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Research Article

A New Approach for Generating the TX Hierarchy as well as Its Integrable Couplings

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Tu Guizhang and Xu Baozhi once introduced an isospectral problem by a loop algebra with degree being λ, for which an integrable hierarchy of evolution equations (called the TX hierarchy) was derived under the frame of zero curvature equations. In the paper, we present a loop algebra whose degrees are 2λ and $2\lambda+1$ to simply represent the above isospectral matrix and easily derive the TX hierarchy. Specially, through enlarging the loop algebra with 3 dimensions to 6 dimensions, we generate a new integrable coupling of the TX hierarchy and its corresponding Hamiltonian structure.

1. Introduction

Since the theory on integrable couplings was proposed [1, 2], some integrable couplings and properties were obtained, such as the results in [3–10]. Tu and Xu [11] employed loop algebra which is subalgebra of the loop algebra $\widetilde{A}_1$ with degree being λ to obtain an integrable hierarchy, which is called by us the TX hierarchy, and its corresponding Hamiltonian structure. In the paper, we would like to extend the loop algebra with 3 dimensions into enlarged loop algebra with 6 dimensions so that an integrable coupling of the TX hierarchy can be derived, the Hamiltonian structure of which is also produced by making use of the variational identity [5].

In paper [7], the Lie algebra was once presented as follows:

$$G = \{g_1, \dots, g_6\}, \tag{1}$$

where

$$g_1 = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}, \quad g_2 = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}, \quad g_3 = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix},$$
$$g_4 = \begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix}, \quad g_6 = \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix}, \tag{2}$$

along with commutative relations as follows:

$$\begin{aligned} &[g_1, g_2] = 2g_2, \quad [g_1, g_3] = -2g_3, \quad [g_2, g_3] = g_1, \\ &[g_1, g_4] = 0, \quad [g_1, g_5] = 2g_5, \quad [g_1, g_6] = -2g_6, \\ &[g_2, g_4] = -2g_5, \quad [g_2, g_5] = 0, \quad [g_2, g_6] = g_4, \\ &[g_3, g_4] = 2g_6, \quad [g_3, g_5] = -g_4, \\ &[g_3, g_6] = [g_4, g_5] = [g_4, g_6] = [g_5, g_6] = 0, \end{aligned} \tag{3}$$

where

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{4}$$

form subalgebra of the Lie algebra A_1, denoted by A_1 again; that is, $A_1 = \{h, e, f\}$. The corresponding loop algebra of G was given by

$$\widetilde{G} = \{g_1(n), \dots, g_6(n)\}, \quad g_i(n) = g_i \lambda^n, \quad i = 1, 2, \dots, 6. \tag{5}$$

Through the $\widetilde{G}$ some integrable couplings were obtained, and some exact solutions of the reduced equations were also produced. In the paper, by redefining the degrees of the Lie algebra G, we give the following loop algebra:

$$\overline{G} = \{g_1(n), \dots, g_6(n)\}, \tag{6}$$

where

$$\begin{aligned} &g_1(n) = g_1\lambda^{2n}, \qquad g_2(n) = g_2\lambda^{2n+1}, \\ &g_3(n) = g_3\lambda^{2n+1}, \\ &g_4(n) = g_4\lambda^{2n}, \qquad g_5(n) = g_5\lambda^{2n+1}, \\ &g_6(n) = g_6\lambda^{2n+1}. \end{aligned} \tag{7}$$

The commutative relations read that

$$\begin{aligned} &[g_1(m), g_2(n)] = 2g_2(m+n), \\ &[g_1(m), g_3(n)] = -2g_3(m+n), \\ &[g_2(m), g_3(n)] = g_1(m+n+1), \\ &[g_1(m), g_4(n)] = 0, \\ &[g_1(m), g_5(n)] = 2g_5(m+n), \\ &[g_1(m), g_6(n)] = -2g_6(m+n), \\ &[g_2(m), g_4(n)] = -2g_5(m+n), \\ &[g_2(m), g_5(n)] = 0, \\ &[g_2(m), g_6(n)] = g_4(m+n+1), \\ &[g_3(m), g_4(n)] = 2g_6(m+n), \\ &[g_3(m), g_5(n)] = -g_4(m+n+1), \\ &[g_3(m), g_6(n)] = [g_4(m), g_5(n)] \\ &\quad = [g_4(m), g_6(n)] = [g_5(m), g_6(n)] = 0. \end{aligned} \tag{8}$$

