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Research Article

Boundary Value Problems for Fourth Order Nonlinear p-Laplacian Difference Equations

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We consider the boundary value problem for a fourth order nonlinear p-Laplacian difference equation containing both advance and retardation. By using Mountain pass lemma and some established inequalities, sufficient conditions of the existence of solutions of the boundary value problem are obtained. And an illustrative example is given in the last part of the paper.

1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ denote the sets of all natural numbers, integers, and real numbers, respectively. For $a, b \in \mathbb{Z}$, define $\mathbb{Z}(a) = \{a, a+1, \ldots\}$ and $\mathbb{Z}(a,b) = \{a, a+1, \ldots, b\}$ when $a \leq b$.

Consider the following fourth order nonlinear difference equation:

$$\Delta^2 \left(r_{n-2}\phi_p \left(\Delta^2 u_{n-2}\right)\right) - f\left(n, u_{n+1}, u_n, u_{n-1}\right) = 0, \quad n \in \mathbb{Z}(1,k), \tag{1}$$

with boundary value conditions

$$u_{-1} = u_0 = u_{k+1} = u_{k+2} = 0, \tag{2}$$

where $k \in \mathbb{N}$, r_j is a positive number for $j \in \mathbb{Z}(-1,k)$, Δ is the forward difference operator defined by $\Delta u_n = u_{n+1} - u_n$, $\Delta^2 u_n = \Delta(\Delta u_n)$, and ϕ_p is the p-Laplacian operator; that is, $\phi_p(s) = |s|^{p-2}s$ $(p > 1)$, $f \in C(\mathbb{Z}(1,k) \times \mathbb{R}^3, \mathbb{R})$.

In the last decade, by using various techniques such as critical point theory, fix point theory, topological degree theory, and coincidence degree theory, a great deal of works have been done on the existence of solutions to boundary value problems of difference equations (see [1–7] and references therein). Among these approaches, the critical point theory seems to be a powerful tool to deal with this problem (see [5, 7–9]). However, compared to the boundary value problems of lower order difference equations ([6, 8, 10–13]), the study of boundary value problems of higher order difference equations is relatively rare (see [9, 14, 15]), especially the works by using the critical point theory [16]. For the background on difference equations, we refer to [17].

In this paper, we will consider the existence of solutions of the boundary value problem of (1) with (2). First, we will construct a functional J such that solutions of the boundary value problem (1) with (2) correspond to critical points of J. Then, by using Mountain pass lemma, we obtain the existence of critical points of J. We mention that (1) is a kind of difference equation containing both advance and retardation. This kind of difference equation has many applications both in theory and practice. For example, in [17], Agarwal considered the following difference equation:

$$-\omega^2 Mu(k) + f(-u(k-1) + 2u(k) - u(k+1)) = 0, \tag{3}$$

with the boundary value conditions

$$u(0) = u(k+1) = 0, \tag{4}$$

as an example. It represents the amplitude of the motion of every particle in the string. And in [7], the authors considered the following second order functional difference equation:

$$Lu_n = f\left(n, u_{n+1}, u_n, u_{n-1}\right), \tag{5}$$

with different boundary value conditions

$$\Delta x_0 = A, \qquad x_{T+1} = B, \tag{6}$$

where the operator L is the Jacobi operator given by

$$Lu_n = a_n u_{n+1} + a_{n-1} u_{n-1} + b_n u_n. \tag{7}$$

In [18], the authors considered the second order p-Laplacian difference equation:

$$\Delta\left(\phi_p\left(\Delta u_{n-1}\right)\right) + f\left(n, u_{n+1}, u_n, u_{n-1}\right) = 0, \quad n \in \mathbb{Z}(1,k), \tag{8}$$

with boundary value conditions

$$u_0 = u_{k+1} = 0. \tag{9}$$

As for the periodic and subharmonic solutions of p-Laplacian difference equations containing both advance and retardation, we refer to [19]. And for the periodic solutions of ϕ-Laplacian difference equations, we refer to [20].

Throughout this paper, we assume that there exists a function $F(n,u,v)$ which is differentiable in (u,v) and $F(n,0,0) = 0$ for each $n \in \mathbb{Z}(0,k)$, satisfying

$$\frac{\partial F(n-1,v,w)}{\partial v} + \frac{\partial F(n,u,v)}{\partial v} = f(n,u,v,w), \quad \forall u,v,w \in \mathbb{R}, \tag{10}$$

for $n \in \mathbb{Z}(1,k)$.

