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Research Article

Global Regularity for the $\overline{\partial}_b$-Equation on *CR* Manifolds of Arbitrary Codimension

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Let M be a $\mathscr{C}^\infty$ compact CR manifold of CR-codimension $\ell \geq 1$ and CR-dimension $n-\ell$ in a complex manifold X of complex dimension $n \geq 3$. In this paper, assuming that M satisfies condition $Y(s)$ for some s with $1 \leq s \leq n-\ell-1$, we prove an L^2-existence theorem and global regularity for the solutions of the tangential Cauchy-Riemann equation for $(0,s)$-forms on M.

1. Introduction and Basic Notations

The tangential Cauchy-Riemann complex (or $\overline{\partial}_b$-complex) was first introduced by Kohn and Rossi [1] for studying the holomorphic extension of CR functions from the boundary of a complex manifold. The closed range property is related to existence and regularity theorems for $\overline{\partial}_b$ and for CR manifolds to a reason of embedding. It is worth then to mention that the $\overline{\partial}_b$-operator has closed range in the L^2-sense on boundaries of smooth bounded pseudoconvex domains in $\mathbb{C}^n$ due to Shaw [2] for all $1 \leq s < n-2$ and Boas and Shaw [3] for $s = n-2$. Later, Kohn [4] obtained results analogue to those of [2, 3] on boundaries of smooth bounded pseudoconvex domains in a complex manifold. Nicoara [5] extended the results of Kohn [4] to compact, orientable, pseudoconvex CR manifold of real dimension $2n-1$, at least five, embedded in $\mathbb{C}^N$, $N \geq n$, leading to global regularity for the $\overline{\partial}_b$-equation on such CR manifolds. The main tool in his proof is that of microlocalizations using a new type of weight functions called strongly CR plurisubharmonic functions (see also [6]).

In addition, Harrington and Raich [7] adapted the microlocal analysis used by Nicoara [5] to establish the closed range property for the $\overline{\partial}_b$-operator on CR manifold of hypersurface type satisfying weak $Y(s)$ condition. More precisely, by using the weighted $\overline{\partial}$-theory, they showed that the complex Green's operator is continuous in the L^2-Sobolev spaces W^k, $k \in \mathbb{N}$, and they further obtained a global solution with $\mathscr{C}^\infty$-regularity for solutions of the $\overline{\partial}_b$-equation for $(0,s)$-forms.

This paper is concerned with proving an L^2-existence theorem for the $\overline{\partial}_b$-Neumann problem on a $\mathscr{C}^\infty$ CR compact manifold M of real dimension $2n-\ell$ ($\ell \geq 1$) that satisfies condition $Y(s)$ for some s with $1 \leq s \leq n-\ell-1$ in an n-dimensional complex manifold X and with establishing the global regularity properties of the $\overline{\partial}_b$-equation. In particular, our $\overline{\partial}_b$-problem is set up in the usual L^2-setting with no weights using our arguments in [8, 9]. Namely, via a partition of unity, we globalize first the local maximal L^2-Sobolev estimates obtained by [10] for $\Box_b$ and patching them together to obtain global ones on M. Further, we explore an L^2-existence theorem for the $\overline{\partial}_b$-equation on M. These L^2 results allow us to prove that the complex Green operator G_b and the Szegö projection operators S_s are continuous in the Sobolev spaces $W^k_{0,s}(M)$ for some s such that $1 \leq s \leq n-\ell-1$ and $k \geq 0$. Furthermore, we obtain a global smooth solution for the $\overline{\partial}_b$-equation given smooth data on M. Before we proceed, we recall first some basic definitions and notations on CR manifolds.

Definition 1. Let M be a $\mathscr{C}^\infty$-manifold of real dimension $2n-\ell$. Then a CR structure on M is given by a complex subbundle $T^{1,0}(M)$ of the complexified tangent bundle $\mathbb{C}T(M) = T(M)\otimes\mathbb{C}$ such that the following conditions are satisfied.

(1) $\dim_{\mathbb{C}} T_z^{1,0}(M) = n-\ell$, where $T_z^{1,0}(M)$ is the fiber at each $z\in M$.

(2) If we define $T^{0,1}(M) = \overline{T^{1,0}(M)}$, then $T^{1,0}(M)\cap T^{0,1}(M)=\{0\}$.

(3) $T^{1,0}(M)$ is involutive (or formally integrable); that is, if L_1 and L_2 are two smooth sections of $T^{1,0}(M)$, defined on an open subset $\mathbb{U}$ of M, then so is their Lie bracket $[L_1,L_2]=L_1L_2-L_2L_1$, for every open subset $\mathbb{U}$ of M.

A $\mathscr{C}^\infty$ manifold M endowed with this CR structure is called a CR manifold of CR-dimension $n-\ell$ and CR codimension ℓ.

Let M be a generic CR manifold of real dimension $2n-\ell$ embedded in an n-dimensional complex manifold X. Such a manifold M can be represented locally in the following form: for each $z\in M$ there exists an open neighborhood U of z in X such that

$$M\cap U=\{\zeta\in U \mid \rho_1(\zeta)=\cdots=\rho_\ell(\zeta)=0\}, \tag{1}$$

where $\{\rho_\nu\}_{\nu=1,\dots,\ell}$ are $\mathscr{C}^\infty$ real-valued functions on U such that

$$\overline{\partial}\rho_1(\zeta)\wedge\cdots\wedge\overline{\partial}\rho_\ell(\zeta)\neq 0 \quad \text{on } M\cap U. \tag{2}$$

The complex subbundle which defines the induced CR structure on M is given by $T^{1,0}(M) = T^{1,0}(X)\cap\mathbb{C}T(M)$. Denote by $\mathscr{C}^\infty_{0,s}(M)$ the space of $(0,s)$-forms with $\mathscr{C}^\infty$-coefficients on M. The involution condition (3) of Definition 1 implies that there is a restriction of the de Rham exterior derivative d to $\mathscr{C}^\infty_{0,s}(M)$, which is defined by $\overline{\partial}_b:\mathscr{C}^\infty_{0,s}(M)\to\mathscr{C}^\infty_{0,s+1}(M)$.

Let us equip X with a Hermitian metric such that $T^{1,0}(X)\perp T^{0,1}(X)$ and consider on M the induced metric, then $T^{1,0}(M)\perp T^{0,1}(M)$. Let $\mathscr{D}_{0,s}(M)$ be the space of $(0,s)$-forms whose coefficients are $\mathscr{C}^\infty$ with compact support in M. We then can define a Hermitian inner product on $\mathscr{D}_{0,s}(M)$ by

$$(\varphi,\psi)=\int_M\langle\varphi,\psi\rangle_z dv, \tag{3}$$

where dv is the volume element associated with the induced metric on M and $\langle\varphi,\psi\rangle_z$ is the pointwise inner product induced on $\mathscr{C}^\infty_{0,s}(M)$ by the metric on $\mathbb{C}T(M)$ at each $z\in M$. Let $\|\varphi\|^2=(\varphi,\varphi)$ be the corresponding norm and $L^2_{0,s}(M)$ the L^2-completion of $\mathscr{D}_{0,s}(M)$ with respect to this norm. Let $\overline{\partial}_b: L^2_{0,s}(M)\to L^2_{0,s+1}(M)$ be the maximal closed extension of the original $\overline{\partial}_b$ on $\mathscr{C}^\infty_{0,s}(M)$. A form $u\in L^2_{0,s}(M)$ is in the domain of $\overline{\partial}_b$ if $\overline{\partial}_b u$, defined in the sense of distributions, belongs to $L^2_{0,s+1}(M)$. In this way, $\overline{\partial}_b$ defines a linear, closed, densely defined operator. Let $\overline{\partial}_b^*: L^2_{0,s+1}(M)\to L^2_{0,s}(M)$ be the L^2-Hilbert space adjoint of $\overline{\partial}_b$ such that $(\varphi,\overline{\partial}_b\psi)=(\overline{\partial}_b^*\varphi,\psi)$ for all ψ in $\mathrm{Dom}(\overline{\partial}_b)$ and φ in $\mathrm{Dom}(\overline{\partial}_b^*)$. The Kohn-Laplacian $\Box_b$ is defined by

$$\Box_b=\overline{\partial}_b\overline{\partial}_b^*+\overline{\partial}_b^*\overline{\partial}_b:\mathrm{Dom}(\Box_b)\longrightarrow L^2_{0,s}(M), \tag{4}$$

where

$$\begin{aligned}&\mathrm{Dom}(\Box_b)\\&=\{\varphi\in\mathrm{Dom}(\overline{\partial}_b)\cap\mathrm{Dom}(\overline{\partial}_b^*)\\&\quad\subset L^2_{0,s}(M)\mid\overline{\partial}_b\varphi\in\mathrm{Dom}(\overline{\partial}_b^*),\overline{\partial}_b^*\varphi\in\mathrm{Dom}(\overline{\partial}_b)\}.\end{aligned} \tag{5}$$

We recall that the Kohn-Laplacian $\Box_b$ is not elliptic, so it has a characteristic set of dimension ℓ. Let $N(M)$ be the ℓ-dimensional bundle such that

$$\mathbb{C}T(M)=T^{1,0}(M)\oplus T^{0,1}(M)\oplus N(M). \tag{6}$$

Let $N^*(M)$ be the dual bundle of $N(M)$. Let $\gamma\in N^*(M)$, then γ annihillates $T^{1,0}(M)\oplus T^{0,1}(M)$. Thus $N^*(M)$ is called the characteristic bundle. The Levi form of M at a point $z\in M$ is defined as the Hermitian form on $T^{1,0}(M)$ with values in $N(M)$ such that

$$\mathscr{L}_z(L_1,L_2)=i\pi_z([L_1,\overline{L}_2]_z),\quad L_1,L_2\in T^{1,0}(M), \tag{7}$$

where π_z is the projection of $\mathbb{C}T_z(M)$ onto $N_z(M)$.