2. The TX Hierarchy and Its Integrable Coupling

In the section, we want to investigate the TX hierarchy and its integrable coupling by employing the loop algebra $\overline{G}$ under the frame of zero curvature equations. Then through the trace identity proposed by Tu [12] and the variational identity [5], we derive the Hamiltonian structure of the TX hierarchy and the Hamiltonian structure of the integrable coupling, respectively.

Obviously, the Lie algebra $A_1 = \{h, e, f\}$ is isomorphic to the subalgebra G_1 of the Lie algebra G, where $G_1 = \{g_1, g_2, g_3\}$, that is,

$$A_1 \cong G_1. \tag{9}$$

The resulting loop algebra $\widetilde{A}_1 = \{h(n), e(n), f(n)\}$ is also isomorphic to $\widetilde{G}_1 = \{g_1(n), g_2(n), g_3(n)\}$, where

$$h(n) = h\lambda^{2n}, \qquad e(n) = e\lambda^{2n+1}, \qquad f(n) = f\lambda^{2n+1}, \tag{10}$$

equipped with

$$\begin{aligned} &[h(m), e(n)] = 2e(m+n), \\ &[h(m), f(n)] = -2f(m+n), \\ &[e(m), f(n)] = h(m+n+1). \end{aligned} \tag{11}$$

Based on the above fact, the isospectral matrix presented in [11] can be written as

$$U = \begin{pmatrix} \lambda^2 + w & \lambda u \\ \lambda v & -\lambda^2 - w \end{pmatrix} = h(1) + ue(0) + vf(0), \tag{12}$$

or

$$U = g_1(1) + ug_2(0) + vg_3(0). \tag{13}$$

Set

$$V = \sum_{m\geq 0} (V_{1m}h(-m) + V_{2m}e(-m) + V_{3m}f(-m)), \tag{14}$$

or

$$V = \sum_{m\geq 0} (V_{1m}g_1(-m) + V_{2m}g_2(-m) + V_{3m}g_3(-m)). \tag{15}$$

It is easy to check that the stationary zero curvature equations

$$\begin{aligned} &\sum_{m\geq 0} (V_{1m,x}h(-m) + V_{2m,x}e(-m) + V_{3m,x}f(-m)) \\ &= \Big[h(1) + ue(0) + vf(0), \\ &\qquad \sum_{m\geq 0} (V_{1m}h(-m) + V_{2m}e(-m) + V_{3m}f(0m))\Big], \\ &\sum_{m\geq 0} (V_{1m,x}g_1(-m) + V_{2m,x}g_2(-m) + V_{3m,x}g_3(-m)) \\ &= \Big[g_1(1) + ug_2(0) + vg_3(0), \\ &\qquad \sum_{m\geq 0} (V_{1m}g_1(-m) + V_{2m}g_2(-m) + V_{3m}g_3(-m))\Big] \end{aligned} \tag{16}$$

have the same solutions for V. That is, starting from (16), we can derive all of the following recursion relations among V_{im} $(i = 1, 2, 3)$:

$$\begin{aligned} &V_{1m,x} = uV_{3,m+1} - vV_{2,m+1}, \\ &V_{2m,x} = 2V_{2,m+1} + 2wV_{2m} - 2uV_{1m}, \\ &V_{3m,x} = -2V_{3,m+1} - 2wV_{3m} + 2vV_{1m}, \end{aligned} \tag{17}$$

which is equivalent to

$$\begin{aligned} &V_{2,m+1} = \frac{1}{2}V_{2m,x} - wV_{2m} + uV_{1m}, \\ &V_{3,m+1} = -\frac{1}{2}V_{3m,x} - wV_{3m} + vV_{1m}, \\ &V_{1m,x} = -\frac{1}{2}uV_{3m,x} - \frac{1}{2}vV_{2m,x} - uwV_{3m} + vwV_{2m}. \end{aligned} \tag{18}$$

