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Research Article

Antiperiodic Solutions for p-Laplacian Systems via Variational Approach

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We establish the existence of solutions for p-Laplacian systems with antiperiodic boundary conditions through using variational methods.

1. Introduction

In this paper we investigate the existence of solutions of the following second order p-Laplacian systems with antiperiodic boundary condition:

$$\begin{gathered} -\left(\left|u'(t)\right|^{p-2}u'(t)\right)' = \nabla F(t,u(t)) \quad \text{a.e. } [0,T], \\ u(0) = -u(T), \qquad u'(0) = -u'(T), \end{gathered} \tag{1}$$

where $T > 0$, $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ $(N \geq 1)$ satisfies the following fundamental assumption:

($\mathscr{H}$) $F(t,x)$ is measurable in t for each $x \in \mathbb{R}^N$, continuous differentiable in x for almost every $t \in [0,T]$, and there exists $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $b \in L^1(0,T;\mathbb{R}^+)$ such that

$$|F(t,x)| \leq a(|x|)\,b(t), \qquad |\nabla F(t,x)| \leq a(|x|)\,b(t) \tag{2}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

In the last few decades, the following second order systems involving periodic boundary condition:

$$\begin{gathered} -\ddot{u} = \nabla F(t,u(t)) \quad \text{a.e. } [0,T], \\ u(0) = u(T), \qquad \dot{u}(0) = \dot{u}(T), \end{gathered} \tag{3}$$

have acted as one of the mainstream research problems in the field of differential equation. Under various assumptions of the potential $F(t,x)$, there have been lots of existence and multiplicity of results in the literatures by using the tool of nonlinear analysis, such as degree theory, minimax methods, and Morse theory. Here we do not even try to review the huge bibliography, but we only list some references for our purpose; for example, we refer the readers to see [1–6] and the references therein.

Comparing problem (1) with problem (3), we observe that the only difference is the boundary conditions. In order to use the variational methods, one of the main difficulties is the variational principle. For this matter, we try to modify some work space such that the variational principle can be established. Thanks to the work of Tian and Henderson [7], we borrow their ideas to give the variational principle for problem (1).

Note that the study of antiperiodic solutions for nonlinear differential systems of the form

$$-\left(\left|u'(t)\right|^{p-2}u'(t)\right)' = \nabla F(t,u(t)) \quad \text{a.e. } t \in \mathbb{R} \tag{4}$$

is closely related to the study of its periodic solutions. Indeed, if we assume that $F(t+T,x) = F(t,x)$ and $\nabla F(t,-x) = -\nabla F(t,x)$, then let

$$v(t) = \begin{cases} u(t), & t \in [0,T], \\ -u(t-T), & t \in [T,2T], \end{cases} \tag{5}$$

and we get that v is a solution of systems (4) with conditions $v(0) = v(2T)$ and $v'(0) = v'(2T)$. Since $F(t,x)$ is $2T$-periodic in t, v can be extended to be $2T$-periodic over $\mathbb{R}$, and hence v is a $2T$-periodic solution of systems (4).

In this paper, we will establish the existence of solutions for problem (1) by variational method. As far as we know, there are few papers studying the second order systems with antiperiodic boundary conditions by variational methods.

This paper is organized as follows. In Section 2, we introduce a variational principle for problem (1). In Section 3, we prove our main results.

2. Variational Principle

In the sequel, we denote by $\|\cdot\|_p$ the L^p-norm ($1 \leq p \leq \infty$). Let $C^\infty(0,T;\mathbb{R}^N)$ be the space of infinitely differentiable functions from $[0,T]$ into $\mathbb{R}^N$. Let $X = \{u \in C^\infty(0,T;\mathbb{R}^N), u(0) = -u(T)\}$; obviously, $\cos((2k+1)\pi/T)t$ and $\sin(((2k+1)\pi/T)t + \pi/2)$ ($k = 0,1,2,\ldots$) belong to X, and thus $X \neq \emptyset$.

The following fundamental lemma proved in [7] is essential to establish the variational principle.

Lemma 1. *Let $u, v \in L^1(0,T;\mathbb{R}^N)$. If, for every $f \in X$,*

$$\int_0^T \left(u(t), f'(t)\right) dt = -\int_0^T \left(v(t), f(t)\right) dt, \tag{6}$$

where $(\cdot,\cdot)$ denotes the inner product on $\mathbb{R}^N$, then

$$\frac{2}{T}\int_0^T [u(t) + tv(t)]\, dt = \int_0^T v(t)\, dt \tag{7}$$

and there exists $c \in \mathbb{R}^N$ such that

$$u(t) = \int_0^t v(s)\, ds + c \quad \text{a.e. on } [0,T]. \tag{8}$$

Remark 2. From Lemma 1, we have the following facts.

