

Research Article

A Necessary and Sufficient Condition for Hardy's Operator in the Variable Lebesgue Space

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The variable exponent Hardy inequality $\|x^{\beta(x)-1} \int_0^x f(t) dt\|_{L^{p(\cdot)}(0,l)} \leq C \|x^{\beta(x)} f\|_{L^{p(\cdot)}(0,l)}$, $f \geq 0$ is proved assuming that the exponents $p : (0, l) \rightarrow (1, \infty)$, $\beta : (0, l) \rightarrow \mathbb{R}$ not rapidly oscillate near origin and $1/p'(0) - \beta > 0$. The main result is a necessary and sufficient condition on p, β generalizing known results on this inequality.

1. Introduction

There are a lot of examples in the theory of differential equations and analysis, in which the boundedness of the Hardy operator in some spaces is used essentially (a weighted Lebesgue space, Lorentz space, their weak spaces, and so on; see, e.g., [1–4]). The aim of present paper is to study a necessary and sufficient condition for the boundedness of Hardy operator in the weighted Lebesgue space $L^{p(\cdot)}(0, l)$ with variable exponent $p : (0, l) \rightarrow (1, \infty)$. The investigation of variable exponent Lebesgue spaces is stimulated with the modeling of electrorheological fluids (see [5, 6]). That led to the development of regularity theory for the nonlinear elliptic and parabolic equations with partial derivatives (see, e.g., the bibliography in [7, 8]). Also, the required mathematical methods of analysis were elaborated to study the boundedness of principal integral operators (maximal operator, fractional operators, singular operator, commutators, and so on) in spaces $L^{p(\cdot)}$ (see the recent monographs [9, 10]).

In all probability, the first investigation of the variable exponent Hardy inequality was started in the works [11–15], subsequently in [16–20]. The variable exponent Hardy inequality was considered also in the recent works [21–25]. Since the Hardy operator is the simplest one among other integral operators, it seems logical to investigate the necessary and sufficient conditions for this operator in the

first place. Though we have not so far succeeded in obtaining appropriate results in the general weighted space and in the case of general exponential functions, the representation of our results in presenting here form seems to us more attractive for comparison with the known results (see below).

More precisely, the subject of present paper is to study the norm inequality

$$\|x^{\beta(x)-1} Hf\|_{L^{p(\cdot)}(0,l)} \leq C \|x^{\beta(x)} f\|_{L^{p(\cdot)}(0,l)}; \quad f \geq 0 \quad (1)$$

for the Hardy operator

$$Hf(x) = \int_0^x f(t) dt. \quad (2)$$

Due to the cited above results, a necessary and sufficient condition for (1) take place if a regularity condition is assumed on p, β at the origin. Namely, let $1 < p^- = \inf p$, $p^+ = \sup p < \infty$, $-\infty < \beta^-, \beta^+ < \infty$, both functions p, β satisfy the condition

$$\limsup_{x \rightarrow 0} |g(x) - g(0)| \log \frac{1}{x} < \infty; \quad (3)$$

then inequality (1) holds if and only if $\beta(0) < 1/p'(0)$.

In our results, the exponent functions $p(\cdot), \beta(\cdot)$ satisfy the following oscillation condition near origin:

$$\limsup_{x \rightarrow +0} \left[\sup_{|y-x| < x/2} |g(x) - g(y)| \right] \ln \frac{1}{x} < \infty. \quad (4)$$

This condition is weaker than known logarithmic condition (3); that is, (3) implies (4). We can give an example of exponential function $p(\cdot)$ for which condition (4) is satisfied but (3) fails: $p(x) = p(0) + (\epsilon/(\ln(1/x))^\alpha); 0 < \alpha < 1, \epsilon > 0$. Also using L'Hopital's rule, it is not difficult to check that such a function p satisfies the condition (5). Therefore, applying the assertion of our results below, we get new results on existence of inequality (1) (compare with the known results in [17] or [24]; the condition (3) was imposed there).

The following main result is obtained in this paper.

