

Research Article

Mean Curvature Type Flow with Perpendicular Neumann Boundary Condition inside a Convex Cone

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We investigate the evolution of hypersurfaces with perpendicular Neumann boundary condition under mean curvature type flow, where the boundary manifold is a convex cone. We find that the volume enclosed by the cone and the evolving hypersurface is invariant. By maximal principle, we prove that the solutions of this flow exist for all time and converge to some part of a sphere exponentially as t tends to infinity.

1. Introduction

Let $\mathbb{N}^{n+1}(K)$ be a space form of sectional curvature $K = 1, 0$, or -1 . It is well known that the Riemannian metric of $\mathbb{N}^{n+1}(K)$ can be defined as

$$dz^2 = d\rho^2 + \phi^2(\rho) dz^2, \quad (1)$$

where dz^2 is the standard induced metric of unit sphere \mathbb{S}^n in Euclidean space \mathbb{R}^{n+1} , and

$$\phi(\rho) = \begin{cases} \sin \rho & K = 1 \\ \rho & K = 0 \\ \sinh \rho & K = -1. \end{cases} \quad (2)$$

Recently, Guan and Li [1] introduced a new type flow in the above space form which was called mean curvature type flow, as follows:

$$\frac{\partial F}{\partial t} = (n\phi'(\rho) - Hu) \nu, \quad (3)$$

where H and ν are the mean curvature and the outward unit normal vector of the evolving hypersurfaces, respectively, and u is the support function of the evolving hypersurfaces defined by $u = \langle \phi(\rho)(\partial/\partial\rho), \nu \rangle$. They proved that this flow evolves closed star-shaped hypersurfaces in space form into some sphere. A natural feature of this flow is that, along

the mean curvature type flow, the volume enclosed by the evolving hypersurface is a constant and its area is always decreasing as long as the solution exists.

Inspired by this result, we focus on the corresponding problem with perpendicular Neumann boundary condition inside a convex cone in \mathbb{R}^{n+1} . Precisely, we suppose that $\Sigma \subset \mathbb{R}^{n+1}$ is a convex cone with outward unit normal vector μ and M^n is a smooth n -dimensional hypersurface with boundary ∂M defined by an initial embedding $F_0 : M^n \rightarrow \mathbb{R}^{n+1}$. We will study how M^n evolves under the flow (3) with boundary conditions: $F_0(\partial M) \subset \Sigma$, and $\langle \mu, \nu \rangle = 0$, where ν is the outward unit normal vector to M^n . Namely, we will consider the following *mean curvature type flow with perpendicular Neumann boundary condition*:

$$\begin{aligned} \frac{dF}{dt} &= (n - Hu) \nu \quad \forall (y, t) \in M^n \times [0, T) \\ F(\cdot, 0) &= M_0 \end{aligned} \quad (4)$$

$$F(y, t) \subset \Sigma \quad \forall (y, t) \in \partial M^n \times [0, T)$$

$$\langle \mu, \nu \rangle(y, t) = 0 \quad \forall (y, t) \in \partial M^n \times [0, T),$$

where u is the Euclidean support function to the hypersurface M_t in \mathbb{R}^{n+1} defined by $u = \langle F, \nu \rangle$.

We take the origin on \mathbb{R}^{n+1} at the vertex of the cone, and $|F(y)|$ denotes the norm of the position vector at some point $y \in M^n$. In this paper, we obtain the following main result.

Theorem 1. *Suppose Σ is a convex cone and M_0 is the initial hypersurface which can be represented as a graph over the intersection of the interior of the cone Σ and a unit sphere \mathbb{S}^n centered at the vertex of the cone; then, a solution to (4) exists for all time and stays always between two spheres with radii $R_1 = \max_{y \in M_0} |F_0(y)|$ and $R_2 = \min_{y \in M_0} |F_0(y)|$. Furthermore, the solution converges exponentially fast to the intersection of the interior of the cone and the sphere with radius*

$$R = (n + 1) \frac{|V|}{|M_\infty|}, \tag{5}$$

where $|M_\infty|$ denotes the area of the intersection of the limit sphere and the interior of the cone and $|V|$ denotes the volume of the domain enclosed by the cone and the evolving hypersurface which is a constant under this flow.