(i) A function v satisfying (6) is called a weak derivative of u. By a Fourier series argument, the weak derivative, if it exists, is unique. We denote by u' the weak derivative of u.

(ii) We will identify the equivalence class u and its continuous representation

$$\hat{u}(t) = \int_0^t u'(s)\, ds + c. \tag{9}$$

(iii) Equations (7) and (8) imply that $u(0) = -u(T) = c$. For this matter, we only show that $u(T) + c = 0$. Indeed, using integration by parts, we have

$$\begin{aligned} u(T) &= \int_0^T v(s)\, ds + c \\ &= \frac{2}{T}\int_0^T [u(t) + tv(t)]\, dt + c \\ &= \frac{2}{T}\int_0^T u(t)\, dt + 2\int_0^T v(t)\, dt - \frac{2}{T}\int_0^T\int_0^t v(s)\, ds\, dt + c \\ &= \frac{2}{T}\int_0^T u(t)\, dt + 2\int_0^T v(t)\, dt - \frac{2}{T}\int_0^T u(t)\, dt + 3c \\ &= 2u(T) + c. \end{aligned} \tag{10}$$

By (9), we have

$$u(t) = u(\tau) + \int_\tau^t u'(s)\, ds, \quad \text{for } t, \tau \in [0,T]. \tag{11}$$

(iv) If u' is continuous on $[0,T]$, then by (9) u' is the classical derivative of $u = \hat{u}$.

(v) It follows from (9) and Rademacher theorem that u' is the classical derivative of u a.e. on $[0,T]$.

The Sobolev space $\widehat{W}_T^{1,p}$ is the space of functions $u \in L^p(0,T;\mathbb{R}^N)$ having a weak derivative $u' \in L^p(0,T;\mathbb{R}^N)$. Obviously, if $u \in \widehat{W}_T^{1,p}$, $u(t) = \int_0^t u'(s)ds + c$ and $u(0) = -u(T) = c$. The norm over $\widehat{W}_T^{1,p}$ is defined by

$$\|u\|_{\widehat{W}_T^{1,p}} = \left(\int_0^T |u(t)|^p dt + \int_0^T \left|u'(t)\right|^p dt\right)^{1/p}. \tag{12}$$

It is easy to see that $\widehat{W}_T^{1,p}$ is a reflexive Banach space and $X \subset \widehat{W}_T^{1,p}$.

With the proof of Lemma 3.3 in [7] and Theorem 8.8 in [8], we have the following embedding theorem.

Lemma 3. *Let $1/p + 1/q = 1$ ($1 < p < +\infty$). Then*

(i) *the embedding $\widehat{W}_T^{1,p} \hookrightarrow L^q(0,T;\mathbb{R}^N)$ is compact;*

(ii) *there exits a constant $c > 0$ such that*

$$\|u\|_\infty \leq c\|u\|_{\widehat{W}_T^{1,p}}. \tag{13}$$

Moreover, the embedding $\widehat{W}_T^{1,p} \hookrightarrow C(0,T;\mathbb{R}^N)$ is compact.

Proof. From Theorem 8.2 in [8], for $u \in \widehat{W}_T^{1,p}$, $u(t) = u(0) + \int_0^t u'(s)ds$ and $u(t) = u(T) - \int_t^T u'(s)ds$. By Hölder's inequality, we have

$$\begin{aligned} u(t) &= \frac{1}{2}\left[u(0) + u(T) + \int_0^t u'(s)\, ds - \int_t^T u'(s)\, ds\right] \\ &\leq \frac{1}{2}T^{1/q}\left(\int_0^T \left|u'(t)\right|^p dt\right)^{1/p}. \end{aligned} \tag{14}$$

On the other hand, letting B be a unit ball in $\widehat{W}_T^{1,p}$, for $u \in B$, we have

$$\begin{aligned} |u(t) - u(s)| &= \left|\int_s^t u'(\tau)\, d\tau\right| \leq \left\|u'\right\|_p |t-s|^{1/q} \\ &\leq |t-s|^{1/q}, \quad \forall t, s \in (0,T). \end{aligned} \tag{15}$$

It follows from the Ascoli-Arzelà theorem that B has a compact closure in $C(0,T;\mathbb{R}^N)$. □

By Lemma 1, the norm $\|\cdot\|_{\widehat{W}_T^{1,p}}$ in $\widehat{W}_T^{1,p}$ is equivalent to the norm defined as

$$\|u\| = \left(\int_0^T \left|u'(t)\right|^p dt\right)^{1/p}. \tag{16}$$

Indeed, by (14), one has

$$\int_0^T |u(t)|^p dt \le T\|u\|_\infty^p \le \frac{T^{1+p/q}}{2^p}\int_0^T \left|u'(t)\right|^p dt. \quad (17)$$

Therefore, we have

$$\frac{2}{T+2}\|u\|_{\widehat{W}_T^{1,p}} \le \|u\| \le \|u\|_{\widehat{W}_T^{1,p}}, \quad \forall u \in \widehat{W}_T^{1,p} \quad (18)$$

and the equivalence of the norm is proved.