Theorem 1. *Let $\beta : (0, l) \rightarrow \mathbb{R}$ and $p : (0, l) \rightarrow (1, \infty)$ be measurable functions such that $\beta(0) < 1/p'(0)$. Suppose there are limits $\lim_{x \rightarrow +0} p(x) = p(0) \in (1, \infty)$ and $\lim_{x \rightarrow +0} \beta(x) = \beta(0) \in \mathbb{R}$ and the functions β, p satisfy condition (4) near origin.*

Then inequality (1) holds if and only if

$$\int_a^l x^{\beta(x)-1/p'(x)} \frac{dx}{x} \leq Ca^{\beta(a)-1/p'(a)}, \quad a \in (0, l). \quad (5)$$

2. Notation

As to the basic properties of spaces $L^{p(\cdot)}$, we refer to [26, 27]. Throughout this paper, it is assumed that $p(x)$ is a measurable function in $(0, l)$ taking its values from the interval $[1, \infty)$ with $p^+ = \sup\{p(x) : x \in (0, l)\} < \infty$. The space of functions $L^{p(\cdot)}(0, l)$ is introduced as the class of measurable functions $f(x)$ in $(0, l)$ which has a finite $I_{p(\cdot)}(f) = \int_0^l |f(x)|^{p(x)} dx$ modular (put also $I_{p(\cdot);E}(f) = \int_E |f(t)|^{p(t)} dt$). A norm in $L^{p(\cdot)}(0, l)$ is given in the form

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : I_{p(\cdot)} \left(\frac{f}{\lambda} \right) \leq 1 \right\}. \quad (6)$$

(By $\|f\|_{p(\cdot);E}$ or simply $\|f\|_{p(\cdot)}$, we denote the $p(\cdot)$ norm over set E .)

For $1 < p^-, p^+ < \infty$, the space $L^{p(\cdot)}(0, l)$ is a reflexive Banach space. The relation between the modular and norm is expressed by the following inequalities (see, e.g., [12]):

$$\begin{aligned} \|f\|_{p(\cdot)}^{p^+} &\leq I_p(f) \leq \|f\|_{p(\cdot)}^{p^-}, & \|f\|_{p(\cdot)} &\leq 1, \\ \|f\|_{p(\cdot)}^{p^-} &\leq I_p(f) \leq \|f\|_{p(\cdot)}^{p^+}, & \|f\|_{p(\cdot)} &> 1. \end{aligned} \quad (7)$$

These inequalities allow us to perform our norm estimates in terms of a modular.

For the function $1 \leq p(x) < \infty$, $p'(x)$ denotes the conjugate function of $p(x)$, $(1/p(x)) + (1/p'(x)) = 1$, and $p'(x) = \infty$ if $p(x) = 1$. We denote by C, C_1, C_2, \dots various positive constants whose values may vary at each appearance. We write $u \sim v$ if there exist positive constants C_1, C_2 such that

$C_1 u(x) \leq v(x) \leq C_2 u(x)$. By χ_E , we denote the characteristic function of set E .

We say a function $u : (0, l) \rightarrow (-\infty, +\infty)$ is almost increasing (decreasing) if there exists a constant $C > 0$ such that $u(t_1) \leq Cu(t_2)$ and $(u(t_2) \leq Cu(t_1))$ for $0 < t_1 \leq t_2 < l$.

3. Proofs

Throughout the section, we assume that $p : (0, l) \rightarrow [1, \infty)$ and $\beta : (0, l) \rightarrow (-\infty + \infty)$ are measurable functions such that $p^+ < \infty, -\infty < \beta^-,$ and $\beta^+ < \infty$.

Lemma 2. *Suppose $s : (0, l) \rightarrow \mathbb{R}$ is a measurable function such that $-\infty < s^-, s^+ < +\infty$. and the condition (4) be satisfied. Then there exists a constant $C > 1$ such that*

$$\frac{1}{C} y^{-s(y)} \leq x^{-s(x)} \leq C y^{-s(y)} \quad (8)$$

for any $0 < x < l, y \in (x/2, 3x/2)$, where the constant C depends on s^-, s^+ , and the constant from the condition (4).

