_i S_k = S_k T_i$, $i \in \{1, 2, \ldots, m\}$, $k \in \{1, 2, \ldots, n\}$.

Lemma 5 (see [19]). *Let X be a nonempty set and $f : X \to X$ a function. Then there exists a subset $E \subset X$ such that $fE = fX$ and $f : E \to X$ is one to one.*

Lemma 6 (see [20]). *Let X be a nonempty set and the mappings $S, T, f : X \to X$ have a unique point of coincidence v in X. If (S, f) and (T, f) are weakly compatible, then fv is a unique common fixed point of S, T, f.*

2. Main Results

Theorem 7. *Let (X, d) be a complete complex valued metric space and $0 \leq h < 1$. If the self-mappings $S, T : X \to X$ satisfy*

$$d(Sx, Ty) \preceq hL(x, y) \tag{8}$$

for all $x, y \in X$, where

$$L(x, y)$$
$$\in \left\{ d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Sx) d(y, Ty)}{1 + d(x, y)} \right\} \tag{9}$$

then S and T have a unique common fixed point.

Proof. We will first show that fixed point of one map is a fixed point of the other. Suppose that $p = Tp$. Then from (8)

$$d(Sp, p) = d(Sp, Tp) \preceq hL(p, p). \tag{10}$$

Case 1

$$d(Sp, p) \preceq hd(p, p) = 0, \quad p = Sp. \tag{11}$$

Case 2

$$d(Sp, p) \preceq hd(p, Sp), \tag{12}$$

which yields that $p = Sp$.

Case 3

$$d(Sp, p) \preceq hd(p, Tp) = 0, \quad p = Sp. \tag{13}$$

Case 4

$$d(Sp, p) \preceq h \left[\frac{d(p, Sp) d(p, Tp)}{1 + d(p, p)} \right], \tag{14}$$

which implies that $d(Sp, p) \preceq 0$, and hence $p = Sp$. In a similar manner it can be shown that any fixed point of S is also the fixed point of T. Let $x_0 \in X$ and define

$$x_{2n+1} = Sx_{2n}$$
$$x_{2n+2} = Tx_{2n+1}, \quad n \geq 0. \tag{15}$$

We will assume that $x_n \neq x_{n+1}$ for each n. Otherwise, there exists an n such that $x_{2n} = x_{2n+1}$. Then $x_{2n} = Sx_{2n}$ and x_{2n} is a fixed point of S, hence a fixed point of T. Similarly, if $x_{2n+1} = x_{2n+2}$ for some n, then x_{2n+1} is common fixed point of T and hence of S. From (8)

$$d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1}) \leq hL(x_{2n}, x_{2n+1}). \quad (16)$$

Case 1

$$\left| d(x_{2n+1}, x_{2n+2}) \right| = \left| d(Sx_{2n}, Tx_{2n+1}) \right| \leq h \left| d(x_{2n}, x_{2n+1}) \right|. \quad (17)$$

Case 2

$$\begin{aligned} \left| d(x_{2n+1}, x_{2n+2}) \right| &= \left| d(Sx_{2n}, Tx_{2n+1}) \right| \\ &\leq h \left| d(x_{2n}, Sx_{2n}) \right| \\ &= h \left| d(x_{2n}, x_{2n+1}) \right|. \end{aligned} \quad (18)$$

Case 3

$$\begin{aligned} \left| d(x_{2n+1}, x_{2n+2}) \right| &= \left| d(Sx_{2n}, Tx_{2n+1}) \right| \\ &\leq h \left| d(x_{2n+1}, Tx_{2n+1}) \right| \\ &= h \left| d(x_{2n+1}, x_{2n+2}) \right|, \end{aligned} \quad (19)$$

which implies that

$$x_{2n+1} = x_{2n+2}, \quad (20)$$

a contradiction to our assumption.