The MCF with boundary conditions has been extensively studied by many mathematicians. Huisken in [2] considered the evolution of a graph over a bounded domain $\Omega \subset \mathbb{R}^n$ with perpendicular Neumann boundary condition and proved that the solution exists for all time and converges to a plane domain at last. More generally, for the hypersurfaces not necessarily represented as graphs, Stahl [3, 4] studied this problem and proved that the flow converges to a round point on the condition that the boundary manifold was umbilic and the initial surface was convex. Buckland in [5] founded boundary monotonicity formulae and classified Type I boundary singularities for $H > 0$ with a perpendicular Neumann boundary condition. Recently, Lambert in [6] considered this problem in a Minkowski space with a timelike cone boundary condition and proved that this flow converges to a homothetically expanding hyperbolic solution. Subsequently, he [7] also considered this problem inside a rotational tori.

However, little is known about the modified MCF, such as volume or area preserving MCF, with perpendicular Neumann boundary condition. Let M^n be a tubular hypersurface between two parallel planes in \mathbb{R}^{n+1} , which can be represented as a graph over some cylinder inside it and meets the parallel planes perpendicularly. Recently, Hartley [8] studied the motion of M^n under the volume preserving mean curvature flow by the center manifold analysis and proved that the solution exists for all time and converges exponentially fast to a cylinder in the C^k topology for any $k \in \mathbb{N}$ as time t tends to infinity.

Generally, for the prescribed contact angle which is not necessarily a right angle, this problem seems more hard. Altschuler and Wu in [9] considered the evolution of 2-dimensional graph over compact convex domain in \mathbb{R}^2 by mean curvature flow under this boundary condition and proved this flow exists for all time and converges to translating solutions at last. Guan [10] extended Altschuler-Wu's result to graphs of high dimensions.

The rest of this paper is organized as follows. In Section 2, we reparametrize the system (4) as a graph and give some primary facts. In Section 3, the evolution equations and boundary derivatives for some useful geometric quantities will be derived. In Section 4, a maximal principle will be

introduced and some basic estimates will be given. In the last section, we prove the convergence and complete the proof of the main theorem.

2. Reparametrization and Notations

Let M^n be a compact hypersurface inside an n -dimensional convex cone Σ with boundary condition $\partial M \subset \Sigma$, given by the embedding $F : \Omega \subset \mathbb{S}^n \rightarrow \mathbb{R}^{n+1}$, where Ω is the intersection of the interior of the cone and unit sphere \mathbb{S}^n centered at the vertex o of the cone; that is, M^n can be expressed as a graph over Ω . Precisely, for any point $x \in \Omega \subset \mathbb{S}^n$, there is only one ray from the vertex o through x intersecting the hypersurface M^n at some point $F(x)$; the position vector to M^n can be expressed as

$$F(x) = \rho(x)x, \quad \rho(x) \in \mathbb{R}^+, \quad x \in \Omega \subset \mathbb{S}^n. \tag{6}$$

Let $\{x_1, \dots, x_n\}$ be the local normal coordinates on \mathbb{S}^n ; e_{ij} denotes the standard spherical metric under the coordinates; the covariant derivative and divergent operator on \mathbb{S}^n with respect to the metric e_{ij} are denoted by ∇ and div , respectively. Then, tangent vectors and the outward unit normal vector on M^n can be expressed as in [11] (see also P28 in [12]):

$$\begin{aligned} \nabla_i F &= \nabla_i \rho x + \rho \nabla_i x, \\ \nu &= \frac{1}{\sqrt{\rho^2 + |\nabla \rho|^2}} (\rho x - e^{ij} \nabla_i \rho \nabla_j x), \end{aligned} \tag{7}$$

where e^{ij} is the inverse of e_{ij} . Thus, the support function, induced metric, and second fundamental form can be given by straightforward calculation as following:

$$\begin{aligned} u &= \frac{\rho^2}{\sqrt{\rho^2 + |\nabla \rho|^2}}, \\ g_{ij} &= \rho^2 e_{ij} + \nabla_i \rho \nabla_j \rho, \quad g^{ij} = \frac{1}{\rho^2} \left(e^{ij} - \frac{\nabla^i \rho \nabla^j \rho}{\rho^2 + |\nabla \rho|^2} \right), \\ h_{ij} &= \frac{1}{\sqrt{\rho^2 + |\nabla \rho|^2}} (-\rho \nabla_i \nabla_j \rho + 2 \nabla_i \rho \nabla_j \rho + \rho^2 e_{ij}). \end{aligned} \tag{8}$$

Furthermore, the system (4) is equivalent to the following parabolic PDE defined on $\Omega \times [0, T)$:

$$\begin{aligned} \frac{d\rho}{dt} &= (n - Hu) \frac{\rho}{u} = \text{div} \left(\frac{1}{\sigma} \nabla \rho \right) + \frac{n}{\rho \sigma} |\nabla \rho|^2 \\ \forall (x, t) &\in \Omega \times [0, T), \end{aligned} \tag{9}$$

$$\nabla \rho \cdot \mu = 0 \quad \forall (x, t) \in \partial \Omega \times [0, T),$$

$$\rho(x, 0) = \rho_0(x),$$

where $\sigma = \sqrt{\rho^2 + |\nabla \rho|^2}$.