Proof. Since s satisfies the condition (4), it follows that, if $s(y) \geq 0$, then

$$\begin{aligned} y^{-s(y)} &\leq \left(\frac{x}{2}\right)^{-s(y)+s(x)} \left(\frac{x}{2}\right)^{-s(x)} \\ &\leq e^{(s(x)-s(y)) \ln(2/x)} 2^{s(x)} x^{-s(x)} \leq C x^{-s(x)}, \end{aligned} \quad (9)$$

and if $s(y) < 0$ then $y^{-s(y)} \leq ((3/2)x)^{-s(y)} \leq C x^{-s(x)}$. By the same arguments,

$$x^{-s(x)} \leq C y^{-s(y)}. \quad (10)$$

□

Proof of Theorem 1. Consider the following.

Necessity. First, show that the function $x^{\beta(x)-(1/p'(x))}$ is almost decreasing if inequality (1) holds and the functions p, β satisfy (4). In this way, we will show that

$$t^{\beta(t)-1/p'(t)} \leq C_1 a^{\beta(a)-1/p'(a)}, \quad 0 < a < t < l < \infty, \quad (11)$$

where the constant C does not depend on a, t . For fixed t, a , there exists an $m \in \mathbb{N}$ such that

$$2^{m-1} a < t \leq 2^m a. \quad (12)$$

Let $n_0 \in \mathbb{N}, n_0 \geq m$, be such that $2^{n_0} a \leq l < 2^{n_0+1} a$. Insert a test function

$$f_0(x) = x^{-\beta(x)-1/p(x)} \chi_{(a/2, a)}(x), \quad x \in (0, l), \quad (13)$$

into inequality (1). Then

$$I_{p(\cdot)}(x^{\beta(x)} f_0) = \int_{a/2}^a x^{-1} dt = \ln 2, \quad (14)$$

and, therefore, $\|x^\beta f_0\|_{p(\cdot)} \leq 1$. It follows from (1) that $\|x^{\beta-1} Hf_0\|_{p(\cdot)} \leq C$. This means that $I_{p(\cdot)}(x^{\beta(x)-1} Hf_0) \leq C_2$. Therefore,

$$\int_a^l \left[x^{\beta(x)-1/p'(x)} \int_{a/2}^a u^{-\beta(u)+1/p'(u)} \frac{du}{u} \right]^{p(x)} \frac{dx}{x} \leq C_2. \quad (15)$$

Hence,

$$\sum_{n=1}^{n_0} \int_{2^{n-1}a}^{2^na} \left[x^{\beta(x)-1/p'(x)} \int_{a/2}^a u^{-\beta(u)+1/p'(u)} \frac{du}{u} \right]^{p(x)} \frac{dx}{x} \leq C_2, \quad (16)$$

$$\int_{2^{m-1}a}^{2^ma} \left[x^{\beta(x)-1/p'(x)} \int_{a/2}^a u^{-\beta(u)+1/p'(u)} \frac{du}{u} \right]^{p(x)} \frac{dx}{x} \leq C_2. \quad (17)$$

In the integral term, it easily follows from Lemma 2 and (4) for p, β that

$$C_3 u^{-\beta(u)+1/p'(u)} \geq a^{-\beta(a)+1/p'(a)}, \quad u \in (a/2, a). \quad (18)$$

Using this estimate and (17), (15), we see that

$$\int_{2^{m-1}a}^{2^ma} \left[\frac{\ln 2}{C_3} a^{1/p'(a)-\beta(a)} x^{\beta(x)-1/p'(x)} \right]^{p(x)} \frac{dx}{x} \leq C_2, \quad (19)$$

$$\int_a^l \left[x^{\beta(x)-1/p'(x)} a^{-\beta(a)+1/p'(a)} \right]^{p(x)} \frac{dx}{x} \leq C_4, \quad (20)$$

respectively.