Remark 4. There is some differences between antiperiodic boundary value problem and periodic boundary value problem. One is the norm on the work space, and the other is the decomposition of space $\widehat{W}_T^{1,p}$ not like $W_T^{1,p} = \mathbb{R}^N \oplus \widetilde{W}_T^{1,p}$, where $\widetilde{W}_T^{1,p} = \{u \in W_T^{1,p} : \int_0^T u(t)dt = 0\}$. And it is well known that the norms $\|u'\|_p$ and $\|u\|_{W_T^{1,p}}$ are equivalent to $\widetilde{W}_T^{1,p}$.

The energy functional $\varphi : \widehat{W}_T^{1,p} \to \mathbb{R}$ corresponding to problem (1) is defined by

$$\varphi(u) = \frac{1}{p}\int_0^T \left|u'(t)\right|^p dt - \int_0^T F(t,u(t))\,dt. \quad (19)$$

Under the assumption of $(\mathscr{H})$, we have the following.

Lemma 5. *Let $(\mathscr{H})$ hold. Then $\varphi \in C^1(\widehat{W}_T^{1,p},\mathbb{R})$ and*

$$\begin{aligned}\left\langle \varphi'(u), v\right\rangle &= \int_0^T \left|u'(t)\right|^{p-2}\left(u'(t), v'(t)\right)dt \\ &\quad - \int_0^T (\nabla F(t,u(t)), v(t))\,dt\end{aligned} \quad (20)$$

for every $v \in \widehat{W}_T^{1,p}$. Moreover, if $\varphi'(u) = 0$, then u is a solution of problem (1)*; that is, $u \in \widehat{W}_T^{1,p}$ satisfies the equation and antiperiodic condition in* (1).

Proof. Similarly as the proof of Theorem 1.4 in [1], we obtain that $\varphi \in C^1(\widehat{W}_T^{1,p},\mathbb{R})$ and (28) holds. If

$$\begin{aligned}0 = \left\langle \varphi'(u), v\right\rangle &= \int_0^T \left|u'(t)\right|^{p-2}\left(u'(t), v'(t)\right)dt \\ &\quad - \int_0^T (\nabla F(t,u(t)), v(t))\,dt = 0,\end{aligned} \quad (21)$$

for all $v \in \widehat{W}_T^{1,p}$ and hence for all $v \in X$, by Lemma 1, there exists a constant $c \in \mathbb{R}^N$ such that

$$-\left|u'(t)\right|^{p-2}u'(t) = \int_0^t \nabla F(s,u(s))\,ds + c, \quad \text{a.e. on } [0,T]. \quad (22)$$

From Remark 2, we know that $|u'(t)|^{p-2}u'(t)$ has a weak derivative

$$-\left(\left|u'(t)\right|^{p-2}u'(t)\right)' = \nabla F(t,u(t)) \quad \text{a.e. on } [0,T]. \quad (23)$$

By Remark 2, we get

$$\left|u'(0)\right|^{p-2}u'(0) = -\left|u'(T)\right|^{p-2}u'(T). \quad (24)$$

By (24), if $u'(0) = u'(T) = 0$, clearly, we have $u'(0) = -u'(T)$. If not, we see that $u'(0)\cdot u'(T) < 0$. Hence, by calculation, we get

$$\left|u'(0)\right| = \left|u'(T)\right|. \quad (25)$$

This implies that $u'(0) = -u'(T)$ and hence the conclusion is proved. □

Lemma 6. *The functional $\varphi : \widehat{W}_T^{1,p} \to \mathbb{R}$ is weakly lower semicontinuous.*

Proof. Assuming $u_n \rightharpoonup u$ in $\widehat{W}_T^{1,p}$, then by (ii) of Lemma 3, we have $u_n \to u$ in $C([0,T];\mathbb{R}^N)$. By hypothesis $(\mathscr{H})$, we have

$$\begin{aligned}&\left|\int_0^T [F(t,u_n(t)) - F(t,u(t))]\,dt\right| \\ &\quad = \left|\int_0^T\int_0^1 (\nabla F(t,u(t)+s(u_n(t)-u(t))), \right. \\ &\qquad \left. u_n(t)-u(t))\,ds\,dt\right| \\ &\quad \le \int_0^T\int_0^1 b(t)\,a(|u+s(u_n-u)|)\,|u_n-u|\,ds\,dt \\ &\quad \le \max_{s\in[0,M]} a(s)\,\|b\|_{L^1}\|u_n-u\|_{L^\infty} \longrightarrow 0,\end{aligned} \quad (26)$$

where $M > 0$ is a constant such that $|u_n + s(u_n-u)| \le M$ for every $n \in \mathbb{N}$, all $s \in [0,1]$, and $t \in [0,T]$. It follows from the weak lower semicontinuity of the norm function in Banach space that

$$\liminf_{n\to\infty}\int_0^T \left|u_n'(t)\right|^p dt \ge \int_0^T \left|u'(t)\right|^p dt. \quad (27)$$