The Levi form of M at a point $z\in M$ in the direction $\gamma\in N^*(M)$ is the scalar Hermitian form denoted $\mathscr{L}_z(\gamma)$ and is given by

$$\begin{aligned}\mathscr{L}_z(\gamma)&=\langle\mathscr{L}_z(L_1,L_2),\gamma\rangle\\&=i\langle[L_1,\overline{L}_2],\gamma\rangle_z,\quad L_1,L_2\in T^{1,0}(M).\end{aligned} \tag{8}$$

Definition 2 (see [10, Definition 1.2]). A CR manifold M of real dimension $2n-\ell$ and codimension $\ell\geq 1$ in a complex manifold of complex dimension n is said to satisfy condition $Z(s)$, $1\leq s\leq n-\ell-1$, at a point $z\in M$ in the direction $\gamma\in N^*(M)$ if the Levi form $\mathscr{L}_z(\gamma)$ has at least $n-\ell-s+1$ positive eigenvalues or at least $s+1$ negative eigenvalues. M is said to satisfy condition $Y(s)$ at $z\in M$ if it satisfies condition $Z(s)$ for all directions $\gamma\in N_z^*(M)$.

Note that in the hypersurface case, that is, $\ell=1$, the condition $Y(s)$ defined above is equivalent to the classical $Y(s)$ condition of Kohn for hypersurfaces (see, e.g., [11] for more details). In particular, if the CR structure is strictly pseudoconvex; that is, the Levi form of M is positive or negative definite, condition $Y(s)$ holds for all $1\leq s\leq n-2$.

2. L^2-Existence Theory for $\overline{\partial}_b$

Let M be a $\mathscr{C}^\infty$ generic CR manifold of real dimension $2n-\ell$ and codimension $\ell\geq 1$ in a complex manifold X

of complex dimension n. For each point $p_0 \in M$, there is then a neighborhood U of p_0 in X and a local orthonormal basis consisting of smooth vector fields $L_1, \dots, L_{n-\ell}$ for $T^{1,0}(U)$ (see, e.g., [12, Section 7.2; Theorem 3]). The collection of vector fields $\{\overline{L}_1, \dots, \overline{L}_{n-\ell}\}$ forms a local orthonormal basis for $T^{0,1}(U)$. Let $T_1, \dots, T_\ell$ be real vector fields on U such that the set $\{L_1, \dots, L_{n-\ell}, \overline{L}_1, \dots, \overline{L}_{n-\ell}, T_1, \dots, T_\ell\}$ forms a local orthonormal basis for $\mathbb{C}T(U)$. Denote by $\{\omega^1, \dots, \omega^{n-\ell}, \overline{\omega}^1, \dots, \overline{\omega}^{n-\ell}, \gamma_1, \dots, \gamma_\ell\}$ the basis for $\mathbb{C}T^\star(U)$ dual to $\{L_1, \dots, \overline{L}_{n-\ell}, T_1, \dots, T_\ell\}$. In terms of this basis, an element φ in $\mathscr{C}^\infty_{0,s}(U)$ can be uniquely expressed as a sum:

$$\varphi = \sum_{|I|=s} \varphi_I \overline{\omega}^I, \tag{9}$$

where $I = (i_1, i_2, \dots, i_s)$ is an s-tuple of integers with $1 \le i_1 < \cdots < i_s \le n-\ell$ and $\overline{\omega}^I = \overline{\omega}^{i_1} \wedge \cdots \wedge \overline{\omega}^{i_s}$.

We then have

$$\begin{aligned} \overline{\partial}_b \varphi &= \sum_{|I|=s} \sum_{j=1}^{n-\ell} \overline{L}_j (\varphi_I) \overline{\omega}^j \wedge \overline{\omega}^I + \cdots \\ &= \sum_{|J|=s+1} \left(\sum_{j,I} \varepsilon_J^{jI} \overline{L}_j (\varphi_I) \right) \overline{\omega}^J + \cdots , \end{aligned} \tag{10}$$

where ε_J^{jI} is zero if $j \cup \{I\} \neq J$ as sets and is the sign of the permutation that reorders jI as J if $j \cup \{I\} = J$, and the $\cdots$ stands for terms of order zero. Using integration by parts, we obtain

$$\begin{aligned} \overline{\partial}_b^* \varphi &= -\sum_{|I|=s} \sum_{j=1}^{n-\ell} L_j (\varphi_{jI}) \overline{\omega}^I + \cdots \\ &= -\sum_{|K|=s-1} \left(\sum_{j,I} \varepsilon_K^{jI} L_j (\varphi_I) \right) \overline{\omega}^K + \cdots . \end{aligned} \tag{11}$$

For φ in $\mathscr{C}^\infty_{0,s}(\overline{U})$, the subspace of smooth $(0,s)$-forms on U that can be extended smoothly up to and including the boundary, we set

$$\begin{aligned} \|\varphi\|^2_{\mathscr{L}(U)} &= \sum_{j=1}^{n-\ell} \left\| L_j(\varphi) \right\|^2 + \|\varphi\|^2, \\ \|\varphi\|^2_{\overline{\mathscr{L}}(U)} &= \sum_{j=1}^{n-\ell} \left\| \overline{L}_j(\varphi) \right\|^2 + \|\varphi\|^2. \end{aligned} \tag{12}$$

If we further assume that M satisfies condition $Y(s)$ for some s with $1 \le s \le n-\ell-1$, for each $p_0 \in M$, we can find a constant $C = C(p_0) > 0$ such that

$$\|\varphi\|^2_{\mathscr{L}(U)} + \|\varphi\|^2_{\overline{\mathscr{L}}(U)} \le C \left(\left\| \overline{\partial}_b \varphi \right\|^2 + \left\| \overline{\partial}_b^* \varphi \right\|^2 + \|\varphi\|^2 \right) \tag{13}$$

uniformly for all $\varphi \in \mathscr{D}_{0,s}(U)$ (see, e.g., [10]).

Set $L_j = X_{2j-1} + iX_{2j}$; $j = 1, \dots, n-\ell$. The condition $Y(s)$ implies that the real vector $X_1, \dots, X_{2n-2\ell}$ and their commutators of length at most two span the tangent space at each point in U. Thus $X_1, \dots, X_{2n-2\ell}$ satisfy Hörmander's finite rank condition of order two. It follows then from [13, Theorem A] (see also [14]) that there is a positive constant $C = C(U)$ satisfying the following 1/2-subelliptic estimate:

$$\|\varphi\|^2_{1/2(U)} \le C \left(\sum_{i=1}^{2n-2\ell} \|X_i \varphi\|^2 + \|\varphi\|^2 \right), \quad \varphi \in \mathscr{D}_{0,s}(U). \tag{14}$$

Here and always $\|\cdot\|_{k(U)}$ denotes the L^2 Sobolev space k-norm, $\|\cdot\|_{-k}$ is the norm of its dual space, and $\|\cdot\|$ is the usual L^2-norm. We may omit the subscript U from the norm notation when there is no danger of confusion.

Combining the above 1/2-subelliptic estimate with (13), as in [10], we get the following theorem.

Theorem 3. *Let M be a $\mathscr{C}^\infty$ CR manifold of real dimension $2n-\ell$ and codimension $\ell \ge 1$ in a complex manifold X of complex dimension n. Suppose that M satisfies condition $Y(s)$ for some s with $1 \le s \le n-\ell-1$. For each point $p_0 \in M$, there is then an open neighborhood U on which the Kohn Laplacian $\square_b$ satisfies the 1/2-subelliptic estimate*

$$\|\varphi\|_{1/2(U)} \le C \left(\left\| \overline{\partial}_b \varphi \right\|^2 + \left\| \overline{\partial}_b^* \varphi \right\|^2 + \|\varphi\|^2 \right) \tag{15}$$

uniformly for all φ in $\mathscr{D}_{0,s}(U)$.

In addition, if M is compact, the estimate (15) holds uniformly on M for all φ in $\mathscr{C}^\infty_{0,s}(M)$.

Theorem 4 (see [10]). *Let M be given as in Theorem 3 and ϕ the unique solution of the equation $(\square_b + Id)\phi = f$ for $f \in L^2_{0,s}(M)$, where Id is the identity operator. Let $U \subset\subset M$ be a relatively compact subset of M. If the restriction of f to U is in $\mathscr{C}^\infty_{0,s}(U)$, the restriction of ϕ to U is then in $\mathscr{C}^\infty_{0,s}(U)$. In addition, suppose that η and η_1 are two cut-off functions supported in U such that $\eta = 1$ on the support of η_1; then if the restriction of f to U is in the L^2-Sobolev space $W^k_{0,s}(U)$ for some nonnegative integer k, the restriction of $\eta_1\phi$ to U is in $W^{k+1}_{0,s}(U)$ and there is a constant $C_k > 0$ (independent of f) such that*

$$\|\eta_1 \phi\|_{k+1(U)} \le C_k \left(\|\eta f\|_{k(U)} + \|f\| \right). \tag{16}$$

Patching the above local estimates, we obtain the following global one.

Theorem 5. *Let M be a $\mathscr{C}^\infty$ compact CR manifold of real dimension $2n-\ell$ and codimension $\ell \ge 1$ in an n-dimensional complex manifold X. Suppose that M satisfies condition $Y(s)$ for some s with $1 \le s \le n-\ell-1$. Let $\phi \in \mathrm{Dom}(\square_b)$ such that $(\square_b + Id)\phi = f$ for f in $W^k_{0,s}(M)$, $k \ge 0$, then ϕ is in $W^{k+1}_{0,s}(M)$ and there exists a constant $C_k > 0$ (independent of f) such that*

$$\|\phi\|_{k+1(M)} \le C_k \|f\|_{k(M)}. \tag{17}$$

Using Theorem 5 and following an induction argument on k, we get the following result.