2. Preliminaries and Main Results

Lemma 1. *Let $p \in (1,\infty)$; then there exist two positive sequences $\{c_*(n)\}_{n\in\mathbb{N}}$ and $\{c^*(n)\}_{n\in\mathbb{N}}$ such that*

$$\begin{aligned} c_*(b-a+1)\left(\sum_{j=a}^{b}\left|u_j\right|^2\right)^{1/2} &\le \left(\sum_{j=a}^{b}\left|u_j\right|^p\right)^{1/p} \\ &\le c^*(b-a+1)\left(\sum_{j=a}^{b}\left|u_j\right|^2\right)^{1/2} \end{aligned} \tag{11}$$

holds for any $a, b \in \mathbb{Z}$ with $a < b$, where $c_*(n) = 1$, $c^*(n) = n^{(2-p)/2p}$ for $p \in (1,2]$ and $c_*(n) = n^{-(p-2)/2p}$, $c^*(n) = 1$ for $p \in (2,\infty)$.

Proof. If $1 < p \le 2$, by Hölder's inequality, we have

$$\begin{aligned} \sum_{j=a}^{b}\left|u_j\right|^p &\le \left(\sum_{j=a}^{b} 1^{2/(2-p)}\right)^{(2-p)/2}\left(\sum_{j=a}^{b}\left|u_j\right|^2\right)^{p/2} \\ &= (b-a+1)^{(2-p)/2}\left(\sum_{j=a}^{b}\left|u_j\right|^2\right)^{p/2}, \end{aligned} \tag{12}$$

which implies that

$$\left(\sum_{j=a}^{b}\left|u_j\right|^p\right)^{1/p} \le (b-a+1)^{(2-p)/2p}\left(\sum_{j=a}^{b}\left|u_j\right|^2\right)^{1/2}, \tag{13}$$

and $c_*(n) = 1$ is obvious. If $p > 2$, then we have

$$\begin{aligned} \sum_{j=a}^{b}\left|u_j\right|^2 &\le \left(\sum_{j=a}^{b} 1^{p/(p-2)}\right)^{(p-2)/p}\left(\sum_{j=a}^{b}\left|u_j\right|^p\right)^{2/p} \\ &= (b-a+1)^{(p-2)/p}\left(\sum_{j=a}^{b}\left|u_j\right|^p\right)^{2/p}, \end{aligned} \tag{14}$$

which implies that

$$\left(\sum_{j=a}^{b}\left|u_j\right|^p\right)^{1/p} \ge (b-a+1)^{-(p-2)/2p}\left(\sum_{j=a}^{b}\left|u_j\right|^2\right)^{1/2}, \tag{15}$$

and $c^*(n) = 1$ is obvious. Now the proof is complete. □

Lemma 2. *There exist two positive sequences $\{\lambda_*(n)\}_{n\in\mathbb{N}}$ and $\{\lambda^*(n)\}_{n\in\mathbb{N}}$ such that*

$$\begin{aligned} \lambda_*(b-a+1)\sum_{j=a}^{b} u_j^2 &\le \sum_{j=a-1}^{b}\left(\Delta u_j\right)^2 \\ &\le \lambda^*(b-a+1)\sum_{j=a}^{b} u_j^2 \end{aligned} \tag{16}$$

holds for any $a, b \in \mathbb{Z}$ with $a < b$ and $u_{a-1} = u_{b+1} = 0$, where

$$\lambda_*(n) = 4\sin^2\frac{\pi}{2(n+1)}, \qquad \lambda^*(n) = 4\sin^2\frac{n\pi}{2(n+1)}. \tag{17}$$

Proof. There is no harm in assuming that $a = 1$, $b = k$. Then

$$\sum_{j=0}^{k}\left(\Delta u_j\right)^2 = \left(u_1, u_2, \ldots, u_k\right) A\left(u_1, u_2, \ldots, u_k\right)^{\mathrm{tr}}, \tag{18}$$

where $(\cdot)^{\mathrm{tr}}$ means the transpose of $(\cdot)$, and A is the $k\times k$ matrix given by