Using the Holder inequality for $p(\cdot)$ -norms, we see that

$$\begin{aligned} & \int_{2^{m-1}a}^{2^ma} \left[a^{1/p'(a)-\beta(a)} x^{\beta(x)-1/p'(x)} \right] \frac{dx}{x} \\ & \leq C_0 \left\| \chi_{(2^{m-1}a, 2^ma)} x^{-1/p'(\cdot)} \right\|_{p'(\cdot)} \\ & \quad \times \left\| \chi_{(2^{m-1}a, 2^ma)} x^{-1/p(\cdot)} C a^{1/p'(a)-\beta(a)} x^{\beta(x)-1/p'(x)} \right\|_{p(\cdot)} \\ & \leq C_0 C \end{aligned} \quad (21)$$

since

$$I_{p'(\cdot)} \left(\chi_{(2^{m-1}a, 2^ma)} x^{-1/p'(\cdot)} \right) = \int_{2^{m-1}a}^{2^ma} \frac{dx}{x} = \ln 2, \quad (22)$$

and by (19)

$$\begin{aligned} & I_{p(\cdot)} \left(x^{-1/p(\cdot)} C a^{1/p'(a)-\beta(a)} x^{\beta(x)-1/p'(x)} \cdot \chi_{(2^{m-1}a, 2^ma)}(x) \right) \\ & \leq C_5. \end{aligned} \quad (23)$$

Hence

$$\int_{2^{m-1}a}^{2^ma} \left[a^{1/p'(a)-\beta(a)} x^{\beta(x)-1/p'(x)} \right] \frac{dx}{x} \leq C, \quad (24)$$

and then

$$\int_{2^{m-1}a}^{2^ma} x^{\beta(x)-1/p'(x)} \frac{dx}{x} \leq C a^{\beta(a)-1/p'(a)}. \quad (25)$$

On the other hand, by Lemma 2 and (4) for p, β , we have

$$x^{\beta(x)-1/p'(x)} \geq C t^{\beta(t)-1/p'(t)}, \quad x \in (2^{m-1}a, 2^ma). \quad (26)$$

Using this inequality from (25), we get (11). Inequity (11) has been proved; that is, the function $x^{\beta(x)-(1/p'(x))}$ is almost decreasing.

Now, it follows from (11) and (20) that

$$\begin{aligned} C & \geq \int_a^l \left[\frac{x^{\beta(x)-1/p'(x)} a^{-\beta(a)+1/p'(a)}}{C_1} \right]^{p(x)} C_1^{p(x)} \frac{dx}{x} \\ & \geq \int_a^l \left[\frac{x^{\beta(x)-1/p'(x)} a^{-\beta(a)+1/p'(a)}}{C_1} \right]^{p^+} C_1 \frac{dx}{x}. \end{aligned} \quad (27)$$

Hence,

$$\int_a^l x^{(\beta(x)-1/p'(x))p^+} \frac{dx}{x} \leq C a^{(\beta(a)-1/p'(a))p^+}. \quad (28)$$

Now, using (28), we will derive a Bari-Stechkin [28] type assertion in order to prove that the function $x^{(\beta(x)-1/p'(x))p^+ + \varepsilon}$ is almost decreasing by some $\varepsilon > 0$.

Indeed, put $g(x) = \int_x^l t^{(\beta(t)-1/p'(t))p^+} (dt/t)$. Then, by (28),

$$-xg'(x) \geq \frac{1}{C} g(x), \quad 0 < x < l. \quad (29)$$

Integrating this inequality,

$$x_1^{1/C} g(x_1) \geq x_2^{1/C} g(x_2), \quad 0 < x_1 \leq x_2 \leq \frac{l}{2}. \quad (30)$$

Using (28),

$$C x_1^{1/C+p^+(\beta(x_1)-1/p'(x_1))} \geq x_2^{1/C} g(x_2). \quad (31)$$

Now, it follows from Lemma 2 and (4) for p, β that

$$g(x_2) \geq \int_{x_2}^{2x_2} x^{(\beta(x)-1/p'(x))p^+} \frac{dx}{x} \geq C x_2^{(\beta(x_2)-1/p'(x_2))p^+}. \quad (32)$$

Therefore,

$$C x_1^{1/C+(\beta(x_1)-1/p'(x_1))p^+} \geq x_2^{1/C+(\beta(x_2)-1/p'(x_2))p^+}; \quad (33)$$

that is, the function $x^{\varepsilon+(\beta(x)-(1/p'(x)))p^+}$ is almost decreasing by $\varepsilon = 1/C$. This implies almost decreasing of the function $x^{\beta(x)-1/p'(x)+\varepsilon_1}$ by $\varepsilon_1 = \varepsilon/p^+$. Then it is easily seen that the condition (5) is satisfied.