Given some initial values $V_{1,0} = \alpha$, $V_{2,0} = V_{3,0} = 0$, (18) admits some explicit solutions as follows:

$$\begin{aligned}
V_{2,1} &= \alpha u, \qquad V_{3,1} = \alpha v, \qquad V_{1,1} = -\frac{\alpha}{2}uv,\\
V_{2,2} &= \frac{\alpha}{2}u_x - \alpha wu - \frac{\alpha}{2}u^2 v,\\
V_{3,2} &= -\frac{\alpha}{2}v_x - \alpha wv - \frac{\alpha}{2}uv^2,\\
V_{1,1} &= \frac{\alpha}{4}(uv_x - u_x v) + \alpha uvw + \frac{3\alpha}{8}u^2v^2,\\
V_{3,2} &= -\frac{\alpha}{2}v_x - \alpha wv - \frac{\alpha}{2}uv^2,\\
V_{1,1} &= \frac{\alpha}{4}(uv_x - u_x v) + \alpha uvw + \frac{3\alpha}{8}u^2v^2,\\
V_{2,3} &= \frac{\alpha}{4}u_{xx} - \alpha wu_x - \frac{\alpha}{2}uw_x - \frac{3\alpha}{4}uu_x v\\
&\quad + \alpha w^2 u + \frac{\alpha}{2}u^2 wv + \frac{3\alpha}{8}u^3v^2,\\
V_{3,3} &= \frac{\alpha}{4}v_{xx} + \frac{\alpha}{2}vw_x + \alpha wv_x + \frac{3\alpha}{4}uvv_x\\
&\quad + \alpha w^2 v + \frac{3\alpha}{2}v^2uw + \frac{3\alpha}{8}u^2v^3, \ldots.
\end{aligned} \tag{19}$$

Denoting

$$V^{(n)} = \sum_{m=0}^{n}\left(V_{1m}h(-m) + V_{2m}e(-m) + V_{3m}f(-m)\right)\lambda^{2n}, \tag{20}$$

we can obtain that

$$\begin{aligned}
-V_x^{(n)} + \left[U, V^{(n)}\right] = &\left(vV_{2,n+1} - uV_{3,n+1}\right)g_1(0)\\
&- 2V_{2,n+1}g_2(0) + 2V_{3,n+1}g_3(0).
\end{aligned} \tag{21}$$

Thus, the compatibility condition of the Lax pair

$$\psi_x = U\psi, \quad \psi_t = V^{(n)}\psi \tag{22}$$

gives rise to

$$\tilde{u}_{t_n} = \begin{pmatrix} w\\ u\\ v\end{pmatrix}_{t_n} = \begin{pmatrix} uV_{3,n+1} - vV_{2,n+1}\\ 2V_{2,n+1}\\ -2V_{3,n+1}\end{pmatrix} = \begin{pmatrix} V_{1n,x}\\ 2V_{2,n+1}\\ -2V_{3,n+1}\end{pmatrix}, \tag{23}$$

and we call (23) the TX hierarchy.

When $n = 2$, $\alpha = 4$, (23) reduces to

$$\begin{aligned}
w_t &= uv_{xx} - u_{xx}v + 3uu_xv^2\\
&\quad + 3vv_xu^2 + 4(uvw)_x,\\
u_t &= 2u_{xx} - 8wu_x - 4uw_x - 6uu_xv\\
&\quad + 8w^2u + 4u^2wv + 3u^3v^2,\\
v_t &= -2v_{xx} - 4vw_x - 8wv_x - 6uvv_x\\
&\quad - 8w^2v - 12v^2uw - 3u^2v^3,
\end{aligned} \tag{24}$$

which is called the TX equation.

