

## Research Article

# A Multiple Iterated Integral Inequality and Applications

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We establish new multiple iterated Volterra-Fredholm type integral inequalities, where the composite function  $w(u(s))$  of the unknown function  $u$  with nonlinear function  $w$  in integral functions in [Ma, QH, Pečarić, J: Estimates on solutions of some new nonlinear retarded Volterra-Fredholm type integral inequalities. *Nonlinear Anal.* 69 (2008) 393–407] is changed into the composite functions  $w_1(u(s)), w_2(u(s)), \dots, w_n(u(s))$  of the unknown function  $u$  with different nonlinear functions  $w_1, w_2, \dots, w_n$ , respectively. By adopting novel analysis techniques, the upper bounds of the embedded unknown functions are estimated explicitly. The derived results can be applied in the study of solutions of ordinary differential equations and integral equations.

## 1. Introduction

The well-known Gronwall-Bellman inequality [1, 2] is the following or can be equivalently regarded as the following:

$$u(t) \leq c + \int_a^t f(s) u(s) ds, \quad t \in [a, a+T], \quad (1)$$

where  $c \geq 0$  is a constant,  $f$  is a given nonnegative continuous function, and  $u$  is the unknown function. It is often used to estimate solutions of differential equations. In 1956 Bihari [3] discussed

$$u(t) \leq c + \int_0^t f(s) w(u(s)) ds. \quad (2)$$

In 1990 Pinto [4] investigated

$$u(t) \leq c(t) + \sum_{i=1}^n \int_a^t g_i(s) w_i(u(s)) ds. \quad (3)$$

Replacing the upper limit  $t$  of the integral with a function  $b(t)$  in (2), in 2000 Lipovan [5] improved Bihari's results by

investigating the following so-called retarded Gronwall-like inequalities:

$$u(t) \leq c + \int_{b(t_0)}^{b(t)} f(s) w(u(s)) ds, \quad (4)$$

$$u(t) \leq c + \int_{t_0}^t f(s) w(u(s)) ds + \int_{b(t_0)}^{b(t)} g(s) w(u(s)) ds. \quad (5)$$

In 2005 Agarwal et al. [6] generally discussed

$$u(t) \leq c(t) + \sum_{i=1}^n \int_{b_i(t_0)}^{b_i(t)} g_i(t, s) w_i(u(s)) ds, \quad t_0 \leq t < t_1. \quad (6)$$

As required in estimation for solutions, invariant sets, and stability, many generalized versions of the Gronwall-Bellman inequality were given with an invariant decomposition [7–9], a singular kernel [10, 11], and maxima [12, 13]. More results about integral inequalities of single variable and multivariables can be found, for example, the books [14, 15].

In order to investigate the behavior of solutions of a linear Volterra-Fredholm type integral equation, a form of

integral inequalities which contains multiple integrals of the unknown,

$$\begin{aligned}
 u(t) &\leq c + \int_{\alpha(t_0)}^{\alpha(T)} b(t, s) u(s) ds \\
 &+ \int_{\alpha(t_0)}^{\alpha(t)} a(t, s) \left[ f(s) u(s) + \int_{\alpha(t_0)}^s c(s, \tau) u(\tau) d\tau \right] ds, \\
 &\forall t \in [t_0, T], \tag{7}
 \end{aligned}$$

called linear Volterra-Fredholm type integral inequality with retardation, is discussed by Pachpatte [16] in 2004.

In 2008 Ma and Pečarić [17] discussed more generally the following inequality:

$$\begin{aligned}
 u(t) &\leq c + \int_{\alpha(t_0)}^{\alpha(t)} \sigma_1(s) \\
 &\times \left[ f(s) w(u(s)) + \int_{\alpha(t_0)}^s \sigma_2(\tau) w(u(\tau)) d\tau \right] ds \\
 &+ \int_{\alpha(t_0)}^{\alpha(T)} \sigma_1(s) \\
 &\times \left[ f(s) w(u(s)) + \int_{\alpha(t_0)}^s \sigma_2(\tau) w(u(\tau)) d\tau \right] ds, \\
 &\forall t \in I, \tag{8}
 \end{aligned}$$

where  $I = [t_0, T]$ . In 2011 Abdeldaim and Yakout [18] investigated the following:

$$\begin{aligned}
 u(t) &\leq c + \int_{\alpha(t_0)}^t f(s) u(s) \\
 &\times \left[ u(s) + \int_{\alpha(t_0)}^s h(\tau) \left[ u(\tau) + \int_{\alpha(t_0)}^\tau g(\xi) u(\xi) d\xi \right] d\tau \right] ds. \tag{9}
 \end{aligned}$$

In 2013 Wang et al. [19] studied a new integral inequality of Gronwall-Bellman-Pachpatte type

$$\begin{aligned}
 \varphi(u(t)) &\leq c(t) \\
 &+ \int_{\alpha(t_0)}^{\alpha(t)} f_1(t, s) w_1(u(s)) ds \\
 &+ \int_{\alpha(t_0)}^{\alpha(t)} f_1(t, s) w_1(u(s)) \\
 &\times \left( \int_{t_0}^s f_2(s, \tau) w_2(u(\tau)) d\tau \right) ds
 \end{aligned}$$

$$\begin{aligned}
 &+ \int_{\alpha(t_0)}^{\alpha(t)} f_1(t, s) w_1(u(s)) \\
 &\times \left( \int_{t_0}^s f_2(s, \tau) w_2(u(\tau)) \right. \\
 &\quad \left. \times \left( \int_{t_0}^\tau f_3(\tau, \xi) w_3(u(\xi)) d\xi \right) d\tau \right) ds. \tag{10}
 \end{aligned}$$