This completes necessity of condition (5).

Sufficiency. Let the functions p, β satisfy (4) and the condition (5). Show that (5) implies almost decreasing of $x^{\beta(x)-1/p'(x)+\varepsilon}$ by some $\varepsilon > 0$. Put $g(x) = \int_x^l t^{\beta(t)-1/p'(t)} dt/t$ and repeat the arguments before.

We have

$$-xg'(x) \geq \frac{1}{C}g(x), \quad \text{integrating,} \tag{34}$$

$$x_1^{1/C}g(x_1) \geq x_2^{1/C}g(x_2), \quad 0 < x_1 \leq x_2 \leq \frac{1}{2}l.$$

Using (5),

$$Cx_1^{1/C+\beta(x)-1/p'(x_1)} \geq x_2^{1/C}g(x_2). \tag{35}$$

By Lemma 2 and (4) for p, β , it follows that

$$g(x_2) \geq \int_{x_2}^{2x_2} x^{\beta(x)-1/p'(x)} \frac{dx}{x} \geq Cx_2^{\beta(x_2)-1/p'(x_2)}. \tag{36}$$

Hence,

$$x_2^{1/C+\beta(x_2)-1/p'(x_2)} \leq Cx_1^{1/C+\beta(x_1)-1/p'(x_1)}, \tag{37}$$

that is, $x^{\varepsilon+\beta(x)-1/p'(x)}$ is almost decreasing by $\varepsilon = 1/C$.

Let $f(x) \geq 0$ be a measurable function such that $\|x^{\beta(x)}f\|_{p(\cdot)} \leq 1$. Then $I_{p(\cdot)}(x^{\beta(x)}f) \leq 1$. We have to prove $\|x^{\beta(x)-1}Hf\|_{p(\cdot)} \leq C_1$. By Minkowskii inequality for $L^{p(\cdot)}$ norms,

$$\begin{aligned} \|x^{\beta(x)-1}Hf\|_{p(\cdot)} &\leq \left\| x^{\beta(x)-1} \int_0^x f(t) dt \right\|_{p(\cdot);(\delta,l)} \\ &+ \sum_{n=0}^{\infty} \left\| x^{\beta(x)-1} \int_{2^{-n-1}x}^{2^{-n}x} f(t) dt \right\|_{p(\cdot);(0,\delta)}, \end{aligned} \tag{38}$$

where $0 < \delta < (1/2)l$ is a fixed number.

We will derive an estimate for every summand in (38). In this way, we will get a modular estimate for the corresponding terms in modular.

Denote

$$p_{x,n}^- = \inf \{p(t) : t \in (2^{-n-1}x, 2^{-n}x)\}; \quad n = 1, 2, \dots \tag{39}$$

By (37),

$$x^{\beta(x)-1/p'(x)+\varepsilon} \leq Ct^{\beta(t)-1/p'(t)+\varepsilon} \tag{40}$$

for any $t \in (2^{-n-1}x, 2^{-n}x)$, $0 < x < \delta$, where C does not depend on n, t, x . From (40) using $2^{-n-1}x < t < 2^{-n}x$, we get

$$\begin{aligned} t^{1/p'(t)-\beta(t)} &= t^\varepsilon t^{1/p'(t)-\beta(t)-\varepsilon} \leq Ct^\varepsilon x^{1/p'(x)-\beta(x)-\varepsilon} \\ &\leq C2^{-n\varepsilon} x^{1/p'(x)-\beta(x)} \end{aligned} \tag{41}$$

or

$$x^{-1/p'(x)+\beta(x)} \leq C2^{-n\varepsilon} t^{-1/p'(t)+\beta(t)}. \tag{42}$$