In what follows, we discuss the Hamiltonian structure of the TX hierarchy (23) by using the loop algebra $\widetilde{A}_1$. Equation (14) can be written as

$$V = \begin{pmatrix} V_1 & \lambda V_2\\ \lambda V_3 & -V_1\end{pmatrix}, \tag{25}$$

where $V_i = \sum_{m\geq 0}\lambda^{-2m}$, $i = 1, 2, 3$.

A direct calculation gives that

$$\begin{aligned}
&\left\langle V, \frac{\partial U}{\partial w}\right\rangle 2V_1, \qquad \left\langle V, \frac{\partial U}{\partial u}\right\rangle \lambda^2 V_3, \qquad \left\langle V, \frac{\partial U}{\partial v}\right\rangle \lambda^2 V_2,\\
&\left\langle V, \frac{\partial U}{\partial \lambda}\right\rangle = 4\lambda V_1 + \lambda v V_2 + \lambda u V_3.
\end{aligned} \tag{26}$$

Substituting these consequences into the trace identity yields

$$\frac{\delta}{\delta\tilde{u}}\left(4\lambda V_1 + \lambda vV_2 + \lambda uV_3\right) = \left(\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\right)\begin{pmatrix}\lambda^2V_3\\ \lambda^2V_2\\ 2V_1\end{pmatrix}. \tag{27}$$

Comparing the coefficients of λ^{-2n-1} in (27), we have

$$\frac{\delta}{\delta\tilde{u}}\left(4V_{1,n+1} + vV_{2,n+1} + uV_{3,n+1}\right) = (-2n+\gamma)\begin{pmatrix}V_{3,n+1}\\ V_{2,n+1}\\ 2V_{1n}\end{pmatrix}. \tag{28}$$

By the previous initial values, we see that $\gamma = 1$. Thus, we get

$$\begin{pmatrix}V_{3,n+1}\\ V_{2,n+1}\\ 2V_{1n}\end{pmatrix} = \frac{\delta H_{n+1}}{\delta\tilde{u}}, \qquad H_{n+1} = \frac{4V_{1,n+1} + vV_{2,n+1} + uV_{3,n+1}}{-2n+1}. \tag{29}$$

Therefore, the TX hierarchy (23) can be written as

$$\begin{aligned}
\tilde{u}_{t_n} &= \begin{pmatrix} w\\ u\\ v\end{pmatrix}_{t_n} = \begin{pmatrix} V_{1n,x}\\ 2V_{2,n+1}\\ -2V_{3,n+1}\end{pmatrix} = \begin{pmatrix}\partial & 0 & 0\\ 0 & 0 & 2\\ 0 & -2 & 0\end{pmatrix}\begin{pmatrix}2V_{1n}\\ V_{3,n+1}\\ V_{2,n+1}\end{pmatrix}\\
&= J_1\begin{pmatrix}2V_{1n}\\ V_{3,n+1}\\ V_{2,n+1}\end{pmatrix} = J_1\frac{\delta H_{n+1}}{\delta\tilde{u}}.
\end{aligned} \tag{30}$$

In order to derive the integrable coupling of the TX hierarchy, we introduce the following Lax matrices:

$$\begin{aligned}
U &= g_1(1) + wg_1(0) + ug_2(0) + vg_3(0)\\
&\quad + pg_4(0) + qg_5(0) + rg_6(0),\\
V &= \sum_{m\geq 0}\left(\sum_{i=1}^{6}V_{im}g_i(-m)\right).
\end{aligned} \tag{31}$$

According to the Tu scheme [12], the stationary zero curvature equation

$$V_x = [\mathrm{U}, V], \tag{32}$$

leads to the following recursion relations:

$$\begin{aligned} V_{4m,x} &= uV_{6,m+1} - vV_{5,m+1} + qV_{3,m+1} - rV_{2,m+1}, \\ V_{5m,x} &= 2V_{5,m+1} + 2wV_{5m} - 2uV_{4m} + 2pV_{2m} - 2qV_{1m}, \\ V_{6m,x} &= -2V_{6,m+1} - 2wV_{6m} + 2vV_{4m} - 2pV_{3m} + 2rV_{1m} \end{aligned} \quad (33)$$