Case 4

$$\begin{aligned} \left| d(x_{2n+1}, x_{2n+2}) \right| &= \left| d(Sx_{2n}, Tx_{2n+1}) \right| \\ &\leq h \left| \frac{d(x_{2n}, Sx_{2n}) \, d(x_{2n+1}, Tx_{2n+1})}{1 + d(x_{2n}, x_{2n+1})} \right| \\ &\leq h \left| \frac{d(x_{2n}, x_{2n+1})}{1 + d(x_{2n}, x_{2n+1})} \right| \left| d(x_{2n+1}, x_{2n+2}) \right| \\ &\leq h \left| d(x_{2n+1}, x_{2n+2}) \right|. \end{aligned} \quad (21)$$

That is

$$x_{2n+1} = x_{2n+2}, \quad (22)$$

a contradiction to our assumption.

Thus, $|d(x_{2n+1}, x_{2n+2})| \leq h|d(x_{2n}, x_{2n+1})|$. Similarly, one can show that $|d(x_{2n+2}, x_{2n+3})| \leq h|d(x_{2n+1}, x_{2n+2})|$. It follows that, for all n,

$$\begin{aligned} \left| d(x_n, x_{n+1}) \right| &\leq h \left| d(x_{n-1}, x_n) \right| \\ &\leq h^2 \left| d(x_{n-2}, x_{n-1}) \right| \leq \cdots \leq h^n \left| d(x_0, x_1) \right|. \end{aligned} \quad (23)$$

Now for any $m > n$,

$$\begin{aligned} \left| d(x_m, x_n) \right| &\leq \left| d(x_n, x_{n+1}) \right| + \left| d(x_{n+1}, x_{n+2}) \right| \\ &\quad + \cdots + \left| d(x_{m-1}, x_m) \right| \\ &\leq \left[h^n + h^{n+1} + \cdots + h^{m-1} \right] \left| d(x_0, x_1) \right| \\ &\leq \left[\frac{h^n}{1-h} \right] \left| d(x_0, x_1) \right| \end{aligned} \quad (24)$$

and so

$$\left| d(x_m, x_n) \right| \leq \frac{h^n}{1-h} \left| d(x_0, x_1) \right| \longrightarrow 0, \quad \text{as } m, n \longrightarrow \infty. \quad (25)$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $u \in X$ such that $x_n \to u$. It follows that $u = Su$; otherwise $d(u, Su) = z \succ 0$ and we would then have

$$\begin{aligned} z &\precsim d(u, x_{2n+2}) + d(x_{2n+2}, Su) \\ &\precsim d(u, x_{2n+2}) + d(Tx_{2n+1}, Su) \\ &\precsim d(u, x_{2n+2}) + hL(u, x_{2n+1}). \end{aligned} \quad (26)$$

Case 1

$$|z| \leq \left| d(u, x_{2n+2}) \right| + h \left| d(u, x_{2n+1}) \right|. \quad (27)$$

That is, $|z| \leq 0$, a contradiction and hence $u = Su$.

Case 2

$$|z| \leq \left| d(u, x_{2n+2}) \right| + h \left| d(u, Su) \right|. \quad (28)$$

That is, $|z| \leq 0$, a contradiction and hence $u = Su$.

Case 3

$$\begin{aligned} |z| &\leq \left| d(u, x_{2n+2}) \right| + h \left| d(x_{2n+1}, Tx_{2n+1}) \right| \\ &\leq \left| d(u, x_{2n+2}) \right| + h \left| d(x_{2n+1}, x_{2n+2}) \right| \\ &\leq \left| d(u, x_{2n+2}) \right| + h^{2n+2} \left| d(x_0, x_1) \right|. \end{aligned} \quad (29)$$

This in turn gives us $|z| \leq 0$, a contradiction and hence $u = Su$.

Case 4

$$\begin{aligned} |z| &\leq \left| d(u, x_{2n+2}) \right| + h \left| \frac{d(u, Su) \, d(x_{2n+1}, Tx_{2n+1})}{1 + d(u, x_{2n+1})} \right| \\ &\leq d(u, x_{2n+2}) + h \frac{\left| d(u, Su) \right| \left| d(x_{2n+1}, x_{2n+2}) \right|}{\left| 1 + d(u, x_{2n+1}) \right|} \\ &\leq \left| d(u, x_{2n+2}) \right| + h^{2n+2} \frac{\left| d(u, Su) \right| \left| d(x_0, x_1) \right|}{\left| 1 + d(u, x_{2n+1}) \right|}. \end{aligned} \quad (30)$$

That is, $|z| \leq 0$ and hence $u = Su$. It follows similarly that $u = Tu$. We now show that S and T have unique common fixed

point. For this, assume that u^* in X is another common fixed point of S and T. Then $d(u, u^*) = d(Su, Tu^*) \precsim hL(u, u^*)$.