In this paper, on the basis of [17, 18], we discuss a new multiple iterated Volterra-Fredholm type integral inequality

$$\begin{aligned}
 u(t) &\leq c + \int_{\alpha(t_0)}^{\alpha(t)} h_1(t_1) \\
 &\times \left[ f_1(t_1) \varphi_1(u(t_1)) + \int_{\alpha(t_0)}^{t_1} h_2(t_2) \right. \\
 &\quad \times \left[ f_2(t_2) \varphi_2(u(t_2)) + \dots \right. \\
 &\quad \left. \left. + \int_{\alpha(t_0)}^{t_{n-2}} h_{n-1}(t_{n-1}) \right. \right. \\
 &\quad \left. \left. \times \left[ f_{n-1}(t_{n-1}) \varphi_{n-1}(u(t_{n-1})) \right. \right. \right. \\
 &\quad \left. \left. \left. + \int_{\alpha(t_0)}^{t_{n-1}} h_n(t_n) \varphi_n(u(t_n)) dt_n \right] \right. \right. \\
 &\quad \left. \left. \times dt_{n-1} \dots \right] dt_2 \right] dt_1 \\
 &+ \int_{\alpha(t_0)}^{\alpha(T)} h_1(t_1) \left[ f_1(t_1) \varphi_1(u(t_1)) + \int_{\alpha(t_0)}^{t_1} h_2(t_2) \right. \\
 &\quad \times \left[ f_2(t_2) \varphi_2(u(t_2)) + \dots \right. \\
 &\quad \left. \left. + \int_{\alpha(t_0)}^{t_{n-2}} h_{n-1}(t_{n-1}) \right. \right. \\
 &\quad \left. \left. \times \left[ f_{n-1}(t_{n-1}) \varphi_{n-1}(u(t_{n-1})) \right. \right. \right. \\
 &\quad \left. \left. \left. + \int_{\alpha(t_0)}^{t_{n-1}} h_n(t_n) \varphi_n(u(t_n)) \right. \right. \right. \\
 &\quad \left. \left. \left. \times dt_n \right] dt_{n-1} \dots \right] dt_2 \right] dt_1. \tag{11}
 \end{aligned}$$

Using monotonicity of some functions, we simplify the above multicomposition in an operator form. The unknown function will be estimated by known functions. Furthermore, we apply our result to retarded nonlinear Volterra-Fredholm type equations for estimation of solutions.

## 2. Preliminaries

Throughout this paper, let  $\mathbb{R}$  denote the set of real numbers,  $\mathbb{R}_+ = [0, +\infty)$  and  $I = [t_0, T]$ . For  $k = 0, 1$ , let  $C^k(M, S)$  denote the class of  $k$ th order continuously differentiable functions defined on the set  $M$  and ranged in the set  $S$ . For simplicity, we use the product  $\prod_{i=1}^n f_i$  and  $\prod_{i=n}^1 f_i$  to present the composition  $f_1 \circ f_2 \circ \dots \circ f_n$  and  $f_n \circ f_{n-1} \circ \dots \circ f_1$ .

**2.1. Monotonization.** First, we monotonize those  $\varphi_i$ s in inequality (11). Define

$$w_1(u) := \max_{\tau \in [0, u]} \{\varphi_1(\tau)\},$$

$$w_{i+1}(u) := \max_{\tau \in [0, u]} \left\{ \frac{\varphi_{i+1}(\tau)}{w_i(\tau)} \right\} w_i(u), \quad i = 1, \dots, n, \tag{12}$$

recursively. One can prove that

- (P1) each  $w_i$  is a nondecreasing nonnegative continuous function,
- (P2)  $\varphi_i(u) \leq w_i(u), i = 1, \dots, n$ ,
- (P3)  $w_{i+1}$  has *stronger monotonicity* than  $w_i$ , denoted by  $w_i \propto w_{i+1}, i = 1, 2, \dots, n - 1$ ; that is, by the definition given in [4, 6], the ratios  $w_{i+1}(u)/w_i(u), i = 1, \dots, n - 1$ , are all nondecreasing.

Thus, the sequence  $\{\varphi_i\}$  can be replaced by a larger but monotonous one  $\{w_i\}$  in (11). For a given constant  $u_i > 0$ , define functions  $W_i(u, u_1), i = 1, 2, \dots, n$ , recursively by

$$W_1(u, u_1) := \int_{u_1}^u \frac{ds}{w_1(s)}, \tag{13}$$

$$W_i(u, u_i) := \int_{u_i}^u \frac{w_{i-1}(\prod_{\ell=1}^{i-1} W_\ell^{-1}(s)) ds}{w_i(\prod_{\ell=1}^{i-1} W_\ell^{-1}(s))}, \quad i = 2, \dots, n, \tag{14}$$

where we use  $W_\ell$  and  $W_\ell^{-1}$  to denote  $W_\ell(u, u_\ell)$  and its inverse when there is no confusion. Clearly, they are all strictly increasing.

For given positive constants  $c_i, i = 1, 2, \dots, n$ , define  $\widehat{W}_i(u)$  by

$$\widehat{W}_i(u) := W_i(u) + c_i, \quad \forall u \in \mathbb{R}_+, \quad i = 1, 2, \dots, n, \tag{15}$$

where  $W_i (i = 1, 2, \dots, n)$  are defined by (13) and (14), respectively.

**Lemma 1.** Suppose that  $\varphi_i (i = 1, 2, \dots, n)$  are nonnegative and integrable on  $[t_0, T]$ . Then  $W_i (i = 1, 2, \dots, n)$  are increasing and continuous differentiable functions, and

$$\frac{d}{d\xi} \left( \prod_{i=n}^2 W_i \circ W_1(\xi) \right) = \frac{1}{w_n(\xi)}, \tag{16}$$

$$\begin{aligned} & \frac{d}{d\xi} \left( \prod_{i=n}^2 \widehat{W}_i \circ \widehat{W}_1(\xi) \right) \\ &= \frac{w_{n-1}(\prod_{i=1}^{n-1} W_i^{-1} \circ (\prod_{i=n-1}^2 \widehat{W}_i \circ \widehat{W}_1(\xi)))}{w_n(\prod_{i=1}^{n-1} W_i^{-1} \circ (\prod_{i=n-1}^2 \widehat{W}_i \circ \widehat{W}_1(\xi)))} \\ & \dots \frac{w_1(W_1^{-1} \circ \widehat{W}_1(\xi))}{w_2(W_1^{-1} \circ \widehat{W}_1(\xi))} \frac{1}{w_1(\xi)}, \quad \forall \xi \geq k(T). \end{aligned} \tag{17}$$