Proposition 6. *Let M be given as in Theorem 5. Then the Kohn Laplacian $\square_b$ is hypoelliptic. Moreover, if $\square_b \phi = f$ for f*

in $W^k_{0,s}(M)$, $k \geq 0$, *then* ϕ *is in* $W^{k+1}_{0,s}(M)$ *and there is a constant* $C_k > 0$ *(independent of* f*) such that*

$$\|\phi\|^2_{k+1(M)} \leq C_k \left(\|f\|^2_{k(M)} + \|\phi\|^2\right). \tag{18}$$

Let

$$\mathscr{H}^b_{0,s}(M) = \left\{\alpha \in \operatorname{Dom}\left(\overline{\partial}_b\right) \cap \operatorname{Dom}\left(\overline{\partial}^\star_b\right) \subset L^2_{0,s}(M) \mid \overline{\partial}_b\alpha = \overline{\partial}^\star_b\alpha = 0\right\} \tag{19}$$

be the closed subspace of $L^2_{0,s}(M)$ consisting of harmonic forms and

$$^\perp\mathscr{H}^b_{0,s}(M) = \left\{\alpha \in L^2_{0,s}(M) \mid (\alpha, \phi) = 0 \ \forall \phi \in \mathscr{H}^b_{0,s}(M)\right\}. \tag{20}$$

The main L^2-result is the following theorem.

Theorem 7. *Let* M *be a* $\mathscr{C}^\infty$ *compact CR manifold of real dimension* $2n - \ell$ *and codimension* $\ell \geq 1$ *in an* n*-dimensional complex manifold* X*. Suppose that* M *satisfies condition* $Y(s)$ *for some* s *such that* $1 \leq s \leq n - \ell - 1$*. Then the following holds.*

(1) *The space of harmonic* $(0, s)$*-forms* $\mathscr{H}^b_{0,s}(M)$ *is of finite dimensional.*

(2) *The operators* $\overline{\partial}_b : L^2_{0,s}(M) \to L^2_{0,s+1}(M)$, $\overline{\partial}^\star_b : L^2_{0,s+1}(M) \to L^2_{0,s}(M)$, *and* $\Box_b = \overline{\partial}_b\overline{\partial}^*_b + \overline{\partial}^*_b\overline{\partial}_b : \operatorname{Dom}(\Box_b) \to L^2_{0,s}(M)$ *have closed ranges.*

(3) *The complex Green operator* $G_b : L^2_{0,s}(M) \to \operatorname{Dom}(\Box_b)$ *exists and is a compact operator in* $L^2_{0,s}(M)$.

(4) *For any* f *in* $L^2_{0,s}(M)$*, we have*

$$f = \overline{\partial}_b\overline{\partial}^\star_b G_b f + \overline{\partial}^\star_b\overline{\partial}_b G_b f + H^b_{0,s} f, \tag{21}$$

where $H^b_{0,s}$ *is the orthogonal projection of* $L^2_{0,s}(M)$ *onto* $\mathscr{H}^b_{0,s}(M)$.

(5) $G_b H^b_{0,s} = H^b_{0,s} G_b = 0$. $G_b\Box_b = \Box_b G_b = Id - H^b_{0,s}$ *on* $\operatorname{Dom}(\Box_b)$.

(6) *If* G_b *is defined on* $L^2_{0,s+1}(M)$ *(resp.,* $L^2_{0,s-1}(M)$*),* $\overline{\partial}_b G_b = G_b\overline{\partial}_b$ *on* $\operatorname{Dom}(\overline{\partial}_b)$ *(resp.,* $\overline{\partial}^\star_b G_b = G_b\overline{\partial}^\star_b$ *on* $\operatorname{Dom}(\overline{\partial}^\star_b)$*).*

(7) *If* f *is in* $L^2_{0,s}(M)$ *such that* $\overline{\partial}_b f = 0$ *and* $f \perp \mathscr{H}^b_{0,s}(M)$*, then* $f = \overline{\partial}_b\overline{\partial}^\star_b G_b f$ *and* $u = \overline{\partial}^\star_b G_b f$ *is the unique solution to the equation* $\overline{\partial}_b u = f$ *which is orthogonal to* $\operatorname{Ker}(\overline{\partial}_b)$ *and satisfies* $\|u\|^2 \leq C\|f\|^2$.

(8) $G_b(\mathscr{C}^\infty_{0,s}(M)) \subseteq \mathscr{C}^\infty_{0,s}(M)$*, and for each* $k \in \mathbb{R}$ *there is a positive constant* C_s *such that the estimate* $\|G_b f\|_{k+1} \leq C_s\|f\|_k$ *holds uniformly for all* f *in* $\mathscr{C}^\infty_{0,s}(M)$.

Proof. Since M is compact, via a partition of unity, the estimate (15) holds globally on M. Suppose that f_k is a sequence in $\operatorname{Dom}(\overline{\partial}_b) \cap \operatorname{Dom}(\overline{\partial}^*_b) \cap L^2_{0,s}(M)$ such that $\|f_k\|$ is bounded, $\overline{\partial}_b f_k \to 0$ in the $L^2_{0,s+1}(M)$-norm and $\overline{\partial}^*_b f_k \to 0$ in the $L^2_{0,s-1}(M)$-norm as $k \to \infty$. Thus, we have $\|f_k\|_{1/2(M)} \leq c$ for some constant c. By Rellich's Lemma, the inclusion map $i_M : W^{1/2}_{0,s}(M) \to L^2_{0,s}(M)$ is compact; we can then extract a subsequence of f_k which converges in $L^2_{0,s}(M)$. Then the hypotheses of Theorem 1.1.3 in Hörmander [15] are satisfied which implies that $\mathscr{H}^b_{0,s}(M)$ is finite dimensional and the estimate

$$\|f\|^2 \leq C\left(\left\|\overline{\partial}_b f\right\|^2 + \left\|\overline{\partial}^*_b f\right\|^2\right) \tag{22}$$

holds for every f in $\operatorname{Dom}(\overline{\partial}_b) \cap \operatorname{Dom}(\overline{\partial}^*_b)$ with $f \perp \mathscr{H}^b_{0,s}(M)$.

By Theorem 1.1.2 in [15], we then conclude that the operators $\overline{\partial}_b : L^2_{0,s}(M) \to L^2_{0,s+1}(M)$ and $\overline{\partial}^\star_b : L^2_{0,s}(M) \to L^2_{0,s-1}(M)$ have closed ranges. We obtain also from (22) that

$$\|f\| \leq C\|\Box_b f\|, \quad f \in \operatorname{Dom}(\Box_b), \ f \perp \mathscr{H}^b_{0,s}(M). \tag{23}$$

This estimate implies that $\Box_b$ is one-to-one and in view of Theorem 1.1.1 in [15] that the range of $\Box_b$ is closed. It forces, since $\Box_b$ is self-adjoint, the strong Hodge decomposition:

$$\begin{aligned} L^2_{0,s}(M) &= \operatorname{Range}(\Box_b) \oplus \mathscr{H}^b_{0,s}(M) \\ &= \overline{\partial}_b\overline{\partial}^*_b \operatorname{Dom}(\Box_b) \oplus \overline{\partial}^*_b\overline{\partial}_b \operatorname{Dom}(\Box_b) \oplus \mathscr{H}^b_{0,s}(M). \end{aligned} \tag{24}$$

Thus $\Box_b : \operatorname{Dom}(\Box_b) \to {}^\perp\mathscr{H}^b_{0,s}(M)$ is one-to-one and onto. This implies the existence of the complex Green operator $G_b : L^2_{0,s}(M) \to \operatorname{Dom}(\Box_b)$ as a unique operator that inverts $\Box_b$ on ${}^\perp\mathscr{H}^b_{0,s}(M)$. The operator G_b is defined as follows: if f is in $\operatorname{Range}(\Box_b)$, we define $G_b f = \phi$, where ϕ is the unique solution of $\Box_b\phi = f$ with $f \perp \mathscr{H}^b_{0,s}(M)$. G_b is extended to the whole $L^2_{0,s}(M)$ space by setting $G_b = 0$ on $\mathscr{H}^b_{0,s}(M)$. The boundedness of G_b in $L^2_{0,s}(M)$ follows from (23).

To show that G_b is compact in $L^2_{0,s}(M)$, it suffices to show compactness on ${}^\perp\mathscr{H}^b_{0,s}(M)$ (since $G_b \equiv 0$ on $\mathscr{H}^b_{0,s}(M)$). When $f \perp \mathscr{H}^b_{0,s}(M)$ (and hence $G_b f \perp \mathscr{H}^b_{0,s}(M)$), the integration by parts, Cauchy-Schwarz inequality ($|(u, v)| \leq \|u\|\|v\|$), and (23) imply

$$\begin{aligned} \left\|\overline{\partial}_b G_b f\right\|^2 + \left\|\overline{\partial}^*_b G_b f\right\|^2 &= \left(\overline{\partial}_b G_b f, \overline{\partial}_b G_b f\right) + \left(\overline{\partial}^*_b G_b f, \overline{\partial}^*_b G_b f\right) \\ &= \left(\overline{\partial}^*_b\overline{\partial}_b G_b f, G_b f\right) + \left(\overline{\partial}_b\overline{\partial}^*_b G_b f, G_b f\right) \\ &= (f, G_b f) \leq \|f\| \, \|G_b f\| \leq C\|f\|^2. \end{aligned} \tag{25}$$

By applying (15) to $G_b f$ and using (23), we get

$$\begin{aligned} \|G_b f\|^2_{1/2(M)} &\leq C\left(\left\|\overline{\partial}_b G_b f\right\|^2 + \left\|\overline{\partial}^*_b G_b f\right\|^2 + \|G_b f\|^2\right) \\ &\leq K\|f\|^2, \end{aligned} \tag{26}$$

where K is a positive constant. Thus the compactness of G_b in $L^2_{0,s}(M)$ follows from Rellich's Lemma.