Case 1

$$d(u, u^*) \precsim hd(u, u^*). \tag{31}$$

Case 2

$$d(u, u^*) \precsim hd(u, Su) \precsim hd(u, u) = 0. \tag{32}$$

This gives us $u = u^*$.

Case 3

$$d(u, u^*) \precsim hd(u^*, Tu^*) = hd(u^*, u^*) = 0. \tag{33}$$

Case 4

$$d(u, u^*) \precsim \frac{hd(u, Su) \, d(u^*, Tu^*)}{1 + d(u, u^*)} = 0. \tag{34}$$

Hence, in all cases $u^* = u$. This completes the proof of the theorem. □

Corollary 8 (see [15]). *Let (X, d) be a complete complex valued metric space and let $S, T : X \rightarrow X$ and $0 < h < 1$. If the self-mappings S, T satisfy*

$$d(Sx, Ty) \precsim hL(x, y) \tag{35}$$

for all $x, y \in X$, where

$$L(x, y) \in \left\{ d(x, y), \frac{d(x, Sx) \, d(y, Ty)}{1 + d(x, y)} \right\}, \tag{36}$$

then S and T have a unique common fixed point.

Corollary 9. *Let (X, d) be a complete complex valued metric space and $0 \le h < 1$. If the self-mapping $T : X \rightarrow X$ satisfies*

$$d(Tx, Ty) \precsim hL(x, y) \tag{37}$$

for all $x, y \in X$, where

$$L(x, y)$$
$$\in \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx) \, d(y, Ty)}{1 + d(x, y)} \right\} \tag{38}$$

then T has a unique fixed point.

Corollary 10 (see [15]). *Let (X, d) be a complete complex valued metric space and let $T : X \rightarrow X$ and $0 \le h < 1$. If the self-mapping T satisfies*

$$d(Tx, Ty) \precsim hL(x, y) \tag{39}$$

for all $x, y \in X$, where

$$L(x, y) \in \left\{ d(x, y), \frac{d(x, Tx) \, d(y, Ty)}{1 + d(x, y)} \right\}, \tag{40}$$

then T has a unique fixed point.

As an application of Theorem 7, we prove the following theorem for two finite families of mappings.

Theorem 11. *If $\{T_i\}_1^m$ and $\{S_i\}_1^n$ are two finite pairwise commuting finite families of self-mapping defined on a complete complex valued metric space (X, d) such that the mappings S and T (with $T = T_1, T_2, \dots, T_m$ and $S = S_1, S_2, \dots, S_n$) satisfy the contractive condition (8), then the component maps of the two families $\{T_i\}_1^m$ and $\{S_i\}_1^n$ have a unique common fixed point.*

Proof. From theorem we can say that the mappings T and S have a unique common fixed point z; that is, $Tz = Sz = z$. Now our requirement is to show that z is a common fixed point of all the component mappings of both families. In view of pairwise commutativity of the families $\{T_i\}_1^m$ and $\{S_i\}_1^n$, (for every $1 \le k \le m$) we can write $T_k z = T_k Tz = TT_k z$ and $T_k z = T_k Sz = ST_k z$ which show that $T_k z$ (for every k) is also a common fixed point of T and S. By using the uniqueness of common fixed point, we can write $T_k z = z$ (for every k) which shows that z is a common fixed point of the family $\{T_i\}_1^m$. Using the same argument one can also show that (for every $1 \le k \le n$) $S_k z = z$. Thus component maps of the two families $\{T_i\}_1^m$ and $\{S_i\}_1^n$ have a unique common fixed point. □

By setting $T_1 = T_2 = \cdots = T_m = F$ and $S_1 = S_2 = \cdots = S_n = G$, in Theorem 11, we get the following corollary.

Corollary 12. *If F and G are two commuting self-mappings defined on a complete complex valued metric space (X, d) satisfying the condition*

$$d(F^m x, G^n y) \precsim hL(x, y) \tag{41}$$

for all $x, y \in X$ and $0 \le h < 1$, where

$$L(x, y) \in \left\{ d(x, y), d(x, F^m x), d(y, G^n y), \right.$$
$$\left. \frac{d(x, F^m x) \, d(y, G^n y)}{1 + d(x, y)} \right\}, \tag{42}$$

then F and G have a unique common fixed point.