plus (17), which are equivalent to

$$\begin{aligned} V_{5,m+1} &= \frac{1}{2}V_{5m,x} - wV_{5m} + uV_{4m} - pV_{2m} + qV_{1m}, \\ V_{6,m+1} &= -\frac{1}{2}V_{6m,x} - wV_{6m} + vV_{4m} - pV_{3m} + rV_{1m}, \\ V_{4m,x} &= -\frac{1}{2}uV_{6m,x} - \frac{1}{2}vV_{5m,x} - \frac{1}{2}qV_{3m,x} - \frac{1}{2}rV_{2m,x} \\ &- uwV_{6m} + vwV_{5m} - (up + qw)V_{3m} + (vp + rw)V_{2m}, \end{aligned} \quad (34)$$

plus (18).

If we set $V_{1,0} = \alpha$, $V_{2,0} = V_{3,0} = V_{4,0} = V_{5,0} = V_{6,0} = 0$, which are the initial values, then we have from (18) and (34) that

$$\begin{aligned} V_{4,1} &= -\frac{\alpha}{2}(ur + qv), \\ V_{5,2} &= \frac{\alpha}{2}q_x - \alpha wq - \frac{\alpha}{2}u^2 r - \alpha quv - \alpha pu, \\ V_{6,2} &= -\frac{\alpha}{2}r_x - \alpha wr - \frac{\alpha}{2}uvr - \frac{\alpha}{2}qv^2 - \alpha pv - \frac{\alpha}{2}uvr, \\ V_{4,2} &= \frac{\alpha}{4}(yr_x - u_x r + qv_x - q_x v) + \alpha qvw \\ &\quad + \alpha rwv + \alpha uvp + \frac{3\alpha}{4}(u^2 vr + uqv^2), \\ V_{5,3} &= \frac{\alpha}{4}q_{xx} - \alpha wq_x - \frac{\alpha}{2}w_x q - \frac{\alpha}{4}(u^2 r)_x - \frac{\alpha}{2}(quv)_x \\ &\quad - \frac{\alpha}{2}(pu)_x + \alpha w^2 q - \frac{\alpha}{2}u^2 rw + 2\alpha wpu + \frac{\alpha}{4}u^2 r_x \\ &\quad - \frac{\alpha}{4}uu_x r - \frac{\alpha}{4}q_x uv + 3\alpha quvw + \alpha uvrw \\ &\quad + \alpha u^2 vp + \frac{3\alpha}{4}(u^3 vr + u^2 v^2 q) \\ &\quad - \frac{\alpha}{2}pu_x + \frac{\alpha}{2}u^2 pv + \frac{\alpha}{2}quv_x - \frac{\alpha}{4}qvu_x + \frac{3\alpha}{8}qu^2 v^2, \\ V_{6,3} &= \frac{\alpha}{4}r_{xx} + \alpha wr_x + \frac{\alpha}{2}w_x r + \frac{\alpha}{4}(uvr)_x + \frac{\alpha}{4}(qv^2)_x \\ &\quad + \frac{\alpha}{2}(pv)_x + \frac{\alpha}{4}(uvr)_x + \alpha w^2 r + 2\alpha wuvr + \frac{3\alpha}{2}wqv^2 \\ &\quad + 2\alpha wpv + \frac{\alpha}{4}(uvr_x - u_x vr + qvv_x - q_x v^2) \\ &\quad + \alpha rwv^2 + \alpha uv^2 p \\ &\quad + \frac{3\alpha}{4}uqv^3 + \frac{\alpha}{2}pv_x + \frac{\alpha}{2}puv^2 + \frac{\alpha}{4}ruv_x \\ &\quad - \frac{\alpha}{4}u_x rv + \frac{9\alpha}{8}u^2 v^2 r, \ldots. \end{aligned} \quad (35)$$