Accordingly, the conclusion is completed. □

Next, we study the eigenvalue problem

$$\begin{aligned}-\left(\left|u'(t)\right|^{p-2}u'(t)\right)' &= \lambda|u(t)|^{p-2}u(t), \quad t \in [0,T], \\ u(0) = -u(T), &\qquad u'(0) = -u'(T).\end{aligned} \quad (28)$$

Definition 7. One says $\lambda \in \mathbb{R}$ is an eigenvalue of problem (28) if there exists $u \in \widehat{W}_T^{1,p}$, $u \not\equiv 0$, such that

$$\begin{aligned}&\int_0^T \left(\left|u'(t)\right|^{p-2}u'(t), v'(t)\right)dt \\ &\quad = \lambda\int_0^T \left(|u(t)|^{p-2}u(t), v(t)\right)dt, \\ &\qquad \text{for every } v \in \widehat{W}_T^{1,p}.\end{aligned} \quad (29)$$

Lemma 8. *The eigenvalue problem* (28) *possesses the following properties:*

(a) *all the eigenvalues are positive real numbers;*

(b) *for each $u \in \widehat{W}_T^{1,p}$, one has $\lambda_1 \int_0^T |u(t)|^p dt \le \int_0^T |u'(t)|^p dt$, where $\lambda_1 = \inf_{u\in \widehat{W}_T^{1,p}, \|u\|_{L^p}=1} \int_0^T |u'(t)|^p dt$.*

Proof. (a) Letting $e_\lambda(t)$ be the eigenfunction corresponding to the eigenvalue λ, we have

$$\int_0^T \left|e'_\lambda(t)\right|^p dt = \lambda \int_0^T \left|e_\lambda(t)\right|^p dt. \tag{30}$$

Hence, $\lambda \ge 0$ and $\lambda = 0$ if and only if $e'_\lambda(t) = 0$, implying that $e_\lambda(t) = C$, a.e. on $[0, T]$, where C is a constant. Since $e_\lambda \in \widehat{W}_T^{1,p}$ ($u(0) = -u(T)$), we see that $e_\lambda(t) \equiv 0$, for all $t \in [0, T]$; this contradicts with the definition of eigenfunction.

(b) By standard minimization arguments, we can prove the conclusion. We omit it. □

Remark 9. Note that the eigenvalue of one dimensional vector p-Laplacian operator under antiperiodic boundary condition possesses some similar properties as the p-Laplacian operator $-\Delta_p$ under Dirichlet boundary condition on bounded domain.

3. Main Results and Proof

In this section, we will give some existence and multiplicity of results for problem (1).

Theorem 10. *Assume that $F(t, u)$ satisfies hypothesis $(\mathscr{H})$ and the following conditions.*

(F_0) *There exist $f \in L^1([0,T];\mathbb{R}^+)$ and $g \in L^1([0,T];\mathbb{R}^+)$ such that*

$$F(t,x) \le f(t)|x|^\alpha + g(t), \quad \forall x \in \mathbb{R}^N, \text{ a.e. } t \in [0,T] \tag{31}$$

with $0 \le \alpha < p$. Then problem (1) *has at least one solution in $\widehat{W}_T^{1,p}$.*

Proof. First we show that φ is coercive on $\widehat{W}_T^{1,p}$. In fact, from (F_0) and Lemma 3, we have

$$\begin{aligned}\varphi(u) &= \frac{1}{p}\int_0^T \left|u'(t)\right|^p dt - \int_0^T F(t,u(t))\,dt \\ &\ge \frac{1}{p}\int_0^T \left|u'(t)\right|^p dt - \|u\|_{L^\infty}^\alpha \int_0^T f(t)\,dt - \int_0^T g(t)\,dt \\ &\ge \frac{1}{p}\int_0^T \left|u'(t)\right|^p dt - C\left(\int_0^T \left|u'(t)\right|^p dt\right)^{\alpha/p} - C.\end{aligned} \tag{32}$$

Therefore, $\varphi(u) \to +\infty$ as $\|u\| \to \infty$. As a result, we get a bounded minimizing sequence in $\widehat{W}_T^{1,p}$. Combining Lemma 6, by Theorem 1.1 and Corollary 1.1 in [1], problem (1) has at least one solution which minimizes φ on $\widehat{W}_T^{1,p}$. □

Remark 11. By hypothesis $(\mathscr{H})$, we see that $F(t, x)$ is summable over $t \in [0, T]$ in the neighborhood of zero, and thus the condition (F_0) can be weaken to hold for $|x|$ large.