Corollary 13. *Let (X, d) be a complete complex valued metric space and let $T : X \rightarrow X$ be a self-mapping satisfying*

$$d(T^n x, T^n y) \precsim hL(x, y) \tag{43}$$

for all $x, y \in X$ and $0 \le h < 1$, where

$$L(x, y) \in \left\{ d(x, y), d(x, T^m x), d(y, T^n y), \right.$$
$$\left. \frac{d(x, T^n x) \, d(y, T^n y)}{1 + d(x, y)} \right\}. \tag{44}$$

Then T has a unique fixed point.

Corollary 14 (see [15]). *Let (X, d) be a complete complex valued metric space and $T : X \rightarrow X$ and $0 \le h < 1$. The self-mapping T satisfies*

$$d(T^n x, T^n y) \precsim hL(x, y) \tag{45}$$

for all $x, y \in X$, where

$$L(x, y) \in \left\{ d(x, y), \frac{d(x, T^n x) d(y, T^n y)}{1 + d(x, y)} \right\}. \tag{46}$$

Then T has a unique fixed point.

Our next example exhibits the superiority of Corollary 13 over Corollary 9.

Example 15. Let $X_1 = \{z \in \mathbb{C} : 0 \le \mathrm{Re}\, z \le 1, \mathrm{Im}\, z = 0\}$ and $X_2 = \{z \in \mathbb{C} : 0 \le \mathrm{Im}\, z \le 1, \mathrm{Re}\, z = 0\}$ and let $X = X_1 \cup X_2$. Then with $z = x + iy$, set $S = T$ and define $T : X \to X$ as follows:

$$T(x, y) = \begin{cases} (0, 0) & \text{if } x, y \in Q \\ (1, 0) & \text{if } x \in Q^c, \ y \in Q \\ (0, 1) & \text{if } x \in Q, \ y \in Q^c \\ (1, 1) & \text{if } x, y \in Q^c. \end{cases} \tag{47}$$

Consider a complex valued metric $d : X \times X \to \mathbb{C}$ as follows:

$$d(z_1, z_2) = \begin{cases} \dfrac{2i}{3} |x_1 - x_2|, & \text{if } z_1, z_2 \in X_1 \\[2mm] \dfrac{i}{3} |y_1 - y_2|, & \text{if } z_1, z_2 \in X_2 \\[2mm] i\left(\dfrac{2}{3} x_1 + \dfrac{1}{3} y_2 \right), & \text{if } z_1 \in X_1, \ z_2 \in X_2 \\[2mm] i\left(\dfrac{1}{3} y_1 + \dfrac{2}{3} x_2 \right) & \text{if } z_1 \in X_2, \ z_2 \in X_1, \end{cases} \tag{48}$$

where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2 \in X$. Then (X, d) is a complete complex valued metric space. By a routine calculation, one can verify that the map T^2 satisfies condition (43) with $\lambda = (1/3)$(say). It is interesting to notice that this example cannot be covered by Corollary 9 as $z_1 = (1, 0)$, $z_2 = (1/2, 0) \in X$ implies

$$\frac{2i}{3} = d(Tz_1, Tz_2) \le d(z_1, z_2) = \frac{i}{3} \tag{49}$$

a contradiction for every choice of λ which amounts to say that condition (37) is not satisfied. Notice that the point $0 \in X$ remains fixed under T and T^2 and is indeed unique.

3. Application

By providing the following result, we establish an existence theorem for the common solution for two Urysohn integral equations.

Theorem 16. *Let $X = C([a, b], \mathbb{R}^n)$, $a > 0$, and $d : X \times X \to \mathbb{C}$ is defined as follows:*

$$d(x, y) = \max_{t \in [a,b]} \|x(t) - y(t)\|_\infty \sqrt{1 + a^2} e^{i \tan^{-1} a}. \tag{50}$$

Consider the Urysohn integral equations

$$x(t) = \int_a^b K_1(t, s, x(s)) \, ds + g(t), \tag{α}$$

$$x(t) = \int_a^b K_2(t, s, x(s)) \, ds + h(t), \tag{β}$$

where $t \in [a, b] \subset \mathbb{R}$, $x, g, h \in X$.