If $p_{x,n}^- \leq p(x)$, then, due to Holder's inequality, for $x \in (0, \delta)$ we have

$$\begin{aligned} &x^{\beta(x)-1} \int_{2^{-n-1}x}^{2^{-n}x} t^{\beta(t)p'(t)} f(t) t^{-\beta(t)p'(t)} dt \\ &\leq x^{\beta(x)-1} \left(\int_{2^{-n-1}x}^{2^{-n}x} t^{1-\beta(t)p'(t)} \frac{dt}{t} \right)^{1/(p_{x,n}^-)'} \\ &\quad \times \left(\int_{2^{-n-1}x}^{2^{-n}x} (t^{\beta(t)} f(t))^{p_{x,n}^-} \cdot t^{-\beta(t)p'(t)} dt \right)^{1/p_{x,n}^-}. \end{aligned} \tag{43}$$

Using (4) for p, β and Lemma 2, it is not difficult to see the following estimate:

$$\left(\int_{2^{-n-1}x}^{2^{-n}x} s^{1-\beta(s)p'(s)} \frac{ds}{s} \right)^{1/(p_{x,n}^-)'} \leq Ct^{1/p'(t)-\beta(t)}, \tag{44}$$

where $t \in (2^{-n-1}x, 2^{-n}x)$ and $0 < x < \delta, n = 1, 2, \dots$, with the constant $C > 0$ not depending on n, x, t .

Indeed, $s^{1-\beta(s)p'(s)} \sim t^{1-\beta(t)p'(t)}$ for p, β and $s, t \in (2^{-n-1}x, 2^{-n}x)$. Then to prove (44), it suffices to show $t^{1/(p_{x,n}^-)'} \sim t^{1/p'(t)}$, which is a simple consequence of (4) for p , Lemma 2, and the fact that there exists a point $y \in (2^{-n-1}x, 2^{-n}x)$ such that $|p_{x,n}^- - p(y)| \leq C/|\ln t|$.

For the second multilayer (43), we have the estimates

$$\begin{aligned} &\int_{2^{-n-1}x}^{2^{-n}x} (t^{\beta(t)p'(t)} f(t))^{p_{x,n}^-} t^{-\beta(t)p'(t)} dt \\ &\leq \int_{2^{-n-1}x}^{2^{-n}x} \chi_{\{t^{\beta(t)p'(t)} f(t) \geq 1\}} (t^{\beta(t)p'(t)} f(t))^{p(t)} t^{-\beta(t)p'(t)} dt \\ &\quad + \int_{2^{-n-1}x}^{2^{-n}x} t^{-\beta(t)p'(t)} dt \\ &\leq \int_{2^{-n-1}x}^{2^{-n}x} \left((t^{\beta(t)p'(t)} f(t))^{p(t)} + 1 \right) t^{-\beta(t)p'(t)} dt. \end{aligned} \tag{45}$$

Combining the estimates (43), (44), and (45) for $0 < x < \delta$ and $t \in (2^{-n-1}x < t \leq 2^{-n}x)$ we have

$$\begin{aligned} &x^{\beta(x)-1} \int_{2^{-n-1}x}^{2^{-n}x} f(t) dt \\ &\leq Cx^{-1/p(x)} \left(t^{1/p'(t)-\beta(t)} x^{\beta(x)-1/p'(x)} \right) \\ &\quad \times \left\{ \int_{2^{-n-1}x}^{2^{-n}x} \left((t^{\beta(t)} f(t))^{p(t)} + t^{-\beta(t)p'(t)} \right) dt \right\}^{1/p_{x,n}^-}. \end{aligned} \tag{46}$$

Now, taking into the account (42), here, we see that the last term is exceeded by

$$\begin{aligned} &\leq C2^{-n\varepsilon} x^{-(1/p(x))} \\ &\quad \times \left[\int_{2^{-n-1}x}^{2^{-n}x} \left((t^{\beta(t)} f(t))^{p(t)} + t^{-\beta(t)p'(t)} \right) dt \right]^{1/p(x)} \left[p(x)/p_{x,n}^- \right]. \end{aligned} \tag{47}$$