Denoting

$$V^{(n)} = \sum_{m=0}^{n}\left(\sum_{i=1}^{6} V_{im} g_i(-m)\right) = \lambda^{2n} V - V_-^{(n)}, \quad (36)$$

then we have

$$\begin{aligned} &-V^{(n)} + [U, V^{(n)}] \\ &= (-uV_{3,n+1} + vV_{2,n+1})\, g_1(0) \\ &\quad - 2V_{2,n+1} g_2(0) + 2V_{3,n+1} g_3(0) \\ &\quad + (-uV_{6,n+1} + vV_{5,n+1} - qV_{3,n+1} + rV_{2,n+1})\, g_4(0) \\ &\quad - 2V_{5,n+1} g_5(0) + 2V_{6,n+1} g_6(0). \end{aligned} \quad (37)$$

Hence, the zero curvature equation

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0 \quad (38)$$

gives that

$$\begin{aligned} \overline{u}_{t_n} &= \begin{pmatrix} w \\ u \\ v \\ p \\ q \\ r \end{pmatrix}_{t_n} \\ &= \begin{pmatrix} uV_{3,n+1} - vV_{2,n+1} \\ 2V_{2,n+1} \\ -2V_{3,n+1} \\ uV_{6,n+1} - vV_{5,n+1} + qV_{3,n+1} - rV_{2,n+1} \\ 2V_{5,n+1} \\ -2V_{6,n+1} \end{pmatrix} \\ &= \begin{pmatrix} V_{1n,x} \\ 2V_{2,n+1} \\ -2V_{3,n+1} \\ V_{4n,x} \\ 2V_{5,n+1} \\ -2V_{6,n+1} \end{pmatrix}. \end{aligned} \quad (39)$$

When $p = q = r = 0$, (39) reduces to the TX hierarchy. According to the theory on integrable couplings, (39) is a kind of integrable coupling of the TX hierarchy.

We consider a reduced case of (39). Set $n = 2$; we get an integrable coupling of the TX equation (24):

$$\begin{aligned} w_t &= uv_{xx} - u_{xx}v + 3uu_x v^2 + 3vv_x u^2 + 4(uvw)_x, \\ u_t &= 2u_{xx} - 8wu_x - 4uw_x - 6uu_x v \\ &\quad + 8w^2 u + 4u^2 wv + 3u^3 v^2, \\ v_t &= -2v_{xx} - 4vw_x - 8wv_x - 6uvv_x \\ &\quad - 8w^2 v - 12v^2 uw - 3u^2 v^3, \end{aligned}$$

$$
\begin{aligned}
p_t &= \frac{\alpha}{4}\left(ur_x - u_x r + qv_x - q_x v\right)_x + \alpha\left(qvw + rwv + uvp\right)_x \\
&\quad + \frac{3\alpha}{4}\left(u^2 vr + uqv^2\right)_x, \\
q_t &= \frac{\alpha}{2}q_{xx} - 2\alpha w q_x - \alpha w_x q - \frac{\alpha}{2}\left(u^2 r\right)_x - \alpha\left(quv\right)_x \\
&\quad - \alpha\left(pu\right)_x + 2\alpha w^2 q + \alpha u^2 rw + 4\alpha wpu + \frac{\alpha}{2}u^2 r_x \\
&\quad - \frac{\alpha}{2}uu_x r + \frac{\alpha}{2}quv_x - \frac{\alpha}{2}q_x uv + 6\alpha quvw + 2\alpha uvrw \\
&\quad + 2\alpha u^2 vp + \frac{3\alpha}{2}\left(u^3 vr + u^2 v^2 q\right) - \alpha pu_x + \alpha u^2 pv \\
&\quad + \frac{\alpha}{2}quv_x - \frac{\alpha}{2}qvu_x + \frac{3\alpha}{4}qu^2 v^2, \\
r_t &= -\frac{\alpha}{2}r_{xx} - 2\alpha w r_x - \alpha w_x r - \frac{\alpha}{2}(uvr)_x - \frac{\alpha}{2}\left(qv^2\right)_x \\
&\quad - \alpha\left(pv\right)_x - \frac{\alpha}{2}(uvr)_x - 2\alpha w^2 r - 4\alpha wuvr - 3\alpha wqv^2 \\
&\quad - 4\alpha wpv - \frac{\alpha}{2}\left(uvr_x - u_x vr + qvv_x - q_x v^2\right) - 2\alpha rwv^2 \\
&\quad - 2\alpha uv^2 p - \frac{3\alpha}{2}uqv^3 - \alpha pv_x - \alpha puv^2 - \frac{\alpha}{2}urv_x \\
&\quad + \frac{\alpha}{2}ruv_x + \frac{\alpha}{2}u_x rv - \frac{9\alpha}{8}u^2 v^2 r.
\end{aligned} \tag{40}
$$

When we set $p = q = r = 0$, the above equation reduces to (24). In addition, if we set $w = u = v = 0$, the above equation reduces to two different heat equations.