The system (9) is a quasi-linear parabolic equation in divergence form, whose long-time existence is equivalent to

the uniform parabolicity, and C^1 bound on ρ by the classical theory of nonlinear parabolic equations (see, e.g., Chapter 12 in [13]).

For simplicity, let $r = \ln \rho$; then,

$$\nabla_i r \rho = \nabla_i \rho, \quad |\nabla r|^2 \rho^2 = |\nabla \rho|^2, \quad u = \frac{\rho}{\omega}, \quad (10)$$

where $\omega = \sqrt{1 + |\nabla r|^2}$. Then, the geometric quantities in (8) can be represented as

$$\begin{aligned} v &= \frac{1}{\omega} (x - e^{ij} \nabla_i r \nabla_j x), \\ g_{ij} &= \rho^2 (e_{ij} + \nabla_i r \nabla_j r), \quad g^{ij} = \frac{1}{\rho^2} \left(e^{ij} - \frac{\nabla^i r \nabla^j r}{\omega^2} \right), \end{aligned} \quad (11)$$

$$h_{ij} = \frac{\rho}{\omega} (-\nabla_i \nabla_j r + \nabla_i r \nabla_j r + e_{ij}),$$

and (9) can be rewritten as

$$\begin{aligned} \frac{dr}{dt} = \frac{n}{u} - H &= \operatorname{div} \left(\frac{1}{\rho \omega} \nabla r \right) + (n+1) \frac{|\nabla r|^2}{\rho \omega} \\ &\quad \forall (x, t) \in \Omega \times [0, T], \end{aligned} \quad (12)$$

$$\nabla r \cdot \mu = 0 \quad \forall (x, t) \in \partial \Omega \times [0, T],$$

$$r(x, 0) = r_0(x).$$

System (12) is also a quasi-linear parabolic equation in divergence form, and the related estimates will be derived in Section 4.

For convenient calculation, we also parametrize the boundary cone as Lambert [6]. Let $\tilde{S} : \mathbb{S}^{n-1} \rightarrow \mathbb{B}^n(0) \subset \mathbb{R}^n$ be a smooth embedding of a sphere into a topological ball centered at the origin with outward unit normal vector \mathbf{n} . Then, we can define the boundary cone Σ by embedding \mathbb{R}^n into \mathbb{R}^{n+1} at height 1, defining Σ to be the set of all rays going through the origin and some point $(\tilde{S}(z), 1) \in \mathbb{R}^{n+1}$, where $\{z_1, \dots, z_{n-1}\}$ denotes the $(n-1)$ -dimensional coordinate for \tilde{S} . So we can parametrize the cone by

$$F_\Sigma(l, \tilde{S}(z)) = \frac{l}{\sqrt{1 + |\tilde{S}|^2}} (\tilde{S}(z) + e_{n+1}), \quad (13)$$

where e_{n+1} is the $(n+1)$ th standard coordinate vector in \mathbb{R}^{n+1} . The second fundamental form of the boundary cone has the following characterization.

Proposition 2. *For the second fundamental form of the boundary cone, one has*

- (i) $A^\Sigma(\partial/\partial l, \cdot) = 0$,
- (ii) $A^\Sigma(\partial \tilde{S}/\partial z_i, \partial \tilde{S}/\partial z_j) = (l/(1 + |\tilde{S}|^2)) A^{\tilde{S}}(\partial \tilde{S}/\partial z_i, \partial \tilde{S}/\partial z_j)$,
 $i = 1, \dots, n-1$,

where A^Σ and $A^{\tilde{S}}$ denote the second fundamental forms, respectively, of Σ in \mathbb{R}^{n+1} and \tilde{S} in \mathbb{R}^n .