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}. \tag{19}$$

We will calculate the eigenvalues of A. Similar to [21], assume that λ is an eigenvalue of A. Since $A - rI$ is positive-definite for $r < 0$ and negative-definite for $r > 4$, where I is the $k \times k$ identity matrix, we see that $\lambda \in [0,4]$. Assume that $\xi = (\xi_1, \xi_2, \ldots, \xi_k)^{\mathrm{tr}}$ is an eigenvector associated to λ and define the sequence $\{y_n\}_{n=0}^{k+1}$ as

$$y_i = \xi_i, \quad i \in \mathbb{Z}(1,k), \qquad y_0 = y_{k+1} = 0. \tag{20}$$

Then $\{y_n\}$ satisfies

$$-y_{n+1}+(2-\lambda)y_n-y_{n-1}=0,\qquad y_0=y_{k+1}=0,\quad n\in\mathbb{Z}(1,k).\tag{21}$$

Since the roots of the equation $-r^2+(2-\lambda)r-1=0$ are

$$r_1=\frac{1}{2}\left(2-\lambda+\sqrt{4-(2-\lambda)^2}\right)i,\qquad r_2=\frac{1}{2}\left(2-\lambda-\sqrt{4-(2-\lambda)^2}\right)i,\tag{22}$$

set

$$\theta=\arccos\frac{1}{2}(2-\lambda).\tag{23}$$

Then

$$y_n=d_1\cos n\theta+d_2\sin n\theta\tag{24}$$

for some constants d_1 and d_2. $y_0=0$ implies that $d_1=0$, and $y_{k+1}=0$ implies that $\sin(k+1)\theta=0$. Therefore, $(k+1)\theta=j\pi$ for $j\in\mathbb{Z}(1,k)$. By (23), we have $\lambda=2(1-\cos\theta)=4\sin^2\theta/2$ which implies that the eigenvalues of A are

$$\lambda_j=4\sin^2\frac{j\pi}{2(k+1)},\qquad j\in\mathbb{Z}(1,k).\tag{25}$$

The maximum eigenvalue of A is λ_k, and the minimal eigenvalue of A is λ_1. Equation (16) follows from (18) and the fact that $\lambda_k=\lambda^*(k)$ and $\lambda_1=\lambda_*(k)$. □

Before we apply the critical point theory, we will establish the corresponding variational framework for (1) with (2).

Let

$$E=\{u=\{u_n\}:\mathbb{Z}(-1,k+2)\longrightarrow\mathbb{R}\mid u_{-1}=u_0=u_{k+1}=u_{k+2}=0\}.\tag{26}$$

Then E is a k-dimensional Hilbert space.

Obviously, E is isomorphic to $\mathbb{R}^k$. In fact, we can find a map $I:E\to\mathbb{R}^k$ defined by

$$I:\{u_n\}\longrightarrow(u_1,u_2,\ldots,u_k)^{\mathrm{tr}}.\tag{27}$$

Define the inner product on E as

$$\langle u,v\rangle=\sum_{j=1}^{k}u_jv_j,\quad\forall u,v\in E.\tag{28}$$

The corresponding norm $\|\cdot\|$ can be induced by

$$\|u\|=\left(\sum_{j=1}^{k}|u_j|^2\right)^{1/2},\quad\forall u\in E.\tag{29}$$

For all $u\in E$, define the functional $J(u)$ on E as follows:

$$J(u)=\frac{1}{p}\sum_{n=-1}^{k}r_n|\Delta^2u_n|^p-\sum_{n=0}^{k}F(n,u_{n+1},u_n).\tag{30}$$

Clearly, $J\in C^1(E,\mathbb{R})$. We can compute the partial derivative as

$$\frac{\partial J(u)}{\partial u_j}=\Delta^2\left(r_{j-2}\phi_p\left(\Delta^2u_{j-2}\right)\right)-f\left(j,u_{j+1},u_j,u_{j-1}\right),\tag{31}$$

for $j\in\mathbb{Z}(1,k)$, $u=\{u_j\}\in E$. Therefore, $u\in E$ is a critical point of J if and only if u is a solution of (1) with (2).