If $\alpha = p$, we may assume the function $f(t)$ in (F_0) satisfies that

$$\int_0^T f(t)\,dt < \frac{2^p}{pT^{p-1}}, \tag{33}$$

as the same proof of Theorem 10, and we can get the following theorem.

Theorem 12. *Under the above assumptions of $F(t, x)$, then problem* (1) *has at least one solution in $\widehat{W}_T^{1,p}$.*

Theorem 13. *Assume that $F(t, u)$ satisfies hypothesis $(\mathscr{H})$ and the following conditions:*

(F_1) *$\lim_{|x|\to 0}(F(t,x)/|x|^p) = 0$, and $\lim_{|x|\to+\infty}(F(t,x)/|x|^p) = +\infty$ uniformly for a.e. $t \in [0,T]$;*

(F_2) *there exist constants $r > p$ and $\mu > r - p$ such that $\limsup_{|x|\to+\infty}(F(t,x)/|x|^r) < +\infty$ uniformly for a.e. $t \in [0,T]$, and $\liminf_{|x|\to+\infty}(((\nabla F(t,x),x) - pF(t,x))/|x|^\mu) > 0$ uniformly for a.e. $t \in [0,T]$.*

Then problem (1) *possesses at least one nontrivial solution in $\widehat{W}_T^{1,p}$.*

In order to prove Theorem 13, we need the following results.

Lemma 14. *Suppose $(\mathscr{H})$, (F_1), and (F_2) hold. Then functional φ satisfies the* (C)*-condition; that is, for every sequence $\{u_n\} \subset \widehat{W}_T^{1,p}$, $\{u_n\}$ has a convergent subsequence if $\varphi(u_n)$is bounded and $(1 + \|u_n\|)\|\varphi'(u_n)\| \to 0$ as $n \to \infty$.*

Proof. Suppose $\{u_n\} \subset \widehat{W}_T^{1,p}$, $\varphi(u_n)$ is bounded, and $(1 + \|u_n\|)\|\varphi'(u_n)\| \to 0$ as $n \to \infty$, and then there exists a constant $M > 0$ such that

$$\left|\varphi(u_n)\right| \le M, \qquad (1+\|u_n\|)\left\|\varphi'(u_n)\right\| \le M \tag{34}$$

for every $n \in \mathbb{N}$. On one hand, by (F_2), there exist constants $C > 0$ and $\delta > 0$ such that

$$F(t,x) \le C|x|^r, \quad \forall |x| \ge \delta, \text{ a.e. } t \in [0,T]. \tag{35}$$

It follows from $(\mathscr{H})$ and (35) that

$$\begin{gathered}F(t,x) \le \max_{s\in[0,\delta]} a(s)\,b(t) + C|x|^r, \\ \forall x \in \mathbb{R}^N, \quad \text{a.e. } t \in [0,T].\end{gathered} \tag{36}$$

Hence, by (34), (36), and Hölder inequality, we have

$$\frac{1}{p}\|u_n\|^p = \varphi(u_n) + \int_0^T F(t, u_n(t))\,dt \le C\int_0^T |u_n(t)|^r dt + C. \tag{37}$$

On the other hand, by (F_2), there are constants $C > 0$ and $\delta_1 > 0$ such that

$$(\nabla F(t,x), x) - pF(t,x) \ge C|x|^\mu, \quad \forall |x| \ge \delta_1, \quad \text{a.e. } t \in [0,T]. \tag{38}$$

By $(\mathscr{H})$, one has

$$|(\nabla F(t,x), x) - pF(t,x)| \le Cb(t), \quad \forall |x| \le \delta_1, \quad \text{a.e. } t \in [0,T], \tag{39}$$

where $C = (p+\delta_1)\max_{s\in[0,\delta_1]} a(s)$. Hence, from (38) and (39), we get

$$(\nabla F(t,x), x) - pF(t,x) \ge C\left(|x|^\mu - \delta_1^\mu\right) - Cb(t), \tag{40}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$. Hence, by (34) and (40), one has

$$\begin{aligned}(p+1)M &\ge p\varphi(u_n) - \left\langle \varphi'(u_n), u_n\right\rangle \\ &= \int_0^T \left[(\nabla F(t,u_n(t)), u_n(t)) - pF(t,u_n(t))\right] dt \\ &\ge C\int_0^T |u_n(t)|^\mu dt - CT\delta_1^\mu - C\int_0^T b(t)\,dt.\end{aligned} \tag{41}$$