*Proof.* By the definition,  $W_i(s) (i = 1, 2, \dots, n)$  are increasing and continuous differentiable functions. From (13) and (14), we have

$$\begin{aligned} & \frac{d}{d\xi} \left( \prod_{i=n}^2 W_i \circ W_1(\xi) \right) \\ &= W_n' \left( \prod_{i=n-1}^1 W_i(\xi) \dots \right) \dots W_2'(W_1(\xi)) W_1'(\xi) \\ &= \frac{w_{n-1}(\prod_{i=1}^{n-1} W_i^{-1}(\prod_{i=n-1}^1 W_i(\xi)))}{w_n(\prod_{i=1}^{n-1} (W_i^{-1}(\prod_{i=n-1}^1 W_i(\xi))))} \\ & \dots \frac{w_1(W_1^{-1}(W_1(\xi)))}{w_2(W_1^{-1}(W_1(\xi)))} \frac{1}{w_1(\xi)} \\ &= \frac{1}{w_n(\xi)}. \end{aligned} \tag{18}$$

Moreover,

$$\begin{aligned} & \frac{d}{d\xi} \left( \prod_{i=n}^2 \widehat{W}_i \circ \widehat{W}_1(\xi) \right) \\ &= \frac{d(W_n(W_{n-1}(\dots(W_2(W_1(\xi) + c_1) + c_2) \dots) + c_{n-1}) + c_n)}{d\xi} \\ &= W_n'(W_{n-1}(\dots(W_2(W_1(\xi) + c_1) + c_2) \dots) + c_{n-1}) \\ & \dots W_2'(W_1(\xi) + c_1) W_1'(\xi) \\ &= w_{n-1} \left( \prod_{i=1}^{n-1} W_i^{-1} \left( W_{n-1}(\dots(W_2(W_1(\xi) + c_1) \right. \right. \right. \\ & \left. \left. \left. + c_2) \dots) + c_{n-1} \right) \right) \end{aligned}$$

$$\begin{aligned} & \times \left( w_n \left( \prod_{i=1}^{n-1} W_i^{-1} \left( W_{n-1} \left( \dots \left( W_2 \left( W_1 (\xi) + c_1 \right) + c_2 \right) \dots \right. \right. \right. \right. \right. \\ & \qquad \qquad \qquad \left. \left. \left. \left. \left. + c_{n-1} \right) \right) \right) \right) \right)^{-1} \\ & \dots \frac{w_1 \left( W_1^{-1} \left( W_1 (\xi) + c_1 \right) \right)}{w_2 \left( W_1^{-1} \left( W_1 (\xi) + c_1 \right) \right)} \frac{1}{w_1 (\xi)}. \end{aligned} \tag{19}$$

By the definitions of  $\widehat{\mathcal{W}}_i(u)$ , we have the relation (17).  $\square$

**2.2. Simplification with Operators.** Let  $h_i, f_i$  be positive continuous functions in (II), where  $i = 1, 2, \dots, n, f_n(t) \equiv 1$ . Define  $\mathcal{H}_i : C([t_0, T]) \rightarrow C([t_0, T])$  by

$$\mathcal{H}_i v(t) := \int_{\alpha(t_0)}^t h_i(s) v(s) ds, \quad i = 1, 2, \dots, n, \tag{20}$$

and define  $\mathcal{F}_i[u] : C([t_0, T]) \rightarrow C([t_0, T])$  by

$$\mathcal{F}_i[u] v(t) := f_i(t) w_i(u(t)) + v(t), \quad i = 1, 2, \dots, n. \tag{21}$$

Having defined those operators, we can enlarge inequality (II) by (12) in the simpler form

$$\begin{aligned} u(t) & \leq k(t) \\ & + \left\{ \prod_{i=1}^{n-1} \left( \mathcal{H}_i \circ \mathcal{F}_i[u] \right) \circ \mathcal{H}_n \circ \mathcal{F}_n[u] \mathbf{0} \right\} (\alpha(t)) \\ & + \left\{ \prod_{i=1}^{n-1} \left( \mathcal{H}_i \circ \mathcal{F}_i[u] \right) \circ \mathcal{H}_n \circ \mathcal{F}_n[u] \mathbf{0} \right\} (\alpha(T)), \\ & \qquad \qquad \qquad \forall t \in [t_0, T], \end{aligned} \tag{22}$$

where  $\mathbf{0}$  denotes a zero function.

**2.3.  $\mathcal{W}$ -Function.** Define  $\mathcal{W}_i(u)(\alpha(T))$  by

$$\begin{aligned} \mathcal{W}_i(u)(\alpha(T)) & := W_i(u) + \left( \prod_{j=1}^{i-1} \mathcal{H}_j \right) \circ (\mathcal{H}_i f_i(t)) (\alpha(T)), \\ & \qquad \qquad \qquad \forall u \in \mathbb{R}_+, \quad i = 1, 2, \dots, n, \end{aligned} \tag{23}$$

where  $W_i$  and  $\mathcal{H}_i$  are defined by (14) and (20), respectively. Since  $W_i (i = 1, 2, \dots, n)$  are continuous functions,  $\mathcal{W}_i (i = 1, 2, \dots, n)$  are also continuous functions.

Define a function

$$\begin{aligned} \widehat{\mathcal{W}}(\xi) & := \prod_{i=n}^2 W_i \circ W_1 (2\xi - c) \\ & \quad - \prod_{i=n}^2 \mathcal{W}_i \circ \mathcal{W}_1 (\xi) (\alpha(T)), \quad \forall \xi > c. \end{aligned} \tag{24}$$

**Lemma 2.** Suppose that  $\varphi_i(u) (i = 1, \dots, n)$  are all continuous such that  $\varphi_i(u) > 0$  for  $u > 0$ ,  $W_i (i = 1, 2, \dots, n)$  defined by

(13) and (14) satisfy  $W_i(+\infty) = +\infty$ . Suppose that  $h_i, f_i$ , and  $\varphi_i (i = 1, \dots, n)$  satisfy

$$\frac{1}{w_n(2\xi - c)} - V(\xi) \geq 0, \quad \forall \xi > c, \tag{25}$$

where

$$\begin{aligned} V(\xi) & := \\ & = \frac{w_{n-1} \left( \prod_{i=1}^{n-1} W_i^{-1} \circ \left( \prod_{i=n-1}^2 \mathcal{W}_i \circ \mathcal{W}_1 (\xi) (\alpha(T)) \right) \right)}{w_n \left( \prod_{i=1}^{n-1} W_i^{-1} \circ \left( \prod_{i=n-1}^2 \mathcal{W}_i \circ \mathcal{W}_1 (\xi) (\alpha(T)) \right) \right)} \\ & \dots \frac{w_1 \left( W_1^{-1} \circ \mathcal{W}_1 (\xi) (\alpha(T)) \right)}{w_2 \left( W_1^{-1} \circ \mathcal{W}_1 (\xi) (\alpha(T)) \right)} \frac{1}{w_1 (\xi)}, \end{aligned} \tag{26}$$

and  $\mathcal{W}_i (i = 1, 2, \dots, n)$  are defined by (23). Then,  $\widehat{\mathcal{W}}(\xi)$  is nondecreasing, and  $\widehat{\mathcal{W}}(\xi) = 0$  has a solution  $\xi = \xi_0$  with  $\xi_0 > c$ .