Proof. From the parametrization for the cone, it is easy to check that

$$\left\langle \frac{\partial \tilde{S}}{\partial z_i}, \mu \right\rangle = 0, \quad \langle \tilde{S} + e_{n+1}, \mu \rangle = 0. \quad (14)$$

Since μ is the outward unit normal vector to the boundary cone, calculating by Gauss equation directly we have

$$\begin{aligned} A^\Sigma \left(\frac{\partial}{\partial l}, \frac{\partial}{\partial l} \right) &= h_{ll}^\Sigma = - \left\langle \frac{\partial^2 F_\Sigma}{\partial l^2}, \mu \right\rangle = \langle 0, \mu \rangle = 0, \\ A^\Sigma \left(\frac{\partial}{\partial l}, \frac{\partial \tilde{S}}{\partial z_i} \right) &= h_{li}^\Sigma \\ &= - \left\langle \frac{1}{\sqrt{1 + |\tilde{S}|^2}} \frac{\partial \tilde{S}}{\partial z_i} - \left(\frac{1}{\sqrt{1 + |\tilde{S}|^2}} \right)_i \right. \\ &\quad \left. \times (\tilde{S}(z) + e_{n+1}), \mu \right\rangle = 0. \end{aligned} \quad (15)$$

This proves the first identity.

Similarly, using (14) again we have

$$\begin{aligned} A^\Sigma \left(\frac{\partial \tilde{S}}{\partial z_i}, \frac{\partial \tilde{S}}{\partial z_j} \right) &= - \left\langle \frac{\partial^2 F_\Sigma}{\partial z_i \partial z_j}, \mu \right\rangle \\ &= - \left\langle \frac{\partial^2 \tilde{S}}{\partial z_i \partial z_j}, \mu \right\rangle \frac{l}{\sqrt{1 + |\tilde{S}|^2}}. \end{aligned} \quad (16)$$

Observing that μ can be decomposed as

$$\mu = \langle \mu, \mathbf{n} \rangle \mathbf{n} + \langle \mu, e_{n+1} \rangle e_{n+1} \quad (17)$$

and $\langle \partial^2 \tilde{S}/\partial z_i \partial z_j, e_{n+1} \rangle = 0$, we, then, have

$$\begin{aligned} A^\Sigma \left(\frac{\partial \tilde{S}}{\partial z_i}, \frac{\partial \tilde{S}}{\partial z_j} \right) &= - \frac{l \cos \angle(\mu, \mathbf{n})}{\sqrt{1 + |\tilde{S}|^2}} \left\langle \mathbf{n}, \frac{\partial^2 \tilde{S}}{\partial z_i \partial z_j} \right\rangle \\ &= \frac{l}{1 + |\tilde{S}|^2} A^{\tilde{S}} \left(\frac{\partial \tilde{S}}{\partial z_i}, \frac{\partial \tilde{S}}{\partial z_j} \right), \end{aligned} \quad (18)$$

where we use the fact that $\cos \angle(\mu, \mathbf{n}) = 1/\sqrt{1 + |\tilde{S}|^2}$. \square

3. Evolution Equations and Boundary Derivatives

In this section, we will derive evolution equations for some useful geometric quantities by straightforward calculation. Let $\{y_1, \dots, y_n\}$ be the local normal coordinates of the evolving hypersurface and let g_{ij} be the corresponding induced

metric; let $\bar{\nabla}$ and $\bar{\Delta}$ be, respectively, the covariant derivative and Laplace operator with respect to the induced metric g_{ij} ; and $|F|$ and $|A|$ denote the norm of the position vector and the second fundamental form to evolving hypersurfaces M^n in \mathbb{R}^{n+1} , respectively.

Lemma 3. *Under the mean curvature type flow (4), we have*

- (i) $(d/dt - u\bar{\Delta})|F|^2 = 0$,
- (ii) $d\nu/dt = -\bar{\nabla}(n - Hu)$,
- (iii) $(d/dt - u\bar{\Delta})u = n - 2Hu + |A|^2u^2 + H\langle\bar{\nabla}u, F\rangle$.

Proof. (i) We can calculate directly as in [14]

$$\frac{d|F|^2}{dt} = 2 \left\langle \frac{dF}{dt}, F \right\rangle = 2 \langle (n - Hu)\nu, F \rangle = 2(n - Hu)u. \quad (19)$$

On the other hand,

$$\begin{aligned} \bar{\Delta}|F|^2 &= g^{ij}\bar{\nabla}_i\bar{\nabla}_j\langle F, F \rangle = 2g^{ij}\bar{\nabla}_i \left\langle \frac{\partial F}{\partial y_j}, F \right\rangle \\ &= 2n - 2g^{ij}\langle h_{ij}\nu, F \rangle = 2(n - Hu). \end{aligned} \quad (20)$$

This proves (i).