Next, we discuss the Hamiltonian structure of the integrable coupling (39), that is, the TX hierarchy. For this reason, we need to construct Lie algebra which is isomorphic to the Lie algebra G. Consider the linear space $R^6 = \{(x_1, \ldots, x_6)^T \mid x_i \in R\}$. Define an operation on R^6 as follows:

$$[a, b]^T = a^T M(b), \tag{41}$$

where

$$
\begin{gathered}
a = (a_1, \ldots, a_6)^T, \qquad b = (b_1, \ldots, b_6)^T \in R^6, \\
M(b) = \begin{pmatrix}
0 & 2b_2 & -2b_3 & 0 & 2b_5 & -2b_6 \\
b_3 & -2b_1 & 0 & b_6 & -2b_4 & 0 \\
-b_2 & 0 & 2b_1 & -b_5 & 0 & 2b_4 \\
0 & 0 & 0 & 0 & 2b_2 & -2b_3 \\
0 & 0 & 0 & b_3 & -2b_1 & 0 \\
0 & 0 & 0 & -b_2 & 0 & 2b_1
\end{pmatrix}.
\end{gathered} \tag{42}
$$

It can be verified that R^6 becomes Lie algebra if equipped with the commutator (41). In addition, we assume a linear map

$$\delta :\longrightarrow R^6, \qquad \sum_{i=1}^{6} a_i g_i \longrightarrow (a_1, a_2, \ldots, a_6)^T. \tag{43}$$

We can prove that δ is an isomorphism between G and R^6. Therefore, (31) can be written as

$$
\begin{aligned}
U &= \left(\lambda^2 + w, \lambda u, \lambda v, p, q\lambda, r\lambda\right)^T, \\
V &= \left(V_1, V_2\lambda, V_3\lambda, V_4, V_5\lambda, V_6\lambda\right)^T,
\end{aligned} \tag{44}
$$

where $V_i = \sum_{m\geq 0} V_{im}\lambda^{-2m}$, $i = 1, 2, \ldots, 6$.

In order to make use of the variational identity to derive the Hamiltonian structure of (39), we need to solve a matrix equation as follows:

$$M(b)F = -(M(b)F)^T, \qquad F^T = F, \tag{45}$$

where $F = (f_{ij})_{6\times 6}$ is a constant matrix independent of x and t.

From (45), we can obtain that

$$
F = \begin{pmatrix}
2\eta_1 & 0 & 0 & 2\eta_2 & 0 & 0 \\
0 & 0 & \eta_1 & 0 & 0 & \eta_2 \\
0 & \eta_1 & 0 & 0 & \eta_2 & 0 \\
2\eta_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \eta_2 & 0 & 0 & 0 \\
0 & \eta_2 & 0 & 0 & 0 & 0
\end{pmatrix}, \tag{46}
$$

from which we construct a linear functional

$$\{a, b\} = \int^x a^T F b\, dx, \quad a, b \in \widetilde{R}^6, \tag{47}$$

where $\widetilde{R}^6$ is the corresponding loop algebra of the Lie algebra R^6. A direct calculation gives, by using (44) and (47), that