*Proof.* Using Lemma 1 and condition (25), we have

$$\mathcal{W}'(\xi) = \frac{1}{w_n(2\xi - c)} - V(\xi) \geq 0. \tag{27}$$

Thus  $\widehat{\mathcal{W}}(\xi)$  is a nondecreasing function. Since  $\widehat{\mathcal{W}}(c) < 0$  and  $\lim_{\xi \rightarrow \infty} \widehat{\mathcal{W}}(\xi) = +\infty$ , we see that  $\widehat{\mathcal{W}}(\xi) = 0$  has a solution  $\xi = \xi_0$  with  $\xi_0 > c$ .  $\square$

### 3. Main Result

The following theorem shows that the unknown function is estimated by the given known functions.

**Theorem 3.** Let  $c$  be a positive constant. Suppose that  $h_n(t), f_i(t)$ , and  $h_i(t) \in C(I, \mathbb{R}_+)$ ,  $i = 1, \dots, n - 1$ . Suppose that  $h_i, f_i, h_i$ , and  $\varphi_i (i = 1, \dots, n)$  satisfy the assumption of Lemma 2. Suppose that  $\alpha \in C^1(I, I)$  is nondecreasing such that  $\alpha(t) \leq t$  on  $I$ . Then the unknown  $u(t)$  in (22) is estimated as

$$u(t) \leq \prod_{i=1}^n W_i^{-1} \left\{ \prod_{i=n}^2 \mathcal{W}_i \circ \mathcal{W}_1 (\xi_0) (\alpha(t)) \right\}, \quad \forall t \in [t_0, T]. \tag{28}$$

*Remark 4.* As explained in Remark 2 in [6], different choices of  $u_i$  in the definitions (13)-(14) of  $W_i$  do not affect our results (28).

*Proof.* For convenience, we cite some definitions in the discussion of our proof as follows: for each fixed positive continuous function  $u$ , define  $\mathcal{F}_{ij}[u] : C([t_0, T]) \rightarrow C([t_0, T])$  by

$$\begin{aligned} \mathcal{F}_{ij}[u] v(t) & := \\ & = f_i(t) \frac{w_i \left( W_1^{-1} \left( \dots \left( W_{j-1}^{-1} (u(t)) \right) \dots \right) \right)}{w_j \left( W_1^{-1} \left( \dots \left( W_{j-1}^{-1} (u(t)) \right) \dots \right) \right)} + v(t), \end{aligned} \tag{29}$$

where  $j < i, i = 2, \dots, n, j = 1, 2, \dots, n - 1$ .

From (22), we have

$$\begin{aligned}
 &u(t) \\
 &\leq c + \left\{ \left( \prod_{i=1}^{n-1} \mathcal{H}_i \circ \mathcal{F}_i [u] \right) \circ \mathcal{H}_n \circ \mathcal{F}_n [u] \mathbf{0} \right\} (\alpha(t)) \\
 &\quad + \left\{ \left( \prod_{i=1}^{n-1} \mathcal{H}_i \circ \mathcal{F}_i [u] \right) \circ \mathcal{H}_n \circ \mathcal{F}_n [u] \mathbf{0} \right\} (\alpha(T)), \tag{30} \\
 &\quad \forall t \in [t_0, T_1].
 \end{aligned}$$

Define a function  $z_1(t)$  by the function on the right-hand side of (30). Then,  $z_1(t)$  is a positive and nondecreasing function on  $[t_0, T]$ . Using (30), we have

$$\begin{aligned}
 &u(t) \leq z_1(t), \quad \forall t \in [t_0, T], \tag{31} \\
 &z_1(t_0) = c + \left\{ \left( \prod_{i=1}^{n-1} \mathcal{H}_i \circ \mathcal{F}_i [u] \right) \circ \mathcal{H}_n \circ \mathcal{F}_n [u] \mathbf{0} \right\} (\alpha(T)). \tag{32}
 \end{aligned}$$

Differentiating  $z_1(t)$  with respect to  $t$ , using (31) we have

$$\begin{aligned}
 &z_1'(t) = \alpha'(t) h_1(\alpha(t)) \\
 &\quad \times \left[ f_1(\alpha(t)) w_1(u(\alpha(t))) \right. \\
 &\quad \left. + \left[ \left( \prod_{i=2}^{n-1} \mathcal{H}_i \circ \mathcal{F}_i [u] \right) \circ \mathcal{H}_n \circ \mathcal{F}_n [u] \mathbf{0} \right] (\alpha(t)) \right] \\
 &\leq \alpha'(t) h_1(\alpha(t)) \\
 &\quad \times \left[ f_1(\alpha(t)) w_1(z_1(\alpha(t))) \right. \\
 &\quad \left. + \left[ \left( \prod_{i=2}^{n-1} \mathcal{H}_i \circ \mathcal{F}_i [z_1] \right) \circ \mathcal{H}_n \circ \mathcal{F}_n [z_1] \mathbf{0} \right] (\alpha(t)) \right], \tag{33}
 \end{aligned}$$

for all  $t \in [t_0, T]$ . From (33), we have

$$\begin{aligned}
 &\frac{z_1'(t)}{w_1(z_1(t))} \leq \alpha'(t) h_1(\alpha(t)) \\
 &\quad \times \left[ f_1(\alpha(t)) + \left[ \left( \prod_{i=2}^{n-1} \mathcal{H}_i \circ \mathcal{F}_i [z_1] \right) \right. \right. \\
 &\quad \left. \left. \circ \mathcal{H}_n \circ \mathcal{F}_n [z_1] \mathbf{0} \right] (\alpha(t)) \right], \tag{34}
 \end{aligned}$$