The assertions in (5) follow immediately from the definition of G_b. For assertion (6), if $f \in \mathrm{Dom}(\overline{\partial}_b)$ and G_b is also defined on $L^2_{0,s+1}(M)$, by (21) and the first assertion of (5), we have

$$\begin{aligned} G_b\overline{\partial}_b f &= G_b\overline{\partial}_b\overline{\partial}_b^*\overline{\partial}_b G_b f \\ &= G_b\left(\overline{\partial}_b\overline{\partial}_b^* + \overline{\partial}_b^*\overline{\partial}_b\right)\overline{\partial}_b G_b f \\ &= G_b\Box_b\overline{\partial}_b G_b f = \overline{\partial}_b G_b f. \end{aligned} \tag{27}$$

A similar equation holds for $\overline{\partial}_b^*$. Assertions (1)–(6) have been established.

To show assertion (7), if $f \perp \mathcal{H}^b_{0,s}(M)$ and $\overline{\partial}_b f = 0$, then $\overline{\partial}_b\overline{\partial}_b^*\overline{\partial}_b G_b f = 0$ as well (from (21)). Consequently, $\|\overline{\partial}_b^*\overline{\partial}_b G_b f\|^2 = (\overline{\partial}_b\overline{\partial}_b^*\overline{\partial}_b G_b f, \overline{\partial}_b G_b f) = 0$, since $\overline{\partial}_b G_b f \in \mathrm{Dom}(\overline{\partial}_b^*)$, and hence $\overline{\partial}_b^*\overline{\partial}_b G_b f = 0$. Thus $f = \overline{\partial}_b(\overline{\partial}_b^* G_b f)$ and $u = \overline{\partial}_b^* G_b f$ is orthogonal to $\mathrm{Ker}(\overline{\partial}_b)$. Following assertion (3) and the fact that G_b is bounded, u satisfies the following L^2-estimate:

$$\begin{aligned} \|u\|^2 &= \left\|\overline{\partial}_b^* G_b f\right\|^2 = \left(\overline{\partial}_b^* G_b f, \overline{\partial}_b^* G_b f\right) \\ &= \left(\overline{\partial}_b\overline{\partial}_b^* G_b f, G_b f\right) = \left(\left(\overline{\partial}_b\overline{\partial}_b^* + \overline{\partial}_b^*\overline{\partial}_b\right) G_b f, G_b f\right) \\ &= \left(f, G_b f\right) \le \|f\|\,\|G_b f\| \le C\|f\|^2. \end{aligned} \tag{28}$$

Finally, we show assertion (8); if $f \in \mathcal{C}^\infty_{0,s}(M)$, then $f - H^b_{0,s} f \in \mathcal{C}^\infty_{0,s}(M)$ and, since M is compact, $f \in \mathrm{Dom}(\Box_b)$. On other hand, from assertion (5), $\Box_b G_b f = f - H^b_{0,s} f$. Since $\Box_b$ is hypoelliptic, by Proposition 6, $G_b f \in \mathcal{C}^\infty_{0,s}(M)$.

Again Proposition 6 implies

$$\begin{aligned} \|G_b f\|_{k+1(M)} &\le C_k\left(\|\Box_b G_b f\|_{k(M)} + \|G_b f\|\right) \\ &\le C_k\left(\|f\|_{k(M)} + \left\|H^b_{0,s} f\right\|_{k(M)} + (\text{const.})\,\|f\|\right) \\ &\le C\|f\|_{k(M)}. \end{aligned} \tag{29}$$

Here we have used the fact that $\mathcal{H}^b_{0,s}(M)$ is of finite dimension to conclude the estimate

$$\left\|H^b_{0,s} f\right\|_{k(M)} \le C_k\left\|H^b_{0,s} f\right\| \le C_k\|f\|_{k(M)} \tag{30}$$

for some constant C_k. The theorem is proved. □

3. Sobolev Space Estimates

In this section, we prove that the complex Green operator G_b, the canonical solution operators $\overline{\partial}_b G_b$ and $\overline{\partial}_b^* G_b$, and the Szegö projection S_s operators enjoy some regularity properties in the L^2-Sobolev spaces $W^k_{0,s}(M)$, $k \ge 0$, for some s with $1 \le s \le n-\ell-1$. Furthermore, we obtain a global regularity for the solutions of the $\overline{\partial}_b$-equation. By the same way for bounded pseudoconvex domains, a differential operator is said to be exactly regular if it maps all L^2-Sobolev spaces $W^k_{0,s}(M)$ $(k \ge 0)$ to themselves and globally regular if it maps the space $\mathcal{C}^\infty_{0,s}(M)$ continuously to itself.

3.1. Continuity of the Complex Green Operator. We prove first the continuity of the complex Green operator G_b on $W^k_{0,s}(M)$, $k \ge 0$.

Theorem 8. *Let M be a $\mathcal{C}^\infty$ compact CR manifold of real dimension $2n-\ell$ and codimension $\ell \ge 1$ in an n-dimensional complex manifold X. Suppose that M satisfies condition $Y(s)$ for some s with $1 \le s \le n-\ell-1$. Then the complex Green operator G_b is continuous on the Sobolev space $W^k_{0,s}(M)$, $k \ge 0$; that is, there is a constant $C = C(k) > 0$ such that*

$$\|G_b f\|_{k(M)} \le C\|f\|_{k(M)}, \quad f \in W^k_{0,s}(M). \tag{31}$$

Proof. We consider the special case when $k = 0, 1, 2, 3, \ldots$. Indeed the general case is then derived by means of interpolation of linear operators. Since M is compact, it is easy to show that $\mathcal{C}^\infty_{0,s}(M)$ is a dense subspace in $W^k_{0,s}(M)$. Further, by Theorem 7 (8), we have $G_b f \in \mathcal{C}^\infty_{0,s}(M)$ for $f \in \mathcal{C}^\infty_{0,s}(M)$. Thus it suffices to establish (31) for $f \in \mathcal{C}^\infty_{0,s}(M)$. For $k = 0$, (31) follows from (23).

For each $k \ge 0$, let $\Lambda^k(\xi)$ be a pseudodifferential operator of order k with symbol $(1+|\xi|^2)^{k/2}$. Let U be an open neighborhood of ζ in M and let η and η_1 be two cutoff functions with supports in U such that $\eta = 1$ on $\mathrm{supp}\,\eta_1$; then $\eta\Lambda^k\eta_1 f \in \mathcal{D}_{0,s}(U)$ whenever $f \in \mathcal{D}_{0,s}(U)$.

Recall that the compactness of G_b in $L^2_{0,s}(U)$ is equivalent to the compactness estimate: for every $\epsilon > 0$ there is a constant $C(\epsilon) > 0$ such that for every $\varphi \in \mathrm{Dom}(\overline{\partial}_b) \cap \mathrm{Dom}(\overline{\partial}_b^*)$

$$\|\varphi\|^2 \le \epsilon Q_b(\varphi, \varphi) + C(\epsilon)\,\|\varphi\|^2_{-1(U)}, \tag{32}$$

where $Q_b(\varphi, \varphi) = (\overline{\partial}_b\varphi, \overline{\partial}_b\varphi) + (\overline{\partial}_b^*\varphi, \overline{\partial}_b^*\varphi)$. For this estimate and further results on the compactness of the complex Green operator see, e.g., [16–19].

Applying (32) for $\eta\Lambda^k\eta_1 G_b f$, we obtain

$$\begin{aligned} \left\|\eta\Lambda^k\eta_1 G_b f\right\|^2 &\le \epsilon Q_b\left(\eta\Lambda^k\eta_1 G_b f, \eta\Lambda^k\eta_1 G_b f\right) \\ &\quad + C(\epsilon)\left\|\eta\Lambda^k\eta_1 G_b f\right\|^2_{-1(U)}. \end{aligned} \tag{33}$$

We sometimes use A for $\eta\Lambda^k\eta_1$ and A^* for its formal adjoint, which is also a tangential operator of order k. We estimate the first term on the right hand side in (33), it is a standard consequence of [20, Corollary 3.1] (or [11, Lemma 2.4.2]) that

$$\begin{aligned} Q_b\left(AG_b f, AG_b f\right) &= \mathrm{Re}\, Q_b\left(G_b f, A^* AG_b f\right) \\ &\quad + \mathcal{O}\left(|||DG_b f|||^2_{k-1(U)}\right) \\ &\le \mathrm{Re}\, Q_b\left(G_b f, A^* AG_b f\right) + C\|G_b f\|^2_{k(U)}. \end{aligned} \tag{34}$$

Here we have used the fact that the tangential derivative D^α of order $|\alpha| = \lambda$ satisfies the tangential Sobolev estimate $|||D^\alpha f|||_r \leq \|f\|_{r+\lambda}$.