$$
\begin{aligned}
&\left\{V, \frac{\partial U}{\partial w}\right\} = 2\eta_1 V_1 + 2\eta_2 V_4, \\
&\left\{V, \frac{\partial U}{\partial u}\right\} = \eta_1\lambda^2 V_3 + \eta_2\lambda^2 V_6, \\
&\left\{V, \frac{\partial U}{\partial v}\right\} = \eta_1\lambda^2 V_2 + \eta_2\lambda^2 V_5, \\
&\left\{V, \frac{\partial U}{\partial p}\right\} = 2\eta_2 V_1, \qquad \left\{V, \frac{\partial U}{\partial q}\right\} = \eta_2\lambda^2 V_3, \\
&\left\{V, \frac{\partial U}{\partial r}\right\} = \eta_2\lambda^2 V_2, \\
&\left\{V, \frac{\partial U}{\partial \lambda}\right\} = \lambda\left(2\eta_1 V_1 + 2\eta_2 V_4 + \eta_1 u V_3 + \eta_2 u V_6\right. \\
&\qquad\qquad \left. + \eta_1 v V_2 + \eta_2 v V_5 + \eta_2 q V_3 + \eta_2 r V_2\right).
\end{aligned} \tag{48}
$$

Substituting the above consequences into the variational identity leads to

$$
\frac{\delta}{\delta \overline{u}}\int^x \{V, U_\lambda\}\, dx = \left(\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\right)
\begin{pmatrix}
2\eta_1 V_1 + 2\eta_2 V_4 \\
\eta_1\lambda^2 V_3 + \eta_2\lambda^2 V_6 \\
\eta_1\lambda^2 V_2 + \eta_2\lambda^2 V_5 \\
2\eta_2 V_1 \\
\eta_2\lambda^2 V_3 \\
\eta_2\lambda^2 V_2
\end{pmatrix}. \tag{49}
$$

Comparing the coefficients of λ^{-2n-1} in (49), we get that

$$\begin{aligned}&\frac{\delta}{\delta\overline{u}}\int^{x}\left(2\eta_1 V_{1,n+1}+2\eta_2 V_{2,n+1}+\left(2\eta_2+\eta_1 v+\eta_2 r\right)V_{2,n+1}\right.\\&\qquad\left.+\left(\eta_1 u+\eta_2 q\right)V_{3,n+1}+\eta_2 vV_{5,n+1}+\eta_2 uV_{6,n+1}\right)dx\\&\equiv\frac{\delta}{\delta\overline{u}}\int^{x}Q_{n+1}dx\\&=\left(-2n+\gamma\right)\begin{pmatrix}2\eta_1 V_1+2\eta_2 V_4\\ \eta_1\lambda^2 V_3+\eta_2\lambda^2 V_6\\ \eta_1\lambda^2 V_2+\eta_2\lambda^2 V_5\\ 2\eta_2 V_1\\ \eta_2\lambda^2 V_3\\ \eta_2\lambda^2 V_2\end{pmatrix}\equiv\left(-2n+\gamma\right)P_n.\end{aligned} \tag{50}$$

It can be determined that $\gamma = 1$ in terms of the initial values of (18) and (34). Thus, we have

$$P_n=\frac{\delta H_{n+1}}{\delta\overline{u}},\qquad H_{n+1}=\frac{1}{-2n+1}\int^{x}Q_{n+1}dx. \tag{51}$$

Therefore, the integrable coupling (39) can be written as the Hamiltonian form

$$\begin{aligned}\overline{u}_{t_n}&=\begin{pmatrix}w\\u\\v\\p\\q\\r\end{pmatrix}_{t_n}\\&=\begin{pmatrix}0&0&0&\frac{\partial}{2\eta_2}&0&0\\0&0&0&0&0&\frac{2}{\eta_2}\\0&0&0&0&-\frac{2}{\eta_2}&0\\\frac{\partial}{2\eta_2}&0&0&-\frac{\eta_1\partial}{2\eta_2^2}&0&0\\0&0&\frac{2}{\eta_2}&0&0&\frac{-2\eta_1}{\eta_2^2}\\0&-\frac{2}{\eta_2}&0&0&\frac{2\eta_1}{\eta_2^2}&0\end{pmatrix}P_n\\&\equiv J_2P_n=J_2\frac{\delta H_{n+1}}{\delta\overline{u}}.\end{aligned} \tag{52}$$

Obviously, J_2 is a Hamiltonian operator. To our knowledge, (52) is completely new consequence.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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