On the other hand, using the assumptions $\beta(0) < 1/p'(0)$ and $I_p(x^\beta f) \leq 1$,

$$\int_{2^{-n-1}x}^{2^{-n}x} \left((t^{\beta(t)} f(t))^{p(t)} + t^{-\beta(t)p'(t)} \right) dt \leq 1 + Ct^{1-\beta(t)p'(t)} \leq C_1, \quad t \in (2^{-n-1}x < t \leq 2^{-n}x). \tag{48}$$

Using (48) and $p_{x,n}^- \leq p(x)$, it follows from (47) that

$$x^{\beta(x)-1} \int_{2^{-n-1}x}^{2^{-n}x} f(t) dt \leq CC_1^{p^+/p^- - 1} 2^{-n\epsilon} x^{-1/p(x)} \left\{ \int_{2^{-n-1}x}^{2^{-n}x} \left((t^{\beta(t)} f(t))^{p(t)} + t^{-\beta(t)p'(t)} \right) dt \right\}^{1/p(x)}. \tag{49}$$

If $p_{x,n}^- > p(x)$, then we repeat all arguments with $p_{x,n}^-$ changed to $p(x)$. Indeed, it follows from the Holder inequality that

$$x^{\beta(x)-1} \int_{2^{-n-1}x}^{2^{-n}x} f(t) dt \leq x^{\beta(x)-1} \left(\int_{2^{-n-1}x}^{2^{-n}x} (t^{\beta(t)p'(t)} f(t))^{p(x)} t^{-\beta(t)p'(t)} dt \right)^{1/p(x)} \times \left(\int_{2^{-n-1}x}^{2^{-n}x} t^{1-\beta(t)p'(t)} \frac{dt}{t} \right)^{1/p'(x)}. \tag{50}$$

By (4), for p, β , we have (see similar arguments after (44))

$$\left(\int_{2^{-n-1}x}^{2^{-n}x} s^{1-\beta(s)p'(s)} \frac{ds}{s} \right)^{1/p'(t)} \leq Ct^{(1/p'(t))-\beta(t)}, \tag{51}$$

$$t \in (2^{-n-1}x < t \leq 2^{-n}x).$$

Also,

$$\int_{2^{-n-1}x}^{2^{-n}x} (t^{\beta(t)p'(t)} f(t))^{p(x)} t^{-\beta(t)p'(t)} dt \leq \int_{2^{-n-1}x}^{2^{-n}x} \chi_{\{t^{\beta(t)p'(t)} f(t) \geq 1\}} (t^{\beta(t)p'(t)} f(t))^{p(t)} t^{-\beta(t)p'(t)} dt + \int_{2^{-n-1}x}^{2^{-n}x} t^{-\beta(t)p'(t)} dt \leq 1 + Ct^{1-\beta(t)p'(t)} \leq C_1, \quad t \in (2^{-n-1}x < t \leq 2^{-n}x) \tag{52}$$

since the conditions $\beta(0)p'(0) < 1$, $p(x) < p_{x,n}^-$, and $I_{p(\cdot);(0,1)}(x^\beta f) \leq 1$ are assumed.

Hence,

$$\int_{2^{-n-1}x}^{2^{-n}x} (t^{\beta(t)p'(t)} f(t))^{p(x)} t^{-\beta(t)p'(t)} dt \leq C_1. \tag{53}$$

Therefore, by use of (42) for $0 < x < \delta$ and $t \in (2^{-n-1}x, 2^{-n})$, it follows that

$$x^{\beta(x)-1} \int_{2^{-n-1}x}^{2^{-n}x} f(t) dt \leq C \left[t^{(1/p'(t)-\beta(t))p'(t)} x^{\beta(x)-1/p'(x)} \right]^{1/p'(x)} x^{-(1/p(x))} \times \left\{ \int_{2^{-n-1}x}^{2^{-n}x} \left((t^{\beta(t)} f(t))^{p(t)} + t^{-\beta(t)p'(t)} \right) dt \right\}^{1/p(x)} \leq C 2^{-n\epsilon/p'(x)} t^{(1/p'(t)-\beta(t))(p'(t)-1)/p'(x)} x^{-(1/p(x))} \times \left\{ \int_{2^{-n-1}x}^{2^{-n}x} \left((t^{\beta(t)} f(t))^{p(t)} dt + t^{-\beta(t)p'(t)} \right) dt \right\}^{1/p(x)}. \tag{54}$$