Taking $v = A^* A f$ in the form $Q_b(G_b u, v) = (u, v)$, we get

$$Q_b(AG_b f, AG_b f) \leq \operatorname{Re}(f, A^* AG_b f) + C\|G_b f\|^2_{k(U)} \leq |(f, A^* AG_b f)| + C\|G_b f\|^2_{k(U)}. \tag{35}$$

The Cauchy-Schwarz inequality implies

$$Q_b(AG_b f, AG_b f) \leq \|Af\| \|AG_b f\| + C\|G_b f\|^2_{k(U)}. \tag{36}$$

Inequality (33) becomes

$$\left\|\eta\Lambda^k \eta_1 G_b f\right\|^2 \leq \epsilon\|f\|_{k(U)}\|G_b f\|_{k(U)} + C(\epsilon)\left\|\eta\Lambda^k \eta_1 G_b f\right\|^2_{-1(U)}. \tag{37}$$

Summing over a partition of unity subordinate to an open covering of M by patches $\{U_i\}_{i=1}^m$, we obtain estimate like (37) on each of these patches and using the interior regularity properties, we get

$$\|G_b f\|^2_{k(M)} \leq \epsilon\|f\|_{k(M)}\|G_b f\|_{k(M)} + C(\epsilon)\|G_b f\|^2_{k-1(M)}. \tag{38}$$

The first term in the right-hand side of (38) is estimated by $\epsilon(\text{s.c.})\|G_b f\|^2_{k(M)} + \epsilon(\text{l.c.})\|f\|^2_{k(M)}$, where s.c. and l.c. denote a small and a large constants, respectively, in the inequality $|ab| \leq (\text{s.c.})a^2 + (\text{l.c.})b^2$. The second term is estimated by interpolation of Sobolev norms ($\|G_b f\|^2_{k-1(M)} \leq \varepsilon\|G_b f\|^2_{k(M)} + C(\varepsilon)\|G_b f\|^2$) and then by using the continuity of G_b in $L^2_{0,s}(M)$ with L^2-bounded norm.

Adding up the analogues terms and absorbing, by choosing ϵ and ε to be small enough, $\|G_b f\|^2_{k(M)}$ into the left, this gives

$$\|G_b f\|^2_{k(M)} \leq C\|f\|^2_{k(M)} + K\|f\|^2, \tag{39}$$

where $C = C(\epsilon, k) > 0$ and $K = K(\epsilon, k) > 0$. The embedding Sobolev space implies (31) for $k = 0, 1, 2, 3, \ldots$. The general case is obtained from interpolation of linear operators. As mentioned above, the density of $\mathscr{C}^\infty_{0,s}(M)$ in $W^k_{0,s}(M)$ passes (31) to forms f in $W^k_{0,s}(M)$. This proves the continuity of G_b in $W^k_{0,s}(M)$. □

Corollary 9. *Let M be given as in Theorem 8, then the canonical solution operators $\overline{\partial}_b G_b$ and $\overline{\partial}^*_b G_b$ are continuous on $W^k_{0,s}(M)$ for all $k \geq 0$.*

Proof. We argue by induction on k. The case when $k = 0$ follows from (25). Suppose that the assertions hold for positive integers less than k and assume that ζ, U, η, and η_1 are given as in the proof of Theorem 8. By the interior elliptic regularity properties, we prove first a priori estimate for $\overline{\partial}_b G_b f$ and $\overline{\partial}^*_b G_b f$ with $f \in \mathscr{D}_{0,s}(U)$ as follows:

$$\begin{aligned}
&\left\|\eta\Lambda^k\eta_1\overline{\partial}_b G_b f\right\|^2 + \left\|\eta\Lambda^k\eta_1\overline{\partial}^*_b G_b f\right\|^2 \\
&= \left(\eta\Lambda^k\eta_1\overline{\partial}_b G_b f, \overline{\partial}_b\eta\Lambda^k\eta_1 G_b f\right) \\
&\quad + \left(\eta\Lambda^k\eta_1\overline{\partial}^*_b G_b f, \overline{\partial}^*_b\eta\Lambda^k\eta_1 G_b f\right) \\
&\quad + \mathscr{O}\left(\left(\left\|\eta\Lambda^k\eta_1\overline{\partial}_b G_b f\right\| + \left\|\eta\Lambda^k\eta_1\overline{\partial}^*_b G_b f\right\|\right)\|G_b f\|_{k(U)}\right) \\
&= \left(\eta\Lambda^k\eta_1\overline{\partial}^*_b\overline{\partial}_b G_b f, \eta\Lambda^k\eta_1 G_b f\right) \\
&\quad + \left(\eta\Lambda^k\eta_1\overline{\partial}_b\overline{\partial}^*_b G_b f, \eta\Lambda^k\eta_1 G_b f\right) \\
&\quad + \mathscr{O}\left(\left(\left\|\eta\Lambda^k\eta_1\overline{\partial}_b G_b f\right\| + \left\|\eta\Lambda^k\eta_1\overline{\partial}^*_b G_b f\right\|\right)\|G_b f\|_{k(U)} \right. \\
&\qquad \left. + \|G_b f\|^2_{k(U)}\right) \\
&= \left(\eta\Lambda^k\eta_1\Box_b G_b f, \eta\Lambda^k\eta_1 G_b f\right) \\
&\quad + \mathscr{O}\left(\left(\left\|\eta\Lambda^k\eta_1\overline{\partial}_b G_b f\right\| + \left\|\eta\Lambda^k\eta_1\overline{\partial}^*_b G_b f\right\|\right)\|G_b f\|_{k(U)} \right. \\
&\qquad \left. + \|G_b f\|^2_{k(U)}\right) \\
&\leq C_1\|f\|_{k(U)}\|G_b f\|_{k(U)} \\
&\quad + C_2\left(\left(\left\|\eta\Lambda^k\eta_1\overline{\partial}_b G_b f\right\| + \left\|\eta\Lambda^k\eta_1\overline{\partial}^*_b G_b f\right\|\right)\|G_b f\|_{k(U)} \right. \\
&\qquad \left. + \|G_b f\|^2_{k(U)}\right).
\end{aligned} \tag{40}$$

Summing over a partition of unity, using the small and large constants for the resulting terms $\|f\|_k\|G_b f\|_k$, $\|\overline{\partial}_b G_b f\|_k\|G_b f\|_k$, and $\|\overline{\partial}^*_b G_b f\|_k\|G_b f\|_k$, using (31) and adding up the analogues terms, we see that the terms on the right-hand side containing $\|\overline{\partial}_b G_b f\|^2_k$ and $\|\overline{\partial}^*_b G_b f\|^2_k$ can be absorbed into the left hand side. We therefore obtain

$$\left\|\overline{\partial}_b G_b f\right\|^2_{k(M)} + \left\|\overline{\partial}^*_b G_b f\right\|^2_{k(M)} \leq C\|f\|^2_{k(M)}, \quad f \in \mathscr{D}_{0,s}(M). \tag{41}$$

This completes the induction on k for the norms of $\overline{\partial}_b G_b$ and $\overline{\partial}^*_b G_b$. By the density of $\mathscr{C}^\infty_{0,s}(M)$ in $W^k_{0,s}(M)$, the estimates extend to forms in $W^k_{0,s}(M)$. As before, the general case is obtained from interpolation of linear operators. Then $\overline{\partial}_b G_b$ and $\overline{\partial}^*_b G_b$ are continuous on $W^k_{0,s}(M)$. □

3.2. Exact and Global Regularity Theorems. We now show the expression of the complex Green operator by Szegö projections.

Theorem 10. *The Szegö projections $S_s : L^2_{0,s}(M) \to \operatorname{Ker}(\overline{\partial}_b)$ are given by the following relations:*

$$S_s = Id - \overline{\partial}_b^* \overline{\partial}_b G_b = Id - G_b \overline{\partial}_b^* \overline{\partial}_b, \quad s \geq 0, \tag{42}$$

$$S_{s-1} = Id - \overline{\partial}_b^* G_b \overline{\partial}_b, \quad s \geq 1. \tag{43}$$

Proof. We first show that $\overline{\partial}_b^* \overline{\partial}_b G_b = G_b \overline{\partial}_b^* \overline{\partial}_b$. For $\alpha, \beta \in^{\perp} \mathscr{H}^b_{0,s}(M)$, we observe that

$$\overline{\partial}_b \alpha = 0 \Longrightarrow \overline{\partial}_b^* \overline{\partial}_b G_b \alpha = 0 \Longrightarrow \alpha = \overline{\partial}_b \overline{\partial}_b^* G_b \alpha = G_b \overline{\partial}_b \overline{\partial}_b^* \alpha, \tag{44}$$

$$\overline{\partial}_b^* \beta = 0 \Longrightarrow \overline{\partial}_b \overline{\partial}_b^* G_b \beta = 0 \Longrightarrow \beta = \overline{\partial}_b^* \overline{\partial}_b G_b \beta = G_b \overline{\partial}_b^* \overline{\partial}_b \beta. \tag{45}$$

As Range $(\overline{\partial}_b) \perp \operatorname{Ker}(\overline{\partial}_b^*)$ and Range $(\overline{\partial}_b^*) \perp \operatorname{Ker}(\overline{\partial}_b)$, one has

$$\overline{\partial}_b \alpha = 0 \Longrightarrow \overline{\partial}_b G_b \alpha = 0, \tag{46}$$

$$\overline{\partial}_b^* \beta = 0 \Longrightarrow \overline{\partial}_b^* G_b \beta = 0. \tag{47}$$

Any $f \perp \mathscr{H}^b_{0,s}(M)$ can then be written as $f = \alpha + \beta$ so that $\overline{\partial}_b \alpha = 0$ and $\overline{\partial}_b^* \beta = 0$. By (45) and (46), we then have

$$\begin{aligned} \overline{\partial}_b^* \overline{\partial}_b G_b f &= \overline{\partial}_b^* \overline{\partial}_b G_b (\alpha + \beta) = \overline{\partial}_b^* \overline{\partial}_b G_b \beta \\ &= G_b \overline{\partial}_b^* \overline{\partial}_b \beta = G_b \overline{\partial}_b^* \overline{\partial}_b f. \end{aligned} \tag{48}$$