So $\{u_n\}$ is bounded in $L^\mu(0,T;\mathbb{R}^N)$. If $\mu \ge r$, by (37) and Hölder inequality, it is easy to obtain that $\{u_n\}$ is bounded in $\widehat{W}_T^{1,p}$. If $\mu < r$, by Lemma 3, we have

$$\int_0^T |u_n(t)|^r dt \le \|u_n\|_{L^\infty}^{r-\mu}\int_0^T |u_n(t)|^\mu dt \le C\|u_n\|^{r-\mu}\int_0^T |u_n(t)|^\mu dt. \tag{42}$$

Hence, by (37) and $\nu > r - p$, we obtain $\{u_n\}$ is bounded in $\widehat{W}_T^{1,p}$ too. Thus, $\{u_n\}$ is bounded in $\widehat{W}_T^{1,p}$. Since $\widehat{W}_T^{1,p}$ is a reflexive Banach space, by Lemma 3, there exists a subsequence, still denoted by $\{u_n\}$, such that

$$u_n \rightharpoonup u \text{ in } \widehat{W}_T^{1,p}, \qquad u_n \longrightarrow u \text{ in } C\left([0,T];\mathbb{R}^N\right), \quad \text{as } n \longrightarrow \infty. \tag{43}$$

Next, we will show that $u_n \to u$ in $\widehat{W}_T^{1,p}$. Indeed, from (43) and hypothesis $(\mathscr{H})$, it is easy to obtain that

$$\begin{gathered}\left\langle \varphi'(u_n) - \varphi'(u), u_n - u\right\rangle \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \\ \int_0^T \left(\nabla F(t,u_n(t)) - \nabla F(t,u(t)), u_n(t) - u(t)\right) dt \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.\end{gathered} \tag{44}$$

Hence, by (44), we get

$$\int_0^T \left(\left|u_n'(t)\right|^{p-2}u_n'(t) - \left|u'(t)\right|^{p-2}u'(t), u_n'(t) - u'(t)\right) dt \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{45}$$

Therefore, it follows from (45) and Lemmas 3.2 and 3.3 in [9] that $u_n \to u$ in $\widehat{W}_T^{1,p}$. The proof of Lemma 14 is completed. □

Proof of Theorem 13. By (F_1), for every $\varepsilon > 0$, there exists $\delta_2 > 0$ such that

$$F(t,x) \le \varepsilon|x|^p, \quad \forall |x| \le \delta_2, \ \text{a.e. } t \in [0,T]. \tag{46}$$

Combined with (35) and (46), we have

$$F(t,x) \le \varepsilon|x|^p + C|x|^r, \quad \forall x \in \mathbb{R}^N, \ \text{a.e. } t \in [0,T]. \tag{47}$$

Thus, by Lemma 3, for $u \in \widehat{W}_T^{1,p}$, one has

$$\varphi(u) = \frac{1}{p}\int_0^T \left|u'(t)\right|^p dt - \int_0^T F(t,u(t))\,dt \ge \left(\frac{1}{p} - \varepsilon C\right)\|u\|^p - C\|u\|^r. \tag{48}$$

Since $r > p$, then there exist $\rho > 0$ and $\alpha > 0$ such that

$$\varphi(u) \ge \alpha, \quad \forall u \in \widehat{W}_T^{1,p} \text{ with } \|u\| = \rho. \tag{49}$$

On the other hand, by (F_1), for any $M_1 > 0$, there exists $\delta_3 > 0$ such that

$$F(t,x) \ge M_1|x|^p, \quad \forall |x| \ge \delta_3, \ \text{a.e. } t \in [0,T]. \tag{50}$$

It follows from $(\mathscr{H})$ that

$$F(t,x) \le \max_{s\in[0,\delta_3]} a(s)\,b(t), \quad \forall |x| \le \delta_3, \ \text{a.e. } t \in [0,T]. \tag{51}$$

Therefore, we obtain

$$F(t,x) \ge M_1\left(|x|^p - \delta_3^p\right) - \max_{s\in[0,\delta_3]} a(s)\,b(t) \tag{52}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