Definition 3. Let E be a real Banach space; the functional $J\in C^1(E,\mathbb{R})$ is said to satisfy the Palais-Smale (P.S. for short) condition if any sequence $\{x_m\}$ in E such that $\{J(x_m)\}$ is bounded and $J'(x_m)\to0$ as $m\to\infty$ contains a convergent subsequence.

Let B_ρ denote the open ball in E with radius ρ and center 0, and let ∂B_ρ denote its boundary.

In order to obtain the existence of critical points of J on E, we need to use the following basic lemma, which is important in the proof of our main results.

Lemma 4 (Mountain pass lemma [22]). *Let E be a real Hilbert space and $J\in C^1(E,\mathbb{R})$ satisfies the P.S. condition, if $J(0)=0$ and the following conditions hold.*

(J_1) *There exist constants $a>0$ and $\rho>0$ such that $J|_{\partial B_\rho}\geq a$.*

(J_2) *There exists $e\in E\setminus B_\rho$ such that $J(e)<0$.*

Then J possesses a critical value $c\geq a$ given by

$$c=\inf_{g\in\Gamma}\max_{s\in[0,1]}J(g(s)),\tag{32}$$

where

$$\Gamma=\{g\in C([0,1],E)\mid g(0)=0,g(1)=e\}.\tag{33}$$

Let

$$r_*=\min\{r_n\mid n\in\mathbb{Z}(-1,k)\},\quad r^*=\max\{r_n\mid n\in\mathbb{Z}(-1,k)\},$$
$$\alpha_0=\frac{r_*}{p}\left(\frac{c_*(k+2)}{c^*(k+1)}\right)^p\left(\frac{\lambda_*(k+1)\lambda_*(k)}{2}\right)^{p/2},\quad\beta_0=\frac{r^*}{p}\left(\frac{c^*(k+2)}{c_*(k+1)}\right)^p\left(\frac{\lambda^*(k+1)\lambda^*(k)}{2}\right)^{p/2}.\tag{34}$$

Then, for $p\in(1,2]$,

$$\alpha_0=\frac{r_*}{p}(k+1)^{-(2-p)/2}8^{p/2}\left(\sin\frac{\pi}{2(k+2)}\sin\frac{\pi}{2(k+1)}\right)^p,$$
$$\beta_0=\frac{r^*}{p}(k+2)^{(2-p)/2}8^{p/2}\left(\sin\frac{(k+1)\pi}{2(k+2)}\sin\frac{k\pi}{2(k+1)}\right)^p.\tag{35}$$

For $p \in (2, \infty)$,

$$\alpha_0 = \frac{r_*}{p}(k+2)^{-(p-2)/2} 8^{p/2} \left(\sin \frac{\pi}{2(k+2)} \sin \frac{\pi}{2(k+1)} \right)^p,$$
$$\beta_0 = \frac{r^*}{p}(k+1)^{(p-2)/2} 8^{p/2} \left(\sin \frac{(k+1)\pi}{2(k+2)} \sin \frac{k\pi}{2(k+1)} \right)^p. \tag{36}$$

Now we state our main results.

Theorem 5. *Assume that $F(n, u, v)$ satisfies the following conditions.*

(F_1) *There exist constants $\delta \in (0, \infty)$ and $\alpha \in (0, \alpha_0)$ such that*

$$F(n, u, v) \le \alpha \left(u^2 + v^2\right)^{p/2}, \quad \text{for } n \in \mathbb{Z}(0, k), \quad u^2 + v^2 \le \delta^2. \tag{37}$$

(F_2) *There exist constants $\beta \in (\beta_0, \infty)$ and $\gamma \in (0, \infty)$ such that*

$$F(n, u, v) \ge \beta \left(u^2 + v^2\right)^{p/2} - \gamma, \quad \text{for } n \in \mathbb{Z}(0, k). \tag{38}$$

Then (1) *with* (2) *possesses at least two nontrivial solutions.*

Remark 6. Comparing our results with the results of the boundary value problems of second order p-Laplacian difference equations in [18], we find that our results are more precisely.

In view of (37) and (38), it is easy to obtain the following corollary.