for all  $t \in [t_0, T]$ . Integrating both sides of the above inequality from  $t_0$  to  $t$ , we have

$$\begin{aligned}
 &W_1(z_1(t)) \\
 &\leq W_1(z_1(t_0)) + \mathcal{H}_1(f_1(s))(\alpha(t)) \\
 &\quad + \left[ \mathcal{H}_1 \circ \left( \prod_{i=2}^{n-1} \mathcal{H}_i \circ \mathcal{F}_i [z_1] \right) \circ \mathcal{H}_n \circ \mathcal{F}_n [z_1] \mathbf{0} \right] (\alpha(t)) \\
 &\leq W_1(z_1(t_0)) + \mathcal{H}_1(f_1(s))(\alpha(T)) \\
 &\quad + \left[ \mathcal{H}_1 \circ \left( \prod_{i=2}^{n-1} \mathcal{H}_i \circ \mathcal{F}_i [z_1] \right) \circ \mathcal{H}_n \circ \mathcal{F}_n [z_1] \mathbf{0} \right] (\alpha(t)), \tag{35}
 \end{aligned}$$

for  $t \in [t_0, T]$ .

Let  $z_2(t)$  denote the function on the right-hand side of (35); we can see that  $z_2(t)$  is a positive and nondecreasing function on  $[t_0, T]$ . From (35), we obtain

$$z_1(t) \leq W_1^{-1}(z_2(t)), \quad \forall t \in [t_0, T], \tag{36}$$

$$z_2(t_0) = W_1(z_1(t_0)) + (\mathcal{H}_1 f_1(s))(\alpha(T)). \tag{37}$$

Differentiating  $z_2(t)$  with respect to  $t$ , using (36) we obtain

$$\begin{aligned}
 &z_2'(t) \\
 &= \alpha'(t) h_1(\alpha(t)) \\
 &\quad \times \left[ \int_{\alpha(t_0)}^{\alpha(t)} h_2(t_2) \left[ f_2(t_2) \frac{w_2(z_1(t_2))}{w_1(z_1(t_2))} \right. \right. \\
 &\quad \left. \left. + \left[ \left( \prod_{i=3}^{n-1} \mathcal{H}_i \circ \mathcal{F}_i [z_1] \right) \right. \right. \right. \\
 &\quad \left. \left. \left. \circ \mathcal{H}_n \circ \mathcal{F}_n [z_1] \mathbf{0} \right] \right] dt_2 \right] \\
 &\leq \alpha'(t) h_1(\alpha(t)) \left[ \int_{\alpha(t_0)}^{\alpha(t)} h_2(t_2) \right. \\
 &\quad \times \left[ f_2(t_2) \frac{w_2(W_1^{-1}(z_2(t_2)))}{w_1(W_1^{-1}(z_2(t_2)))} \right. \\
 &\quad \left. \left. + \left[ \left( \prod_{i=3}^{n-1} \mathcal{H}_i \circ \mathcal{F}_i [W_1^{-1} \circ z_2] \right) \right. \right. \right. \\
 &\quad \left. \left. \left. \circ \mathcal{H}_n [W_1^{-1} \circ z_2] \mathbf{0} \right] \right] dt_2 \right], \\
 &\quad \forall t \in [t_0, T]. \tag{38}
 \end{aligned}$$

From (38), we have

$$\begin{aligned} & \frac{z_2'(t) w_1(W_1^{-1}(z_2(t)))}{w_2(W_1^{-1}(z_2(t)))} \\ & \leq \alpha'(t) h_1(\alpha(t)) \int_{\alpha(t_0)}^{\alpha(t)} h_2(t_2) f_2(t_2) dt_2 \\ & \quad + \alpha'(t) h_1(\alpha(t)) \left[ \mathcal{H}_2 \circ \left( \prod_{i=3}^{n-1} \mathcal{H}_i \circ \mathcal{F}_{i2} [z_2] \right) \right. \\ & \quad \left. \circ \mathcal{H}_n \circ \mathcal{F}_{n2} [z_2] \mathbf{0} \right] (\alpha(t)), \end{aligned} \tag{39}$$

for all  $t \in [t_0, T]$ . From (39), we have

$$\begin{aligned} W_2(z_2(t)) & \leq W_2(z_2(t_0)) \\ & \quad + [\mathcal{H}_1 \circ (\mathcal{H}_2 f_2(s))] (\alpha(t)) \\ & \quad + \left[ \mathcal{H}_1 \circ \mathcal{H}_2 \circ \left( \prod_{i=3}^{n-1} \mathcal{H}_i \circ \mathcal{F}_{i2} [z_2] \right) \right. \\ & \quad \left. \circ \mathcal{H}_n \circ \mathcal{F}_{n2} [z_2] \mathbf{0} \right] (\alpha(t)) \\ & \leq W_2(z_2(t_0)) + [\mathcal{H}_1 \circ (\mathcal{H}_2 f_2(s))] (\alpha(T)) \\ & \quad + \left[ \mathcal{H}_1 \circ \mathcal{H}_2 \circ \left( \prod_{i=3}^{n-1} \mathcal{H}_i \circ \mathcal{F}_{i2} [z_2] \right) \right. \\ & \quad \left. \circ \mathcal{H}_n \circ \mathcal{F}_{n2} [z_2] \mathbf{0} \right] (\alpha(t)), \end{aligned} \tag{40}$$

for all  $t \in [t_0, T]$ . Proceeding with the same derivation as in (36) to (40) and so on, we obtain

$$\begin{aligned} W_{n-2}(z_{n-2}(t)) & \leq W_{n-2}(z_{n-2}(t_0)) \\ & \quad + \left[ \left( \prod_{i=1}^{n-2} \mathcal{H}_i \right) f_{n-2}(s) \right] (\alpha(t)) \\ & \quad + \left[ \left( \prod_{i=1}^{n-2} \mathcal{H}_i \right) \circ \mathcal{H}_{n-1} \circ \mathcal{F}_{(n-1)(n-2)} [z_{n-2}] \right. \\ & \quad \left. \circ \mathcal{H}_n \circ \mathcal{F}_{n(n-2)} [z_{n-2}] \mathbf{0} \right] (\alpha(t)) \\ & \leq W_{n-2}(z_{n-2}(t_0)) \\ & \quad + \left[ \left( \prod_{i=1}^{n-2} \mathcal{H}_i \right) f_{n-2}(s) \right] (\alpha(T)) \end{aligned}$$