Suppose that $K_1, K_2 : [a, b] \times [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ are such that $F_x, G_x \in X$ for each $x \in X$, where,

$$F_x(t) = \int_a^b K_1(t, s, x(s)) \, ds,$$

$$G_x(t) = \int_a^b K_2(t, s, x(s)) \, ds, \quad \forall t \in [a, b]. \tag{51}$$

If there exists $0 \le h < 1$ such that for every $x, y \in X$

$$\|F_x(t) - G_y(t) + g(t) - h(t)\|_\infty \sqrt{1 + a^2} e^{i \tan^{-1} a}$$
$$\lesssim hL(x, y)(t), \tag{52}$$

where

$$L(x, y)(t)$$
$$\in \{A(x, y)(t), B(x, y)(t), C(x, y)(t), D(x, y)(t)\},$$

$$A(x, y)(t) = \|x(t) - y(t)\|_\infty \sqrt{1 + a^2} e^{i \tan^{-1} a},$$

$$B(x, y)(t) = \|F_x(t) + g(t) - x(t)\|_\infty \sqrt{1 + a^2} e^{i \tan^{-1} a},$$

$$C(x, y)(t) = \|G_y(t) + h(t) - y(t)\|_\infty \sqrt{1 + a^2} e^{i \tan^{-1} a},$$

$$D(x, y)(t)$$
$$= \frac{\|F_x(t) + g(t) - x(t)\|_\infty \|G_y(t) + h(t) - y(t)\|_\infty}{1 + \max_{t \in [a,b]} A(x, y)(t)}$$
$$\times \sqrt{1 + a^2} e^{i \tan^{-1} a}, \tag{53}$$

then the system of integral equations (α) and (β) has a unique common solution.

Proof. Define $S, T : X \to X$ by

$$Sx = F_x + g, \qquad Tx = G_x + h. \tag{54}$$

Then

$$d\left(Sx, Ty\right) = \max_{t\in[a,b]}\left\|F_x\left(t\right) - G_y\left(t\right) + g\left(t\right) - h\left(t\right)\right\|_\infty$$

$$\times \sqrt{1 + a^2}\, e^{i\tan^{-1}a},$$

$$d\left(x, y\right) = \max_{t\in[a,b]} A\left(x, y\right)\left(t\right),$$

$$d\left(x, Sx\right) = \max_{t\in[a,b]} B\left(x, y\right)\left(t\right), \tag{55}$$

$$d\left(y, Ty\right) = \max_{t\in[a,b]} C\left(x, y\right)\left(t\right),$$

$$\frac{d\left(x, Sx\right) d\left(y, Ty\right)}{1 + d\left(x, y\right)} = \max_{t\in[a,b]} D\left(x, y\right)\left(t\right).$$

It is easily seen that $d(Sx, Ty) \precsim hL(x, y)$, where

$$L\left(x, y\right)$$

$$\in \left\{d\left(x, y\right), d\left(x, Sx\right), d\left(y, Ty\right), \frac{d\left(x, Sx\right) d\left(y, Ty\right)}{1 + d\left(x, y\right)}\right\}$$

$$\tag{56}$$

for every $x, y \in X$. By Theorem 7, the Urysohn integral equations (α) and (β) have a unique common solution. $\square$

Remark 17. Now we will apply techniques of [6] to obtain the common fixed points of three and four mappings by using a common fixed point result for two mappings.

Theorem 18. *Let (X, d) be a complete complex valued metric space and $0 \le h < 1$. Let $S, T, f : X \to X$ by the self-mappings such that $SX \cup TX \subset fX$. Assume that the following holds:*

$$d\left(Sx, Ty\right) \precsim hL\left(x, y\right) \tag{57}$$

for all $x, y \in X$, where

$$L\left(x, y\right) \in \left\{d\left(fx, fy\right), d\left(fx, Sx\right), d\left(fy, Ty\right), \right.$$

$$\left. \frac{d\left(fx, Sx\right) d\left(fy, Ty\right)}{1 + d\left(fx, fy\right)}\right\}. \tag{58}$$

If (S, f) and (T, f) are weakly compatible and fX is closed, then $S, T,$ and f have a unique common fixed point in X.