Corollary 7. *Assume that $F(n, u, v)$ satisfies*

$$\lim_{u^2+v^2 \to 0} \frac{F(n, u, v)}{(u^2 + v^2)^{p/2}} = 0, \quad \forall n \in \mathbb{Z}(0, k),$$
$$\lim_{u^2+v^2 \to +\infty} \frac{F(n, u, v)}{(u^2 + v^2)^{p/2}} = +\infty, \quad \forall n \in \mathbb{Z}(0, k). \tag{39}$$

Then (1) *with* (2) *possesses at least two nontrivial solutions.*

For the case when $p = 2$, we have the following corollary for the boundary value problems of the fourth order nonlinear difference equations.

Corollary 8. *Assume that $F(n, u, v)$ satisfies the following conditions.*

(F_3) *There exist constants $\delta_1 > 0$ and $0 < \alpha_1 < (8r_*/p)(\sin(\pi/2(k+2)) \sin(\pi/2(k+1)))^2$ such that*

$$F(n, u, v) \le \alpha_1 \left(u^2 + v^2\right), \quad \text{for } n \in \mathbb{Z}(0, k), \quad u^2 + v^2 \le \delta_1^2. \tag{40}$$

(F_4) *There exist constants $\beta_1 > (8r^*/p)(\sin((k+1)\pi/2(k+2)) \sin(k\pi/2(k+1)))^2$ and $\gamma_1 > 0$ such that*

$$F(n, u, v) \ge \beta_1 \left(u^2 + v^2\right) - \gamma_1, \quad \text{for } n \in \mathbb{Z}(0, k). \tag{41}$$

Then the following fourth order nonlinear difference equation

$$\Delta^2 \left(r_{n-2} \Delta^2 u_{n-2}\right) - f(n, u_{n+1}, u_n, u_{n-1}) = 0, \quad n \in \mathbb{Z}(1, k), \tag{42}$$

with the boundary value conditions (2) *possesses at least two nontrivial solutions.*

3. Proof of Theorem 5

In order to prove Theorem 5, we first establish the following lemma.

Lemma 9. *Assume that F satisfies (F_2); then the functional J satisfies the P.S. condition.*

Proof. Let $\{u^{(s)}\}_{s \in \mathbb{N}}$ be a sequence in E such that $\{J(u^{(s)})\}$ is bounded and $J'(u^{(s)}) \to 0$ as $s \to \infty$. Then there exists a positive constant C such that $|J(u^{(s)})| \le C$ for $s \in \mathbb{N}$.

By (11) and (16), we have

$$\begin{aligned}
&\frac{1}{p} \sum_{n=-1}^{k} r_n \left|\Delta^2 u_n^{(s)}\right|^p \\
&\le \frac{r^*}{p} (c^*(k+2))^p \left(\sum_{n=-1}^{k} \left|\Delta^2 u_n^{(s)}\right|^2 \right)^{p/2} \\
&\le \frac{r^*}{p} (c^*(k+2))^p \left(\lambda^*(k+1) \sum_{n=0}^{k} (\Delta u_n^{(s)})^2 \right)^{p/2} \\
&\le \frac{r^*}{p} (c^*(k+2))^p \left(\lambda^*(k+1) \lambda^*(k) \sum_{n=1}^{k} (u_n^{(s)})^2 \right)^{p/2} \\
&= \frac{r^*}{p} (c^*(k+2))^p (\lambda^*(k+1)\lambda^*(k))^{p/2} \left\|u^{(s)}\right\|^p \\
&= (c_*(k+1))^p 2^{p/2} \beta_0 \left\|u^{(s)}\right\|^p.
\end{aligned} \tag{43}$$

And by (F_2), (11), and (16), we have

$$\begin{aligned}
&\sum_{n=0}^{k} F\left(n, u_{n+1}^{(s)}, u_n^{(s)}\right) \\
&\ge \beta \sum_{n=0}^{k} \left(\left(u_{n+1}^{(s)}\right)^2 + \left(u_n^{(s)}\right)^2 \right)^{p/2} - (k+1)\gamma \\
&\ge \beta \left[c_*(k+1) \left(\sum_{n=0}^{k} \left(\left(u_{n+1}^{(s)}\right)^2 + \left(u_n^{(s)}\right)^2 \right) \right)^{1/2} \right]^p \\
&\quad - (k+1)\gamma \\
&= (c_*(k+1))^p 2^{p/2} \beta \left\|u^{(s)}\right\|^p - (k+1)\gamma.
\end{aligned} \tag{44}$$