$$\begin{aligned} & + \left[ \left( \prod_{i=1}^{n-2} \mathcal{H}_i \right) \circ \mathcal{H}_{n-1} \circ \mathcal{F}_{(n-1)(n-2)} [z_{n-2}] \right. \\ & \quad \left. \circ \mathcal{H}_n \circ \mathcal{F}_{n(n-2)} [z_{n-2}] \mathbf{0} \right] (\alpha(t)), \end{aligned} \tag{41}$$

for all  $t \in [t_0, T]$ , where  $W_{n-2}$  is defined by (14).

Define a function  $z_{n-1}(t)$  by the function on the right-hand side of (41). Then  $z_{n-1}(t)$  is a positive and nondecreasing function on  $[t_0, T]$ . From (41), we get

$$z_{n-2}(t) \leq W_{n-2}^{-1}(z_{n-1}(t)), \quad \forall t \in [t_0, T], \tag{42}$$

$$\begin{aligned} z_{n-1}(t_0) & = W_{n-2}(z_{n-2}(t_0)) + \left[ \left( \prod_{i=1}^{n-2} \mathcal{H}_i \right) f_{n-2}(s) \right] (\alpha(T)). \end{aligned} \tag{43}$$

Differentiating  $z_{n-1}(t)$  with respect to  $t$ , we have

$$\begin{aligned} z_{n-1}'(t) & = \alpha'(t) h_1(\alpha(t)) \\ & \quad \times \left[ \left( \prod_{i=2}^{n-1} \mathcal{H}_i \right) \circ \mathcal{F}_{(n-1)(n-2)} [z_{n-2}] \right. \\ & \quad \left. \circ \mathcal{H}_n \circ \mathcal{F}_{n(n-2)} [z_{n-2}] \mathbf{0} \right] (\alpha(t)) \\ & \leq \alpha'(t) h_1(\alpha(t)) \\ & \quad \times \left[ \left( \prod_{i=2}^{n-1} \mathcal{H}_i \right) \circ \mathcal{F}_{(n-1)(n-2)} [W_{n-2}^{-1} \circ z_{n-1}] \right. \\ & \quad \left. \circ \mathcal{H}_n \circ \mathcal{F}_{n(n-2)} [W_{n-2}^{-1} \circ z_{n-1}] \mathbf{0} \right] (\alpha(t)), \end{aligned} \tag{44}$$

for all  $t \in [t_0, T]$ . Then (44) is equivalent to

$$\begin{aligned} & \frac{z_{n-1}'(t) w_{n-2}(W_1^{-1}(W_2^{-1}(\dots(W_{n-2}^{-1}(z_{n-1}(t_{n-1})))\dots)))}{w_{n-1}(W_1^{-1}(W_2^{-1}(\dots(W_{n-2}^{-1}(z_{n-1}(t_{n-1})))\dots)))} \\ & \leq \alpha'(t) h_1(\alpha(t)) \left[ \left( \prod_{i=2}^{n-1} \mathcal{H}_i \right) f_{n-1}(s) \right] (\alpha(t)) \\ & \quad + \alpha'(t) h_1(\alpha(t)) \\ & \quad \times \left[ \left( \prod_{i=2}^{n-1} \mathcal{H}_i \right) \circ \mathcal{H}_n \circ \mathcal{F}_{n(n-1)} [z_{n-1}] \mathbf{0} \right] (\alpha(t)), \end{aligned} \tag{45}$$

for all  $t \in [t_0, T]$ . Integrating both sides of (45) from  $t_0$  to  $t$ , we have

$$\begin{aligned} &W_{n-1}(z_{n-1}(t)) \\ &\leq W_{n-1}(z_{n-1}(t_0)) + \left[ \left( \prod_{i=1}^{n-1} \mathcal{H}_i \right) f_{n-1}(s) \right] (\alpha(t)) \\ &\quad + \left[ \left( \prod_{i=1}^{n-1} \mathcal{H}_i \right) \circ \mathcal{H}_n \circ \mathcal{F}_{n(n-1)} [z_{n-1}] \mathbf{0} \right] (\alpha(t)) \quad (46) \\ &\leq W_{n-1}(z_{n-1}(t_0)) + \left[ \left( \prod_{i=1}^{n-1} \mathcal{H}_i \right) f_{n-1}(s) \right] (\alpha(T)) \\ &\quad + \left[ \left( \prod_{i=1}^{n-1} \mathcal{H}_i \right) \circ \mathcal{H}_n \circ \mathcal{F}_{n(n-1)} [z_{n-1}] \mathbf{0} \right] (\alpha(t)), \end{aligned}$$

for all  $t \in [t_0, T]$ .

Define a function  $z_n(t)$  by the function on the right-hand side of (46); then,  $z_n(t)$  is a positive and nondecreasing function on  $[t_0, T]$ . From (46), we have

$$z_{n-1}(t) \leq W_{n-1}^{-1}(z_n(t)), \quad \forall t \in [t_0, T], \quad (47)$$

$$z_n(t_0) = W_{n-1}(z_{n-1}(t_0)) + \left[ \left( \prod_{i=1}^{n-1} \mathcal{H}_i \right) f_{n-1}(s) \right] (\alpha(T)). \quad (48)$$