Let us note that

$$t^{(1/p'(t)-\beta(t))(p'(t)-1)/p'(x)} \leq C, \quad 2^{-n-1}x < t < 2^{-n}x \tag{55}$$

since $\beta(0) < 1/p'(0)$. Therefore,

$$x^{\beta(x)-1} \int_{2^{-n-1}x}^{2^{-n}x} f(t) dt \leq C_1 2^{-n\epsilon_1} x^{-1/p(x)} \left\{ \int_{2^{-n-1}x}^{2^{-n}x} \left((t^{\beta(t)} f(t))^{p(t)} + t^{-\beta(t)p'(t)} \right) dt \right\}^{1/p(x)}, \tag{56}$$

with $\epsilon_1 = \epsilon/p'(0)$.

It follows from (56) and (49) that, in both cases $p(x) < p_{x,n}^-$ and $p(x) \geq p_{x,n}^-$, we have the same estimates (with different C, ϵ not depending on x, n). Denote again by ϵ the minimum of ϵ and ϵ_1 and, taking into account (56) and (49), we get

$$I_{p(\cdot);(0,\delta)} \left(x^{\beta(x)-1} \int_{2^{-n-1}x}^{2^{-n}x} f(t) dt \right) \leq CC_1^{p^+/p^- - 1} 2^{-n\epsilon p^-} \times \int_0^\delta x^{-1} \left(\int_{2^{-n-1}x}^{2^{-n}x} \left((t^{\beta(t)} f(t))^{p(t)} + t^{-\beta(t)p'(t)} \right) dt \right) dx. \tag{57}$$

Due to Fubini's theorem, this is exceeded by

$$\begin{aligned} & C C_1^{p^+/p^- - 1} 2^{-n\epsilon p^-} \int_0^{\delta 2^{-n}} \left(\int_{2^{n-1}t}^{2^n t} x^{-1} dx \right) \\ & \quad \times \left((t^{\beta(t)} f(t))^{p(t)} + t^{-\beta(t)p'(t)} \right) dt \\ & = C_1^{p^+/p^- - 1} 2^{-n\epsilon p^-} \ln 2 \int_0^{\delta 2^{-n}} \left((t^\beta f(t))^{p(t)} + t^{-\beta(t)p'(t)} \right) dt \\ & \leq C_6 2^{-n\epsilon p^-}. \end{aligned} \quad (58)$$

We have used that $I_{p(\cdot)}(t^{\beta(x)} f(\cdot)) \leq 1$ and the estimate

$$\int_0^{\delta 2^{-n}} t^{-\beta(t)p'(t)} dt \leq C, \quad (59)$$

which easily follows from $\beta(0) < 1/p'(0)$.

Therefore,

$$\left\| x^{\beta(x)-1} \int_{2^{n-1}x}^{2^{-n}x} f(t) dt \right\|_{L^{p(\cdot)}(0,\delta)} \leq C 2^{-n\epsilon p^- / p^+}. \quad (60)$$

From this estimate and (38), it follows that

$$\left\| x^{\beta(x)-1} Hf \right\|_{p(\cdot);(0,\delta)} \leq C \sum_{n=0}^{\infty} 2^{-n\epsilon p^- / p^+} \leq C_1. \quad (61)$$

It remains to get an estimate of $\left\| x^{\beta(x)-1} \int_0^x f(t) dt \right\|_{p(\cdot);(\delta,l)}$ far from origin. Since $x^{\beta(x)-1}$ is separated from zero and infinity in (δ, l) , it suffices to note the estimate

$$\int_0^x f(t) dt \leq \left\| t^{\beta(t)} f \right\|_{p(\cdot);(0,l)} \left\| t^{-\beta(t)} \right\|_{p'(\cdot);(0,l)} \leq C. \quad (62)$$

Here, the boundedness of first multiplier follows from the assumption. Boundedness of second multiplier follows from the condition $\beta(0) < 1/p'(0)$ and the assertion of Lemma 2.

Theorem 1 has been proved. \square

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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