Therefore, by (30), we obtain

$$\begin{aligned} J\left(u^{(s)}\right) &= \frac{1}{p}\sum_{n=-1}^{k} r_n\left|\Delta^2 u_n^{(s)}\right|^p - \sum_{n=0}^{k} F\left(n, u_{n+1}^{(s)}, u_n^{(s)}\right) \\ &\leq \left(c_*(k+1)\right)^p 2^{p/2}\left(\beta_0 - \beta\right)\left\|u^{(s)}\right\|^p \\ &\quad + (k+1)\gamma. \end{aligned} \tag{45}$$

Noticing that $J(u^{(s)}) \geq -C$ and $\beta > \beta_0$, by (45), we have

$$\left\|u^{(s)}\right\|^p \leq \frac{(k+1)\gamma + C}{\left(c_*(k+1)\right)^p 2^{p/2}\left(\beta - \beta_0\right)}. \tag{46}$$

Since E is a finite-dimensional space, (46) implies that $\{u^{(s)}\}$ is bounded and has a convergent subsequence. Thus P.S. condition is verified. □

Now we give the proof of Theorem 5.

Proof. For any $u \in E$ with $\|u\| \leq \delta$, according to (11) and (16), we have

$$\begin{aligned} &\frac{1}{p}\sum_{n=-1}^{k} r_n\left|\Delta^2 u_n\right|^p \\ &\quad\geq \frac{r_*}{p}\left(c_*(k+2)\right)^p\left(\sum_{n=-1}^{k}\left|\Delta^2 u_n\right|^2\right)^{p/2} \\ &\quad\geq \frac{r_*}{p}\left(c_*(k+2)\right)^p\left(\lambda_*(k+1)\sum_{n=0}^{k}\left(\Delta u_n\right)^2\right)^{p/2} \\ &\quad\geq \frac{r_*}{p}\left(c_*(k+2)\right)^p\left(\lambda_*(k+1)\lambda_*(k)\sum_{n=1}^{k}\left(u_n\right)^2\right)^{p/2} \\ &\quad= \frac{r_*}{p}\left(c_*(k+2)\right)^p\left(\lambda_*(k+1)\lambda_*(k)\right)^{p/2}\|u\|^p. \end{aligned} \tag{47}$$

By (F_1), (11), and (16), we have

$$\begin{aligned} &\sum_{n=0}^{k} F\left(n, u_{n+1}, u_n\right) \\ &\qquad\leq \alpha\sum_{n=0}^{k}\left(u_{n+1}^2 + u_n^2\right)^{p/2} \\ &\qquad\leq \alpha\left[c^*(k+1)\left(\sum_{n=0}^{k}\left(u_{n+1}^2 + u_n^2\right)\right)^{1/2}\right]^p \\ &\qquad= \alpha\left(c^*(k+1)\right)^p 2^{p/2}\|u\|^p. \end{aligned} \tag{48}$$

So, by (30), we get

$$\begin{aligned} J(u) &= \frac{1}{p}\sum_{n=-1}^{k} r_n\left|\Delta^2 u_n\right|^p - \sum_{n=0}^{k} F\left(n, u_{n+1}, u_n\right) \\ &\geq \left[\frac{r_*}{p}\left(c_*(k+2)\right)^p\left(\lambda_*(k+1)\lambda_*(k)\right)^{p/2}\right. \\ &\qquad\left. - \alpha\left(c^*(k+1)\right)^p 2^{p/2}\right]\|u\|^p \\ &= \left(c^*(k+1)\right)^p 2^{p/2}\left(\alpha_0 - \alpha\right)\|u\|^p. \end{aligned} \tag{49}$$

Since $\alpha < \alpha_0$, we let $a = (c^*(k+1))^p 2^{p/2}(\alpha_0 - \alpha)\delta^p$ and $\rho = \delta$. Then by (49),

$$J(u) \geq a, \quad \forall u \in \partial B_\rho, \tag{50}$$

which means that J satisfies the condition (J_1) of the Mountain pass lemma.

By our assumptions, it is clear that $J(0) = 0$. In order to use Mountain pass lemma, it suffices to verify that condition (J_2) holds. In fact, similar to the proof of (45), we have

$$J(u) \leq \left(c_*(k+1)\right)^p 2^{p/2}\left(\beta_0 - \beta\right)\|u\|^p + (k+1)\gamma, \tag{51}$$

for any $u \in E$. Since $\beta_0 < \beta$, it is easy to see that there exists an $e \in E$ with $\|e\| > \rho$ such that $J(\pm e) < 0$. Thus (J_2) holds.