Now, we choose $e_0 \in \widehat{W}_T^{1,p}$ being the eigenfunction corresponding to λ_1 which is defined in Lemma 8. For $\zeta > 0$, from (52), we have

$$\begin{aligned}\varphi(\zeta e_0) &= \frac{\zeta^p}{p}\int_0^T \left|e_0'(t)\right|^p dt - \int_0^T F(t,\zeta e_0(t))\,dt \\ &\le \frac{\zeta^p}{p}\int_0^T \left|e_0'(t)\right|^p dt - M_1\zeta^p \times \int_0^t \left|e_0(t)\right|^p dt + M_1\delta_3^p T + C \\ &= \zeta^p\left(\frac{1}{p} - \frac{M_1}{\lambda_1}\right)\|e_0\|^p + M_1\delta_3^p T + C.\end{aligned} \tag{53}$$

We choose $M_1 = 2(\lambda_1/p)$, and the above inequality implies that

$$\varphi(\zeta e_0) \longrightarrow -\infty, \quad \text{as } \zeta \longrightarrow +\infty. \tag{54}$$

In view of Lemma 14, (49), and (54), noting that $\varphi(0) = 0$ and applying the mountain pass theorem under the (C)-condition, there exists a critical point $u \in \widehat{W}_T^{1,p}$ of φ, such that $\varphi(u) \geq \alpha$. Hence u is a nontrivial solution of problem (1), and this completes the proof. □

We can weaken the condition (F_1) in the following condition (F_1'):

$$\limsup_{|x|\to 0} \frac{pF(t,x)}{|x|^p} < \lambda_1, \qquad \liminf_{|x|\to+\infty} \frac{pF(t,x)}{|x|^p} > \lambda_1 \tag{55}$$

uniformly for a.e. $t \in [0,T]$.

Theorem 15. *Suppose that $F(t,x)$ satisfies hypotheses $(\mathscr{H})$, (F_1'), and (F_2). Then problem* (1) *has at least one nontrivial solution in $\widehat{W}_T^{1,p}$.*

Proof. Checking the proof of Theorem 13, we only need to verify that (49) and (54) hold. In fact, by (F_1'), there exist two constants $\gamma < \lambda_1$ and $\delta_4 > 0$ such that

$$F(t,x) \leq \frac{\gamma}{p}|x|^p, \quad \forall |x| \leq \delta_4, \text{ a.e. } t \in [0,T]. \tag{56}$$

From (35) and (56), we have

$$F(t,x) \leq \frac{\gamma}{p}|x|^p + C|x|^r, \quad \forall x \in \mathbb{R}^N, \text{ a.e. } t \in [0,T]. \tag{57}$$

Thus, for $u \in \widehat{W}_T^{1,p}$, one has

$$\begin{aligned}\varphi(u) &= \frac{1}{p}\int_0^T \left|u'(t)\right|^p dt - \int_0^T F(t,u(t))\,dt \\ &\geq \frac{1}{p}\left(1 - \frac{\gamma}{\lambda_1}\right)\|u\|^p - C\|u\|^r.\end{aligned} \tag{58}$$

Since $r > p$ and $1 - \gamma/\lambda_1 > 0$, then (49) holds.

By (F_1'), there exist two constants $\gamma_1 > \lambda_1$ and $\delta_5 > 0$ such that

$$F(t,x) \geq \frac{\gamma_1}{p}|x|^p, \quad \forall |x| \geq \delta_5, \text{ a.e. } t \in [0,T]. \tag{59}$$

Hence, by $(\mathscr{H})$ and (59), we get

$$F(t,x) \geq \frac{\gamma_1}{p}\left(|x|^p - \delta_5^p\right) - \max_{s\in[0,\delta_5]} a(s)\,b(t) \tag{60}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$. Therefore, (60) and $\gamma_1 > \lambda_1$ imply that

$$\varphi(\zeta e_0) \longrightarrow -\infty, \quad \text{as } \zeta \longrightarrow +\infty. \tag{61}$$

Hence, we complete the proof. □

Next, we consider the asymptotically quadratic case. For this purpose, we suppose F satisfies the following conditions:

(F_3) $\limsup_{|x|\to+\infty}(F(t,x)/|x|^p) = +\infty$ uniformly for a.e. $t \in [0,T]$;

(F_4) there exists $l(t) \in L^1(0,T;\mathbb{R}^+)$ such that $(\nabla F(t,x),x) - pF(t,x) \geq l(t)$ for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$ and

$$\lim_{|x|\to+\infty}\left[(\nabla F(t,x),x) - pF(t,x)\right] = +\infty \tag{62}$$

uniformly for a.e. $t \in [0,T]$.