Proof. By Lemma 5, there exists $E \subset X$ such that $fE = fX$ and $f : E \to X$ is one to one. Now define the self-mappings $g, h : fE \to fE$ by $g(fx) = Sx$ and $h(fx) = Tx$, respectively. Since f is one to one on E, then g, h are well defined. Note that

$$d\left(g\left(fx\right), h\left(fx\right)\right) \precsim hL\left(x, y\right), \tag{59}$$

where

$$L\left(x, y\right) \in \left\{d\left(fx, fy\right), d\left(fx, g\left(fx\right)\right), d\left(fy, h\left(fy\right)\right), \right.$$

$$\left. \frac{d\left(fx, g\left(fx\right)\right) d\left(fy, h\left(fy\right)\right)}{1 + d\left(fx, fy\right)}\right\}. \tag{60}$$

By Theorem 7 as fE is complete, we deduce that there exists a unique common fixed point $fz \in fE$ of g and h; that is, $fz = g(fz) = h(fz)$. Thus, z is a coincidence point of S, T, and f. Now we show that S, T, and f have unique point of coincidence. Now let $x \in X$ such that $Sx = Tx = fx$, $fx \ne fz$. Then fx is another common fixed point of g and h, which is a contradiction, which implies that S, T, and f have a unique point of coincidence. Since (S, f) and (T, f) are weakly compatible by Lemma 6, we deduce that fz is a unique common fixed point of S, T, and f. $\square$

Theorem 19. *Let (X, d) be a complete complex valued metric space and $0 \le h < 1$. Let $S, T, f, g : X \to X$ by the self-mappings such that $SX, TX \subset fX = gX$. Assume that the following holds:*

$$d\left(Sx, Ty\right) \precsim hL\left(x, y\right) \tag{61}$$

for all $x, y \in X$, where

$$L\left(x, y\right) \in \left\{d\left(fx, gy\right), d\left(fx, Sx\right), d\left(gy, Ty\right), \right.$$

$$\left. \frac{d\left(fx, Sx\right) d\left(gy, Ty\right)}{1 + d\left(fx, gy\right)}\right\}. \tag{62}$$

If (S, f) and (T, g) are weakly compatible and fX is closed in X, then $S, T, f,$ and g have a unique common fixed point in X.

Proof. By Lemma 5, there exists $E_1, E_2 \subset X$ such that $fE_1 = fX = gX = gE_2$, $f : E_1 \to X, g : E_2 \to X$ are one to one. Now define the mappings $A, B : fE_1 \to fE_1$ by $Af(x) = Sx$ and $Bg(x) = Tx$, respectively. Since f, g are one to one on E_1 and E_2, respectively, then the mappings A, B are well-defined. Now

$$d\left(Sx, Ty\right) = d\left(A\left(fx\right), B\left(gy\right)\right) \precsim hL\left(x, y\right), \tag{63}$$

where

$$L\left(x, y\right)$$

$$\in \left\{d\left(fx, gy\right), d\left(fx, A\left(fx\right)\right), d\left(gy, B\left(gy\right)\right), \right.$$

$$\left. \frac{d\left(fx, A\left(fx\right)\right) d\left(gy, B\left(gy\right)\right)}{1 + d\left(fx, gy\right)}\right\} \tag{64}$$

for all $fx, gy \in fE_1$. By Theorem 7, as fE_1 is complete subspace of X, we deduce that there exists a unique common fixed point $fz \in fE_1$ of A and B; that is, $A(fz) = Bf(z) = fz$.

This implies that $Sz = fz$; let $v \in X$ such that $fz = gv$. We have $B(gv) = gv \Rightarrow Tv = gv$. We show that S and f have a unique point of coincidence. If $Sw = fw$ then fw is a fixed point of A. By the proof of Theorem 7 fw is another common fixed point of A and B which is a contradiction. Hence, S and f have a unique point of coincidence. By Lemma 6, it follows that fz is a unique common fixed point of S and f. Similarly, gv is the unique common fixed point for T and g. This proves that $fz = gv$ is the unique common fixed point for S, T, f, and g. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

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