Differentiating  $z_n(t)$  with respect to  $t$ , using (47) we have

$$\begin{aligned} &z'_n(t) \\ &= \alpha'(t) h_1(\alpha(t)) \left[ \left( \prod_{i=2}^n \mathcal{H}_i \right) \circ \mathcal{F}_{n(n-1)} [z_{n-1}] \mathbf{0} \right] (\alpha(t)) \\ &\leq \alpha'(t) h_1(\alpha(t)) \\ &\quad \times \left[ \left( \prod_{i=2}^n \mathcal{H}_i \right) \circ \mathcal{F}_{n(n-1)} [W_{n-1}^{-1} \circ z_n] \mathbf{0} \right] (\alpha(t)), \quad (49) \end{aligned}$$

for all  $t \in [t_0, T]$ . From (49), we have

$$\begin{aligned} &\frac{z'_n(t) w_{n-1}(W_1^{-1}(W_2^{-1}(\dots(W_{n-1}^{-1}(z_n(t_n))))\dots))}{w_n(W_1^{-1}(W_2^{-1}(\dots(W_{n-1}^{-1}(z_n(t_n))))\dots))} \\ &= \alpha'(t) h_1(\alpha(t)) \left[ \left( \prod_{i=2}^n \mathcal{H}_i \right) \mathbf{1} \right] (\alpha(t)), \quad (50) \end{aligned}$$

for all  $t \in [t_0, T]$ , where  $\mathbf{1}$  denote the constant function  $v(t) \equiv 1$ . Integrating both sides of the above inequality from  $t_0$  to  $t$ , we obtain

$$W_n(z_n(t)) - W_n(z_n(t_0)) \leq \left[ \left( \prod_{i=1}^n \mathcal{H}_i \right) \mathbf{1} \right] (\alpha(t)), \quad (51)$$

for all  $t \in [t_0, T]$ . From (36), (42), (47), and (51), we have

$$\begin{aligned} z_1(t) &\leq W_1^{-1}(W_2^{-1}(\dots(W_{n-1}^{-1}(z_n(t))))\dots)) \\ &\leq \prod_{i=1}^n W_i^{-1} \left\{ W_n(z_n(t_0)) + \left[ \left( \prod_{i=1}^n \mathcal{H}_i \right) \mathbf{1} \right] (\alpha(t)) \right\}, \quad (52) \end{aligned}$$

for all  $t \in [t_0, T]$ . Substituting (37), (43), and (48) into (52), we have

$$\begin{aligned} z_1(t) &\leq \prod_{i=1}^n W_i^{-1} \left\{ W_n \left( \prod_{i=n-1}^2 \mathcal{W}_i \circ \mathcal{W}_1(z_1(t_0)) \right) (\alpha(T)) \right. \\ &\quad \left. + \left[ \left( \prod_{i=1}^n \mathcal{H}_i \right) \mathbf{1} \right] (\alpha(t)) \right\}, \quad (53) \end{aligned}$$

for all  $t \in [t_0, T]$ . Since  $T$  is chosen arbitrarily, we have

$$z_1(t) \leq \prod_{i=1}^n W_i^{-1} \left\{ W_n \left( \prod_{i=n}^2 \mathcal{W}_i \circ \mathcal{W}_1(z_1(t_0)) \right) (\alpha(t)) \right\}, \quad (54)$$

for all  $t \in [t_0, T]$ . By the definition of  $z_1$  and (32), we have

$$\begin{aligned} &2z_1(t_0) - c \\ &= c + 2 \left\{ \left( \prod_{i=1}^{n-1} \mathcal{H}_i \circ \mathcal{F}_i[u] \right) \circ \mathcal{H}_n \circ \mathcal{F}_n[u] \mathbf{0} \right\} (\alpha(T)) \\ &= z_1(T). \quad (55) \end{aligned}$$

From (54) and (55), we have

$$2z_1(t_0) - c \leq \prod_{i=1}^n W_i^{-1} \left\{ \prod_{i=n}^2 \mathcal{W}_i \circ \mathcal{W}_1(z_1(t_0)) (\alpha(T)) \right\}, \quad (56)$$

or

$$\prod_{i=n}^2 W_i \circ W_1(2\xi - c) - \prod_{i=n}^2 \mathcal{W}_i \circ \mathcal{W}_1(\xi) (\alpha(T)) \leq 0. \quad (57)$$

By the definition of  $\mathcal{W}(\xi)$ , the assumption of Theorem 3, and (57), we observe that

$$\mathcal{W}(z_1(t_0)) \leq 0 = \mathcal{W}(\xi_0). \quad (58)$$

By Lemma 2,  $\mathcal{W}(\xi)$  is increasing. From the last inequality and (31) we have the desired estimation (28).  $\square$

We define the following functions:

$$\begin{aligned} E(u) &= W_2 \{W_1(2u - c)\} \\ &\quad - W_2 \{W_1(u) + (\mathcal{H}_1 f_1(t)) (\alpha(T))\} \quad (59) \\ &\quad - \mathcal{H}_1 \circ (\mathcal{H}_2 \mathbf{1}) (\alpha(T)), \end{aligned}$$

for all  $u > c$ , where  $W_i, i = 1, 2$  are defined by (13) and (14), respectively.

*Example 5.* Let  $n = 2$ ,  $f_1(t)$ ,  $h_i(t)$ ,  $\varphi_i$ ,  $W_i$ ,  $i = 1, 2$ ,  $\alpha$  be as in Theorem 3;  $c$  is a positive constant. Suppose that the function  $E(u)$  is increasing and  $E(u) = 0$  has a solution  $c$  for  $u > c$ . If  $u(t)$  satisfies (22), then

$$\begin{aligned}
 u(t) &\leq W_1^{-1} \left\{ W_2^{-1} \left\{ W_2 \left\{ W_1(c) + (\mathcal{H}_1 f_1)(t) (\alpha(t)) \right. \right. \right. \\
 &\quad \left. \left. \left. - \mathcal{H}_1 \circ (\mathcal{H}_2 \mathbf{1})(\alpha(t)) \right\} \right\}, \quad \forall t \in [t_0, T], \tag{60}
 \end{aligned}$$

where  $W_i^{-1}(i = 1, 2)$  are inverse functions of  $W_i$ , respectively.

*Remark 6.* If  $w_1 = w_2$  in Example 5, then the result in Example 5 will yield the conclusion that appeared in Theorem 2.1' in [17]. Since if  $w_1 = w_2$ , then  $W_2(u) = u - u_2$ ,  $W_2^{-1}(u) = u + u_2$ , from (60), we have  $u(t) \leq W_1^{-1} \{W_1(c) + H_1(t) + H_2(t)\}$  for all  $t \in [t_0, T]$ .