Theorem 16. *Suppose $(\mathscr{H})$, (F_1), (F_3), and (F_4) hold. Then problem* (1) *possesses at least one nontrivial solution in $\widehat{W}_T^{1,p}$.*

Proof. Paralleling to the proof of Theorem 13, we only need to verify that φ satisfies the (C)-condition. Suppose $\{u_n\} \subset \widehat{W}_T^{1,p}$, $\varphi(u_n)$ is bounded, and $(1 + \|u_n\|)\|\varphi'(u_n)\| \to 0$ as $n \to \infty$. Hence, it is easy to get that

$$\begin{aligned}&\int_0^T \left[(\nabla F(t,u_n(t)),u_n(t)) - pF(t,u_n(t))\right] dt \\ &\quad = p\varphi(u_n) - \left\langle \varphi'(u_n),u_n\right\rangle \leq C.\end{aligned} \tag{63}$$

We next show that $\{u_n\}$ is bounded in $\widehat{W}_T^{1,p}$. If not, without loss of generality, we can assume that $\|u_n\| \to +\infty$ as $n \to \infty$. Letting $v_n = u_n/\|u_n\|$, then $\|v_n\| = 1$, and so going to a sequence if necessary, we assume that $v_n \rightharpoonup v$ in $\widehat{W}_T^{1,p}$ and $v_n \to v$ in $C(0,T;\mathbb{R}^N)$.

By (F_3), there exist two constants $a_1 > 0$ and $R_1 > 0$ such that

$$F(t,x) \leq a_1|x|^p, \quad \forall |x| \geq R_1, \text{ a.e. } t \in [0,T]. \tag{64}$$

From $(\mathscr{H})$ and (64), we get

$$F(t,x) \leq a_1|x|^p + b(t)\max_{s\in[0,R_1]} a(s) \tag{65}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$. Hence, by (65), one has

$$\begin{aligned}\frac{\varphi(u_n)}{\|u_n\|^p} &= \frac{1}{p} - \frac{1}{\|u_n\|^p}\int_0^T F(t,u_n(t))\,dt \\ &\geq \frac{1}{p} - a_1\int_0^T |v_n(t)|^p dt - \frac{C}{\|u_n\|^p},\end{aligned} \tag{66}$$

which implies that

$$\|v\|_{L^p} \geq C > 0 \tag{67}$$

and thus $v \not\equiv 0$. Therefore, there exists a subset $E \subset [0,T]$ with $\operatorname{meas}(E) > 0$ such that $v(t) \neq 0$ on E.

By (F_4), from Lemma 2 in [5], then for $\varepsilon = (1/2)\operatorname{meas}(E) > 0$, there exists a subset $E_\varepsilon \subset [0,T]$ with $\operatorname{meas}([0,T]\setminus E_\varepsilon) < \varepsilon$ such that

$$\lim_{|x|\to+\infty}\left[(\nabla F(t,x),x) - pF(t,x)\right] = +\infty \tag{68}$$

uniformly for $t \in E_\varepsilon$. Obviously, $\operatorname{meas}(E \cap E_\varepsilon) > 0$. If not, we assume $\operatorname{meas}(E \cap E_\varepsilon) = 0$. Since $E = (E \cap E_\varepsilon) \cup (E \setminus E_\varepsilon)$, thus we have

$$\begin{aligned}0 &< \operatorname{meas}(E) \leq \operatorname{meas}(E \cap E_\varepsilon) + \operatorname{meas}([0,T] \setminus E_\varepsilon) \\ &< \varepsilon = \frac{1}{2}\operatorname{meas}(E),\end{aligned} \tag{69}$$

which leads to a contradiction. Hence, we have proved that

$$u_n(t) \longrightarrow +\infty, \quad \text{as } n \longrightarrow \infty, \text{ for a.e. } t \in E \cap E_\varepsilon. \tag{70}$$

From (F_4), we have

$$\begin{aligned}
&\int_0^T \left[\left(\nabla F\left(t, u_n(t)\right), u_n(t) \right) - pF\left(t, u_n(t)\right) \right] dt \\
&\quad = \int_{E\cap E_\varepsilon} \left[\left(\nabla F\left(t, u_n(t)\right), u_n(t) \right) - pF\left(t, u_n(t)\right) \right] dt \\
&\qquad + \int_{[0,T]\setminus E\cap E_\varepsilon} \left[\left(\nabla F\left(t, u_n(t)\right), u_n(t) \right) - pF\left(t, u_n(t)\right) \right] dt \\
&\quad \geq \int_{E\cap E_\varepsilon} \left[\left(\nabla F\left(t, u_n(t)\right), u_n(t) \right) - pF\left(t, u_n(t)\right) \right] dt \\
&\qquad + \int_{[0,T]\setminus E\cap E_\varepsilon} l(t)\, dt.
\end{aligned} \tag{71}$$

By (68) and (70), we get

$$\int_0^T \left[\left(\nabla F\left(t, u_n(t)\right), u_n(t) \right) - pF\left(t, u_n(t)\right) \right] dt \longrightarrow +\infty \tag{72}$$

as $n \to \infty$. This contradicts with (63). The proof is completed. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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