According to Mountain pass lemma, J possesses a critical value $c \geq a$ given by

$$c = \inf_{g\in\Gamma_1}\max_{s\in[0,1]} J\left(g(s)\right), \tag{52}$$

where

$$\Gamma_1 = \left\{g \in C\left([0,1], E\right) \mid g(0) = 0,\ g(1) = e\right\}. \tag{53}$$

Let $\overline{u} \in E$ be a critical point of J corresponding to the critical value c; then $\overline{u}$ is nontrivial and $J(\overline{u}) = c$.

On the other hand, by (51), we have

$$\lim_{\|u\|\to\infty} J(u) = -\infty. \tag{54}$$

Since E is a k-dimensional space, by the continuity of $J(u)$ on u, we see that there exists $\widehat{u} \in E$ such that

$$J(\widehat{u}) = \max\left\{J(u) \mid u \in E\right\}. \tag{55}$$

Clearly, $\widehat{u}$ is a nonzero critical point of J, and $J(\widehat{u}) \geq c \geq a > 0$.

If $\widehat{u} \neq \overline{u}$, then the proof is finished. Otherwise, $\widehat{u} = \overline{u}$. Since $\|-e\| > \rho$ and $J(-e) < 0$, then by Mountain pass lemma again, J possesses a critical value $\widetilde{c} \geq a$ given by

$$\widetilde{c} = \inf_{g\in\Gamma_2}\max_{s\in[0,1]} J\left(g(s)\right), \tag{56}$$

where

$$\Gamma_2 = \left\{g \in C\left([0,1], E\right) \mid g(0) = 0,\ g(1) = -e\right\}. \tag{57}$$

Let $\widetilde{u} \in E$ be a critical point of J corresponding to the critical value $\widetilde{c}$. If $\widetilde{u} \neq \widehat{u}$, then the proof is finished. Otherwise $\widetilde{u} = \widehat{u} = \overline{u}$. Let $g_1(s) = se$ for $s \in [0,1]$; then $g_1 \in \Gamma_1$. By the definition of c, we see that there exists $s_1 \in (0,1)$ such that $J(s_1 e) = J(\widehat{u})$. Thus $s_1 e$ is a critical point of J. Similar, let $g_2(s) = -se$ for $s \in [0,1]$; then $g_2 \in \Gamma_2$. By the definition of $\widetilde{c}$, we see that there exists $s_2 \in (0,1)$ such that $J(-s_2 e) = J(\widehat{u})$. And $-s_2 e$ is a critical point of J. Clearly $s_1 e \neq -s_2 e$. The proof is now completed. □

In the last part of this paper, we give an example to illustrate our results.

Example 10. Consider (1) with (2), where f is defined by

$$\begin{aligned} f(n,u,v,w) &= \frac{2\mu_n v\left(u^2+v^2\right)^{p/2}}{1+u^2+v^2} \\ &\quad + \mu_n p v \ln\left(1+u^2+v^2\right)\left(u^2+v^2\right)^{(p/2)-1} \\ &\quad + \frac{2\mu_{n-1} v\left(v^2+w^2\right)^{p/2}}{1+v^2+w^2} \\ &\quad + \mu_{n-1} p v \ln\left(1+v^2+w^2\right)\left(v^2+w^2\right)^{(p/2)-1}, \end{aligned} \tag{58}$$

for $n \in \mathbb{Z}(1,k)$. Here $\mu_n > 0$ for $n \in \mathbb{Z}(0,k)$. Define

$$F(n,u,v) = \mu_n \ln\left(1+u^2+v^2\right)\left(u^2+v^2\right)^{p/2}, \quad n \in \mathbb{Z}(0,k). \tag{59}$$

Then $F(n,0,0)=0$ for $n \in \mathbb{Z}(0,k)$ and (10) holds. Moreover, it is easy to see that $F(n,u,v)$ satisfies (39) for $n \in \mathbb{Z}(0,k)$. By Corollary 7, we see that (1) with (2) when f is defined by (58) has at least two nontrivial solutions.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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