This implies the second equality in (42). Now, If $f \in \operatorname{Ker}(\overline{\partial}_b)$, then $(Id - G_b \overline{\partial}_b^* \overline{\partial}_b) f = f$, so the expression for S_s holds. Next, if $f \perp \operatorname{Ker}(\overline{\partial}_b)$ and hence $f \perp \mathscr{H}^b_{0,s}(M)$, so $f = \overline{\partial}_b \overline{\partial}_b^* G_b f + \overline{\partial}_b^* \overline{\partial}_b G_b f$ and $u = \overline{\partial}_b^* \overline{\partial}_b G_b f$ is the canonical solution to the equation $\overline{\partial}_b u = \overline{\partial}_b f$. Thus $\overline{\partial}_b (f - u) = 0$, that is, $f - u \in \operatorname{Ker}(\overline{\partial}_b)$. We claim that $u \perp \operatorname{Ker}(\overline{\partial}_b)$. Indeed, for all $g \in \operatorname{Ker}(\overline{\partial}_b)$ one has $(u, g) = (\overline{\partial}_b^* \overline{\partial}_b G_b f, g) = (\overline{\partial}_b G_b f, \overline{\partial}_b g) = 0$. Since $f \perp \operatorname{Ker}(\overline{\partial}_b)$, it turns out that $f - u \perp \operatorname{Ker}(\overline{\partial}_b)$ so $f - u = 0$ and then $0 = f - u = (Id - \overline{\partial}_b^* \overline{\partial}_b G_b f)$. This proves (42). Similarly, we get (43). □

Theorem 11. *Let M be given as in Theorem 8. Then the Szegö projections operators S_{s-1} and S_s are continuous in the Sobolev spaces $W^k_{0,s-1}(M)$ and $W^k_{0,s}(M)$ for all $k \geq 0$, respectively.*

Proof. We investigate first the continuity of S_{s-1}. For the case $k = 0$, when $f \in L^2_{0,s}(M)$, we have

$$\begin{aligned} \left\| \overline{\partial}_b^* G_b \overline{\partial}_b f \right\|^2 &= \left(\overline{\partial}_b^* G_b \overline{\partial}_b f, \overline{\partial}_b^* G_b \overline{\partial}_b f \right) \\ &= \left(G_b \overline{\partial}_b f, \overline{\partial}_b \overline{\partial}_b^* G_b \overline{\partial}_b f \right) \\ &= \left(G_b \overline{\partial}_b f, \overline{\partial}_b f \right) = \left(\overline{\partial}_b^* G_b \overline{\partial}_b f, f \right) \\ &\leq \left\| \overline{\partial}_b^* G_b \overline{\partial}_b f \right\| \| f \| . \end{aligned} \tag{49}$$

Here we have used the fact that $\overline{\partial}_b \overline{\partial}_b^* G_b \overline{\partial}_b f = \overline{\partial}_b f$, because $\overline{\partial}_b^2 = 0$. The relation (43) thus implies that $\| S_{s-1} f \| \leq C \| f \|$. This proves the continuity in $L^2_{0,s-1}(M)$.

The case $k \geq 1$. Applying (32) for $\varphi = \eta \Lambda^k \eta_1 G_s \overline{\partial}_b f$ on U, we obtain

$$\begin{aligned} \left\| \eta \Lambda^k \eta_1 G_b \overline{\partial}_b f \right\|^2 &\leq \epsilon Q_b \left(\eta \Lambda^k \eta_1 G_b \overline{\partial}_b f, \eta \Lambda^k \eta_1 G_b \overline{\partial}_b f \right) \\ &\quad + C(\epsilon) \left\| \eta \Lambda^k \eta_1 G_b \overline{\partial}_b f \right\|^2_{-1(U)}. \end{aligned} \tag{50}$$

The first term on the right-hand side of (50) is estimated as

$$\begin{aligned} Q_b \left(A G_b \overline{\partial}_b f, A G_b \overline{\partial}_b f \right) &= \left\| \overline{\partial}_b A G_b \overline{\partial}_b f \right\|^2 + \left\| \overline{\partial}^* A G_b \overline{\partial}_b f \right\|^2 \\ &= \left(\overline{\partial}_b A G_b \overline{\partial}_b f, \overline{\partial}_b A G_b \overline{\partial}_b f \right) \\ &\quad + \left(\overline{\partial}_b^* A G_b \overline{\partial}_b f, \overline{\partial}_b^* A G_b \overline{\partial}_b f \right) \\ &= \left(A \overline{\partial}_b G_b \overline{\partial}_b f, \overline{\partial}_b A G_b \overline{\partial}_b f \right) \\ &\quad + \left(A \overline{\partial}_b^* G_b \overline{\partial}_b f, \overline{\partial}_b^* A G_b \overline{\partial}_b f \right) \\ &\quad + \left(\left[\overline{\partial}_b, A \right] G_b \overline{\partial}_b f, \overline{\partial}_b A G_b \overline{\partial}_b f \right) \\ &\quad + \left(\left[\overline{\partial}_b^*, A \right] G_b \overline{\partial}_b f, \overline{\partial}_b^* A G_b \overline{\partial}_b f \right). \end{aligned} \tag{51}$$

The sum of the last two terms on the right-hand side of the preceding equality is estimated by

$$\begin{aligned} &\left\| \left[\overline{\partial}_b, A \right] G_b \overline{\partial}_b f \right\| \left\| \overline{\partial}_b A G_b \overline{\partial}_b f \right\| \\ &\quad + \left\| \left[\overline{\partial}_b^*, A \right] G_b \overline{\partial}_b f \right\| \left\| \overline{\partial}_b^* A G_b \overline{\partial}_b f \right\| \\ &\leq \left\| D G_b \overline{\partial}_b f \right\|_{k-1(U)} \left\| \overline{\partial}_b A G_b \overline{\partial}_b f \right\| \\ &\quad + \left\| D G_b \overline{\partial}_b f \right\|_{k-1(U)} \left\| \overline{\partial}_b^* A G_s \overline{\partial}_b f \right\| \\ &\leq \left\| G_b \overline{\partial}_b f \right\|_{k(U)} \left(\left\| \overline{\partial}_b A G_b \overline{\partial}_b f \right\| + \left\| \overline{\partial}_b^* A G_b \overline{\partial}_b f \right\| \right) \\ &= \mathcal{O} \left((\text{l.c.}) \left\| G_b \overline{\partial}_b f \right\|^2_{k(U)} \right. \\ &\quad \left. + (\text{s.c.}) \left(\left\| \overline{\partial}_b A G_b \overline{\partial}_b f \right\| + \left\| \overline{\partial}_b^* A G_b \overline{\partial}_b f \right\| \right)^2 \right) \\ &= \mathcal{O} \left(\left\| G_b \overline{\partial}_b f \right\|^2_{k(U)} \right). \end{aligned} \tag{52}$$

We then have

$$\begin{aligned} Q_b \left(A G_b \overline{\partial}_b f, A G_b \overline{\partial}_b f \right) &\leq \left(\overline{\partial}_b G_b \overline{\partial}_b f, A^* \overline{\partial}_b A G_b \overline{\partial}_b f \right) \\ &\quad + \left(A \overline{\partial}_b^* G_b \overline{\partial}_b f, \overline{\partial}_b^* A G_b \overline{\partial}_b f \right) \\ &\quad + \mathcal{O} \left(\left\| G_b \overline{\partial}_b f \right\|^2_{k(U)} \right). \end{aligned} \tag{53}$$

The first term on the right-hand side of (53) equals zero due to the fact that $\bar{\partial}_b G_b \bar{\partial}_b f = \bar{\partial}_b^2 G_b f = 0$.

We now analyze the second term as follows:

$$\begin{aligned}
&\left(A\bar{\partial}_b^* G_b\bar{\partial}_b f, \bar{\partial}_b^* A G_b\bar{\partial}_b f\right) \\
&= \left(\bar{\partial}_b A\bar{\partial}_b^* G_b\bar{\partial}_b f, A G_b\bar{\partial}_b f\right) \\
&= \left(A\bar{\partial}_b\bar{\partial}_b^* G_b\bar{\partial}_b f, A G_b\bar{\partial}_b f\right) + \left(\left[\bar{\partial}_b, A\right]\bar{\partial}_b^* G_b\bar{\partial}_b f, A G_b\bar{\partial}_b f\right) \\
&= \left(A\bar{\partial}_b f, A G_b\bar{\partial}_b f\right) + \left(\left[\bar{\partial}_b, A\right]\bar{\partial}_b^* G_b\bar{\partial}_b f, A G_b\bar{\partial}_b f\right) \\
&= \left(\bar{\partial}_b A f, A G_b\bar{\partial}_b f\right) + \left(\left[A, \bar{\partial}_b\right] f, A G_b\bar{\partial}_b f\right) \\
&\quad + \left(\left[\bar{\partial}_b, A\right]\bar{\partial}_b^* G_b\bar{\partial}_b f, A G_b\bar{\partial}_b f\right) \\
&= \left(A f, \bar{\partial}_b^* A G_b\bar{\partial}_b f\right) + \cdots \\
&= \left(A f, A\bar{\partial}_b^* G_b\bar{\partial}_b f\right) + \left(A f, \left[\bar{\partial}_b^*, A\right] G_b\bar{\partial}_b f\right) \\
&\quad + \left(\left[A, \bar{\partial}_b\right] f, A G_b\bar{\partial}_b f\right) + \left(\left[\bar{\partial}_b, A\right]\bar{\partial}_b^* G_b\bar{\partial}_b f, A G_b\bar{\partial}_b f\right).
\end{aligned} \tag{54}$$