### 4. Application

In this section, we apply our result in Theorem 3 to investigate the retarded Volterra-Fredholm integral equations

$$\begin{aligned}
 x(t) &= x_0 \\
 &+ \int_{t_0}^t F_1 \left\{ s, x(s-d(s)), \int_{t_0}^s F_2 [\tau, x(\tau-d(\tau))] d\tau \right\} ds \\
 &+ \int_{t_0}^T F_1 \left\{ s, x(s-d(s)), \int_{t_0}^s F_2 [\tau, x(\tau-d(\tau))] d\tau \right\} ds, \tag{61}
 \end{aligned}$$

for  $t \in [t_0, T]$ , where  $x \in C(I, \mathbf{R})$ ,  $d \in C^1(I, I)$  is non-decreasing with  $t - d(t) \geq t_0$ ,  $d(t_0) = 0$ ,  $d'(t) < 1$ ,  $F_1 \in C(I \times \mathbf{R}^2, \mathbf{R})$ , and  $F_2 \in C(I \times \mathbf{R}, \mathbf{R})$ . Let  $\beta(t) = t - d(t)$ ; then  $\beta(t) \in C^1(I, I)$ ,  $\beta(t) \leq t$ . Since  $\beta'(t) = 1 - d'(t) > 0$ ,  $\beta(t)$  is an increasing and invertible function.

The following corollary gives the bound on the solution of (61).

**Corollary 7.** *Suppose that the  $F_1, F_2$  in (61) satisfy the conditions*

$$|F_1(s, x, y)| \leq h_1(s) [f_1(s) w_1(|x|) + |y|], \tag{62}$$

$$|F_2(s, x)| \leq h_2(s) w_2(|x|), \tag{63}$$

where  $f_1(s)$ ,  $h_1(s)$ ,  $h_2(s)$ ,  $w_1(s)$ , and  $w_2(s)$  are as in Theorem 3; let  $M = \max_{t \in [t_0, T]} (1/\beta'(t)) < \infty$ . Assume that the function

$$\begin{aligned}
 E^*(u) &= W_2 [W_1(2u - |x_0|)] \\
 &- W_2 \left[ W_1(u) + \int_{\beta(t_0)}^{\beta(T)} M h_1(s) f_1(s) ds \right] \\
 &- \int_{\beta(t_0)}^{\beta(T)} M h_1(s) \times \left[ \int_{\beta(t_0)}^s M h_2(\tau) d\tau \right] ds \tag{64}
 \end{aligned}$$

is increasing and  $E^*(u) = 0$  has a solution  $c$  for  $u > \xi_0$ . If  $x(t)$  is a solution of (61), then

$$\begin{aligned}
 |x(t)| &\leq W_1^{-1} \\
 &\times \left\{ W_2^{-1} \left[ W_2 \left[ W_1(c) \right. \right. \right. \\
 &\quad \left. \left. \left. + \int_{\beta(t_0)}^{\beta(t)} M h_1(\beta^{-1}(s)) f_1(\beta^{-1}(s)) ds \right] \right. \right. \\
 &\quad \left. \left. + \int_{\beta(t_0)}^{\beta(t)} M h_1(\beta^{-1}(s)) \right. \right. \\
 &\quad \left. \left. \times \left[ \int_{\beta(t_0)}^s M h_2(\beta^{-1}(\tau)) d\tau \right] ds \right] \right\}, \\
 &\quad \forall t \in [t_0, T], \tag{65}
 \end{aligned}$$

where  $W_1, W_2, W_1^{-1}$ , and  $W_2^{-1}$  are as in Theorem 3.

*Proof.* Using the conditions (62)-(63) we have

$$\begin{aligned}
 |x(t)| &\leq |x_0| \\
 &+ \int_{t_0}^t h_1(s) \left[ f_1(s) w_1(|x(s-d(s))|) \right. \\
 &\quad \left. + \int_{t_0}^s h_2(\tau) w_2(|x(\tau-d(\tau))|) d\tau \right] ds \\
 &+ \int_{t_0}^T h_1(s) \left[ f_1(s) w_1(|x(s-d(s))|) \right. \\
 &\quad \left. + \int_{t_0}^s h_2(\tau) w_2(|x(\tau-d(\tau))|) d\tau \right] ds \\
 &= |x_0| + \int_{t_0}^t h_1(s) \left[ f_1(s) w_1(|x(\beta(s))|) \right. \\
 &\quad \left. + \int_{t_0}^s h_2(\tau) w_2(|x(\beta(\tau))|) d\tau \right] ds \\
 &+ \int_{t_0}^T h_1(s) \left[ f_1(s) w_1(|x(\beta(s))|) \right. \\
 &\quad \left. + \int_{t_0}^s h_2(\tau) w_2(|x(\beta(\tau))|) d\tau \right] ds \\
 &\leq |x_0| \\
 &+ \int_{t_0}^t h_1(s) \left[ f_1(s) w_1(|x(\beta(s))|) \right. \\
 &\quad \left. + \int_{\beta(t_0)}^{\beta(s)} M h_2(\beta^{-1}(\tau)) w_2(|x(\tau)|) d\tau \right] ds
 \end{aligned}$$



$$\begin{aligned}
& + \int_{t_0}^T h_1(s) \left[ f_1(s) w_1(|x(\beta(s))|) \right. \\
& \quad \left. + \int_{\beta(t_0)}^{\beta(s)} Mh_2(\beta^{-1}(\tau)) w_2(|x(\tau)|) d\tau \right] ds \\
\leq & |x_0| + \int_{\beta(t_0)}^{\beta(t)} Mh_1(\beta^{-1}(s)) \\
& \times \left[ f_1(\beta^{-1}(s)) w_1(|x(s)|) \right. \\
& \quad \left. + \int_{\beta(t_0)}^s Mh_2(\beta^{-1}(\tau)) w_2(|x(\tau)|) d\tau \right] ds \\
& + \int_{\beta(t_0)}^{\beta(T)} Mh_1(\beta^{-1}(s)) \\
& \times \left[ f_1(\beta^{-1}(s)) w_1(|x(s)|) \right. \\
& \quad \left. + \int_{\beta(t_0)}^s Mh_2(\beta^{-1}(\tau)) w_2(|x(\tau)|) d\tau \right] ds,
\end{aligned} \tag{66}$$

for  $t \in [t_0, T]$ , where several changes of variables are made. Applying the result of Theorem 3 to the last inequality, we obtain the desired estimation (65).  $\square$

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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