Thus

$$\begin{aligned}
&Q_b\left(A G_b\bar{\partial}_b f, A G_b\bar{\partial}_b f\right) \\
&\quad\le \left(A f, A\bar{\partial}_b^* G_b\bar{\partial}_b f\right) + E + \mathcal{O}\left(\left\|G_b\bar{\partial} f\right\|_{k(U)}^2\right) \\
&\quad\le \left|\left(A f, A\bar{\partial}_b^* G_b\bar{\partial}_b f\right)\right| + |E| + \mathcal{O}\left(\left\|G_b\bar{\partial}_b f\right\|_{k(U)}^2\right),
\end{aligned} \tag{55}$$

where

$$\begin{aligned}
E = &\left(A f, \left[\bar{\partial}_b^*, A\right] G_b\bar{\partial}_b f\right) + \left(\left[A, \bar{\partial}_b\right] f, A G_b\bar{\partial}_b f\right) \\
&+ \left(\left[\bar{\partial}_b, A\right]\bar{\partial}_b^* G_b\bar{\partial}_b f, A G_b\bar{\partial}_b f\right).
\end{aligned} \tag{56}$$

As above, the three terms on the right-hand side of (56) are estimated, respectively, by

$$\begin{aligned}
&\|A f\| \left\|\left[\bar{\partial}_b^*, A\right] G_b\bar{\partial}_b f\right\| \\
&\quad\le \|f\|_{k(U)} \left\|G_b\bar{\partial}_b f\right\|_{k(U)} \\
&\quad\le (\text{s.c.}) \|f\|_{k(U)}^2 + (\text{l.c.}) \left\|G_b\bar{\partial}_b f\right\|_{k(U)}^2, \\
&\|f\|_{k(U)} \left\|A G_b\bar{\partial}_b f\right\| \\
&\quad\le (\text{s.c.}) \|f\|_{k(U)}^2 + (\text{l.c.}) \left\|A G_b\bar{\partial}_b f\right\|^2 \\
&\quad= \mathcal{O}\left(\left\|G_b\bar{\partial}_b f\right\|_{k(U)}^2\right), \\
&\left\|\left[\bar{\partial}_b, A\right]\bar{\partial}_b^* G_b\bar{\partial}_b f\right\| \left\|A G_b\bar{\partial}_b f\right\| \\
&\quad\le (\text{s.c.}) \left\|\bar{\partial}_b^* G_b\bar{\partial}_b f\right\|_{k(U)}^2 + (\text{l.c.}) \left\|A G_b\bar{\partial}_b f\right\|^2.
\end{aligned} \tag{57}$$

Now we are left with the first term in the right-hand side of (55) which, by applying the Cauchy-Schwarz inequality, is estimated by $\|f\|_{k(U)}\|\bar{\partial}_b^* G_b\bar{\partial}_b f\|_{k(U)}$. By choosing the s.c. small enough we can absorb the first term in the right-hand side of the last inequality into $\|f\|_{k(U)}\|\bar{\partial}_b^* G_b\bar{\partial}_b f\|_{k(U)}$. This completes the estimation of the first term on the right-hand side of (50). Therefore (50) becomes

$$\begin{aligned}
&\left\|\eta\Lambda^k\eta_1 G_b\bar{\partial}_b f\right\|^2 \\
&\quad\le \epsilon\|f\|_{k(U)} \left\|\bar{\partial}_b^* G_b\bar{\partial}_b f\right\|_{k(U)} \\
&\qquad + \epsilon\,(\text{s.c.}) \|f\|_{k(U)}^2 + \epsilon C\left\|G_b\bar{\partial}_b f\right\|_{k(U)}^2 \\
&\qquad + C(\epsilon) \left\|\eta\Lambda^k\eta_1 G_b\bar{\partial}_b f\right\|_{-1(U)}^2 \\
&\quad\le \epsilon\|f\|_{k(U)} \left\|\bar{\partial}_b^* G_b\bar{\partial}_b f\right\|_{k(U)} \\
&\qquad + \epsilon\,(\text{s.c.}) \|f\|_{k(U)}^2 + \epsilon C\left\|G_b\bar{\partial}_b f\right\|_{k(U)}^2 \\
&\qquad + C'(\epsilon) \left\|G_b\bar{\partial}_b f\right\|_{k-1(U)}^2.
\end{aligned} \tag{58}$$

By summing over a partition of unity subordinate to an open covering of M by patches $\{U_i\}_{i=1}^m$ so that on each of these patches an estimate like (58) is satisfied, using the interior regularity properties, we get

$$\begin{aligned}
\left\|G_b\bar{\partial}_b f\right\|_{k(M)}^2 \le{}& \epsilon\|f\|_{k(M)} \left\|\bar{\partial}_b^* G_b\bar{\partial}_b f\right\|_{k(M)} + \epsilon\,\text{s.c.}\|f\|_{k(M)}^2 \\
&+ \epsilon C\left\|G_b\bar{\partial}_b f\right\|_{k(M)} + C'(\epsilon) \left\|G_b\bar{\partial}_b f\right\|_{k-1(M)}^2.
\end{aligned} \tag{59}$$

By using the small and large constants, the first term on the right-hand side in (59) is estimated as

$$\epsilon\left((\text{s.c.}) \|f\|_{k(M)}^2 + (\text{l.c.}) \left\|\bar{\partial}_b^* G_b\bar{\partial}_b f\right\|_{k(M)}^2\right). \tag{60}$$

Then adding and choosing ϵ and the s.c. small enough we can absorb the third term on the right-hand side of (59) into the left-hand side; we obtain

$$\begin{aligned}
\left\|G_b\bar{\partial}_b f\right\|_{k(M)}^2 \le{}& \epsilon C\left\|\bar{\partial}_b^* G_b\bar{\partial}_b f\right\|_{k(M)}^2 \\
&+ C'(\epsilon)\left(\|f\|_{k(M)}^2 + \left\|G_b\bar{\partial}_b f\right\|_{k-1(M)}^2\right).
\end{aligned} \tag{61}$$

Applying this inequality with k replaced by $k-1$ to the last term on the right-hand side and repeating, we obtain

$$\begin{aligned}
\left\|G_b\bar{\partial}_b f\right\|_{k(M)}^2 \le{}& \epsilon C\left\|\bar{\partial}_b^* G_b\bar{\partial}_b f\right\|_{k(M)}^2 \\
&+ C'(\epsilon)\left(\|f\|_{k(M)}^2 + \left\|G_b\bar{\partial}_b f\right\|^2\right).
\end{aligned} \tag{62}$$

We have

$$\begin{aligned}
&\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|^2\\
&\quad=\left(\eta\Lambda^k\eta_1\bar{\partial}_b^*G_b\bar{\partial}_b f,\eta\Lambda^k\eta_1\bar{\partial}_b^*G_b\bar{\partial}_b f\right)\\
&\quad=\left(\bar{\partial}_b^*\eta\Lambda^k\eta_1G_b\bar{\partial}_b f,\eta\Lambda^k\eta_1\bar{\partial}_b^*G_b\bar{\partial}_b f\right)\\
&\qquad+\mathcal{O}\left(\left\|G_b\bar{\partial}_b f\right\|_{k(U)}\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|\right)\\
&\quad=\left(\eta\Lambda^k\eta_1G_b\bar{\partial}_b f,\eta\Lambda^k\eta_1\bar{\partial}_b\bar{\partial}_b^*G_b\bar{\partial}_b f\right)\\
&\qquad+\mathcal{O}\left(\left\|G_b\bar{\partial}_b f\right\|_{k(U)}\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|\right)\\
&\quad=\left(\eta\Lambda^k\eta_1G_b\bar{\partial}_b f,\eta\Lambda^k\eta_1\bar{\partial}_b f\right)\\
&\qquad+\mathcal{O}\left(\left\|G_b\bar{\partial}_b f\right\|_{k(U)}\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|\right)\\
&\quad=\left(\eta\Lambda^k\eta_1G_b\bar{\partial}_b f,\bar{\partial}_b\eta\Lambda^k\eta_1 f\right)\\
&\qquad+\mathcal{O}\left(\left\|G_b\bar{\partial}_b f\right\|_{k(U)}\left(\|f\|_{k(U)}+\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|\right)\right)\\
&\quad=\left(\bar{\partial}_b^*\eta\Lambda^k\eta_1G_b\bar{\partial}_b f,\eta\Lambda^k\eta_1 f\right)\\
&\qquad+\mathcal{O}\left(\left\|G_b\bar{\partial}_b f\right\|_{k(U)}\left(\|f\|_{k(U)}+\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|\right)\right)\\
&\quad=\left(\eta\Lambda^k\eta_1\bar{\partial}_b^*G_b\bar{\partial}_b f,\eta\Lambda^k\eta_1 f\right)\\
&\qquad+\mathcal{O}\left(\left\|G_b\bar{\partial}_b f\right\|_{k(U)}\left(\|f\|_{k(U)}+\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|\right)\right)\\
&\quad\le\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|\left\|\eta\Lambda^k\eta_1 f\right\|\\
&\qquad+\mathcal{O}\left(\left\|G_b\bar{\partial}_b f\right\|_{k(U)}\left(\|f\|_{k(U)}+\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|\right)\right).
\end{aligned}\tag{63}$$

Again summing over a partition of unity, using the interior regularity properties and the small and large constants technique, we obtain

$$\left\|\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|_{k(M)}^2\le C\left(\left\|G_b\bar{\partial}_b f\right\|_{k(M)}^2+\|f\|_{k(M)}^2\right).\tag{64}$$

Substituting (62) into (64), we obtain

$$\begin{aligned}\left\|\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|_{k(M)}^2&\le K\epsilon\left\|\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|_{k(M)}^2\\&\quad+C'(\epsilon)\left(\|f\|_{k(M)}^2+\left\|G_b\bar{\partial}_b f\right\|^2\right).\end{aligned}\tag{65}$$

Choosing $\epsilon>0$ small enough allows us to absorb the first term on the right-hand side into the left, we then get

$$\left\|\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|_{k(M)}^2\le C'(\epsilon)\left(\|f\|_{k(M)}^2+\left\|G_b\bar{\partial}_b f\right\|^2\right).\tag{66}$$

As the operator $\bar{\partial}_b^*$ has $L^2(M)$-closed range, it follows from Theorem 1.1.1 in Hörmander [15] that there is a positive constant C such that

$$\left\|G_b\bar{\partial}_b f\right\|\le C\left\|\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|.\tag{67}$$

Then, by (49), we obtain

$$\left\|G_b\bar{\partial}_b f\right\|\le C\|f\|.\tag{68}$$

Substituting (68) into (66), we get

$$\left\|\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|_{k(M)}^2\le C\|f\|_{k(M)}^2.\tag{69}$$

By (43), the Szegö projection S_{s-1} is therefore continuous on $W^k_{0,s-1}(M)$ for each $k=0,1,2\ldots$. The general case is obtained from interpolation of linear operators.

For the continuity of the Szegö projection S_s, in view of (42), it suffices to show that

$$\left\|\bar{\partial}_b^*\bar{\partial}_b G_b f\right\|_{k(M)}^2\le C\|f\|_{k(M)}^2,\quad k\ge 0.\tag{70}$$

For $k=0$, we have

$$\begin{aligned}\left\|\bar{\partial}_b^*\bar{\partial}_b G_b f\right\|^2&=\left(\bar{\partial}_b\bar{\partial}_b^*\bar{\partial}_b G_b f,\bar{\partial}_b G_b f\right)=\left(\bar{\partial}_b f,\bar{\partial}_b G_b f\right)\\&=\left(f,\bar{\partial}_b^*\bar{\partial}_b G_b f\right)\le C\|f\|\left\|\bar{\partial}_b^*\bar{\partial}_b G_b f\right\|.\end{aligned}\tag{71}$$

For $k\ge 1$, as before, an elliptic regularity argument implies

$$\begin{aligned}
&\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*\bar{\partial}_b G_b f\right\|^2\\
&\quad=\left(\eta\Lambda^k\eta_1\bar{\partial}_b^*\bar{\partial}_b G_b f,\eta\Lambda^k\eta_1\bar{\partial}_b^*\bar{\partial}_b G_b f\right)\\
&\quad=\left(\eta\Lambda^k\eta_1\bar{\partial}_b\bar{\partial}_b^*\bar{\partial}_b G_b f,\eta\Lambda^k\eta_1\bar{\partial}_b G_b f\right)\\
&\qquad+\left(\eta\Lambda^k\eta_1\bar{\partial}_b^*\bar{\partial}_b G_b f,\left[\eta\Lambda^k\eta_1,\bar{\partial}_b^*\right]\bar{\partial}_b G_b f\right)\\
&\qquad+\left(\left[\bar{\partial}_b,\eta\Lambda^k\eta_1\right]\bar{\partial}_b^*\bar{\partial}_b G_b f,\eta\Lambda^k\eta_1\bar{\partial}_b G_b f\right)\\
&\quad=\left(\eta\Lambda^k\eta_1\bar{\partial}_b f,\eta\Lambda^k\eta_1\bar{\partial}_b G_b f\right)\\
&\qquad+\mathcal{O}\left(\left\|\eta\Lambda^k\eta_1\bar{\partial}_b G_b f\right\|\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*\bar{\partial}_b G_b f\right\|\right)\\
&\quad=\left(\eta\Lambda^k\eta_1 f,\eta\Lambda^k\eta_1\bar{\partial}_b^*\bar{\partial}_b G_b f\right)\\
&\qquad+\mathcal{O}\left(\left\|\eta\Lambda^k\eta_1\bar{\partial}_b G_b f\right\|\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*\bar{\partial}_b G_b f\right\|\right)\\
&\qquad+\mathcal{O}\left(\|f\|_{k(U)}\left\|\eta\Lambda^k\eta_1\bar{\partial}_b G_b f\right\|\right)\\
&\quad\le\left\|\eta\Lambda^k\eta_1 f\right\|\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*\bar{\partial}_b G_b f\right\|\\
&\qquad+\mathcal{O}\left(\left(\|f\|_{k(U)}+\left\|\eta\Lambda^k\eta_1\bar{\partial}_b G_b f\right\|\right)\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*\bar{\partial}_b G_b f\right\|\right).
\end{aligned}\tag{72}$$

Summing over a partition of unity, using the small and large constants argument, absorbing the terms containing $\|\overline{\partial}_b^* \overline{\partial}_b G_b f\|_{k(M)}$, and finally using the fact that $\overline{\partial}_b G_b$ is continuously bounded on $W^k_{0,s}(M)$, we conclude (70) which proves the continuity of S_s on $W^k_{0,s}(M)$. □

Corollary 12. *Let M be a $\mathscr{C}^\infty$ compact CR manifold of real dimension $2n-\ell$ and codimension $\ell \geq 1$ in an n-dimensional complex manifold X. Suppose that M satisfies condition $Y(s)$ for some s with $1 \leq s \leq n-\ell-1$. Then for any f in $W^k_{0,s}(M)$ ($k \geq 0$) such that $\overline{\partial}_b f = 0$ and $f \perp \mathscr{H}^b_{0,s}(M)$, there exists u in $W^k_{0,s-1}(M)$ which solves the equation $\overline{\partial}_b u = f$.*

Theorem 13. *Let M be a $\mathscr{C}^\infty$ compact CR manifold of real dimension $2n-\ell$ and codimension $\ell \geq 1$ in an n-dimensional complex manifold X. Suppose that M satisfies condition $Y(s)$ for some s with $1 \leq s \leq n-\ell-1$. Then for any f in $\mathscr{C}^\infty_{0,s}(M)$, with $\overline{\partial}_b f = 0$ and $f \perp \mathscr{H}^b_{0,s}(M)$, there exists a global solution u in $\mathscr{C}^\infty_{0,s-1}(M)$ to the equation $\overline{\partial}_b u = f$.*

Proof. By Corollary 12, for each $k \geq 0$, there exists some $u_k \in W^k_{0,s-1}(M)$ such that $\overline{\partial}_b u_k = f$. We modify each u_k by an element of $\mathrm{Ker}(\overline{\partial}_b)$ in order to construct a telescoping series that belongs to $W^k_{0,s}(M)$ for each $k \geq 1$. To conclude the proof, we first claim that $W^k_{0,s}(M) \cap \mathrm{Ker}(\overline{\partial}_b)$ is dense in $W^m_{0,s}(M) \cap \mathrm{Ker}(\overline{\partial}_b)$ for any $k > m \geq 0$. Since $\mathscr{C}^\infty_{0,s}(M)$ is dense in $W^m_{0,s}(M)$, $m \geq 0$, in the W^m-norm, then for a given $\eta \in W^m_{0,s}(M) \cap \mathrm{Ker}(\overline{\partial}_b)$ there is a sequence $\eta_j \in \mathscr{C}^\infty_{0,s}(M)$ converging to η in the $W^m_{0,s}(M)$-norm; that is, $\|\eta_j - \eta\|_{m(M)} \to 0$ as $j \to \infty$. $\overline{\partial}_b \eta = 0$ implies that $\eta - S_s\eta = \overline{\partial}_b^* G_b \overline{\partial}_b \eta = 0$, so $\eta = S_s u$. Let $\widehat{\eta}_j = S_s \eta_j$. $\widehat{\eta}_j \in W^k_{0,s}(M) \cap \mathrm{Ker}(\overline{\partial}_b)$ since the Szegö projection S_s is a bounded operator on $W^k_{0,s}(M)$. By the same reason we have $\|\widehat{\eta}_j - \eta\|_{m(M)} = \|S_s(\eta_j - \eta)\|_{m(M)} \leq C\|\eta_j - \eta\|_{m(M)} \to 0$ as $j \to \infty$. This implies that $\widehat{\eta}_j \to \eta$ in the W^m-norm. Thus, indeed, $W^k_{0,s}(M) \cap \mathrm{Ker}(\overline{\partial}_b)$ is dense in $W^m_{0,s}(M) \cap \mathrm{Ker}(\overline{\partial}_b)$ for any $k > m \geq 0$.

Next, using this result and following the inductive argument due to [21, page 230], we can construct a sequence $\widetilde{u}_k \in W^k_{0,s-1}(M)$, $\overline{\partial}_b \widetilde{u}_k = f$, and $\|\widetilde{u}_{k+1} - u_k\|_{k(M)} \leq 2^{-k}$ as follows:

$$\widetilde{u}_1 = u_1, \qquad \widetilde{u}_2 = u_2 + v_2, \tag{73}$$

where $v_2 \in W^2_{0,s-1}(M) \cap \mathrm{Ker}(\overline{\partial}_b)$ is such that

$$\|\widetilde{u}_2 - u_1\|_{1(M)} \leq 2^{-1} \tag{74}$$

and in general

$$\widetilde{u}_{k+1} = u_{k+1} + v_{k+1}, \tag{75}$$

where $v_{k+1} \in W^{k+1}_{0,s}(M) \cap \mathrm{Ker}(\overline{\partial}_b)$ is such that

$$\|\widetilde{u}_{k+1} - u_k\|_{k(M)} \leq 2^{-k}. \tag{76}$$

Clearly $\overline{\partial}_b \widetilde{u}_k = f$, so set

$$u = \widetilde{u}_j + \sum_{k=j}^{\infty} \left(\widetilde{u}_{k+1} - \widetilde{u}_k\right), \quad j \in \mathbb{N}. \tag{77}$$

It follows that $u \in W^k_{0,s-1}(M)$ for each $k \in \mathbb{N}$, and hence $u \in \mathscr{C}^\infty_{0,s-1}(M)$ and $\overline{\partial}_b u = f$. The general case is obtained from interpolation of linear operators. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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