(ii) As $\langle\nu, \nu\rangle = 1$, we have $\langle d\nu/dt, \nu \rangle = 0$, and then,

$$\begin{aligned} \frac{d\nu}{dt} &= \left\langle \frac{d\nu}{dt}, \frac{\partial F}{\partial y_i} \right\rangle g^{ij} \frac{\partial F}{\partial y_j} = - \left\langle \nu, \frac{\partial}{\partial y_i} \frac{\partial F}{\partial t} \right\rangle g^{ij} \frac{\partial F}{\partial y_j} \\ &= - \frac{\partial}{\partial y_i} (n - Hu) g^{ij} \frac{\partial F}{\partial y_j} = -\bar{\nabla}(n - Hu). \end{aligned} \quad (21)$$

(iii) Similarly, as in (i),

$$\begin{aligned} \frac{du}{dt} &= \frac{d}{dt} \langle F, \nu \rangle = \left\langle \frac{dF}{dt}, \nu \right\rangle + \left\langle F, \frac{d\nu}{dt} \right\rangle \\ &= (n - Hu) - \langle F, \bar{\nabla}(n - Hu) \rangle, \\ \bar{\Delta}u &= g^{ij}\bar{\nabla}_i\bar{\nabla}_j \langle F, \nu \rangle = g^{ij}\bar{\nabla}_i \left(\left\langle \frac{\partial F}{\partial y_j}, \nu \right\rangle + \left\langle F, h_{jk} \frac{\partial F}{\partial y_k} \right\rangle \right) \\ &= g^{ij}g_{ik}h_{jk} + \langle F, \bar{\nabla}H \rangle - h_{jk}h_{il}g^{ij}g^{kl}u \\ &= H + \langle F, \bar{\nabla}H \rangle - |A|^2u. \end{aligned} \quad (22)$$

Then, (iii) follows by combining the above two formulae. \square

The following relationship between A^Σ and A was proved by Stahl in [3].

Lemma 4 (see [3]). *For $p \in \partial M \times [0, T)$, one has*

$$A^\Sigma(\xi, \nu) = -A(\xi, \mu), \quad (23)$$

where $\xi \in T_pM \cap T_p\Sigma$.

In order to apply the Hopf maximal principle to obtain the basic estimates, we also need the following boundary derivatives.

Lemma 5. *For $(y, t) \in \partial M \times [0, T)$, one has*

- (i) $\langle \bar{\nabla}|F|^2, \mu \rangle = 0$,
- (ii) $\langle \bar{\nabla}u, \mu \rangle = uA^\Sigma(\nu, \nu)$.

Proof. (i) Denote by $\tilde{\nabla}$ the Euclidean covariant derivative in \mathbb{R}^{n+1} . Obviously, $\tilde{\nabla}|F|^2 \in T_p\Sigma$; combining the boundary condition in (4), we have

$$\langle \bar{\nabla}|F|^2, \mu \rangle = \langle \tilde{\nabla}|F|^2 - \langle \tilde{\nabla}|F|^2, \nu \rangle \nu, \mu \rangle = \langle \tilde{\nabla}|F|^2, \mu \rangle = 0. \quad (24)$$

(ii) Calculating directly, we obtain

$$\bar{\nabla}u = \left\langle F, \frac{\partial \nu}{\partial y_i} \right\rangle g^{ij} \frac{\partial}{\partial y_j} = A \left(F^\top, \frac{\partial}{\partial y_i} \right) g^{ij} \frac{\partial}{\partial y_j}, \quad (25)$$

$$\langle \bar{\nabla}u, \mu \rangle = \left\langle A \left(F^\top, \frac{\partial}{\partial y_i} \right) g^{ij} \frac{\partial}{\partial y_j}, \mu \right\rangle = A(F^\top, \mu).$$

Observing $F^\top = F - u\nu$, combination of Lemma 4 and (i) in Proposition 2 yields

$$\begin{aligned} \langle \bar{\nabla}u, \mu \rangle &= -A^\Sigma(F - u\nu, \nu) = -A^\Sigma(F, \nu) + uA^\Sigma(\nu, \nu) \\ &= uA^\Sigma(\nu, \nu). \end{aligned} \quad (26)$$

\square

4. Gradient Estimate

Let $a_{ij}(y, t) \in L^\infty(M^n \times [0, T))$ be a positive definite matrix such that

$$\mathcal{L}f = \frac{df}{dt} - a^{ij}\bar{\nabla}_i\bar{\nabla}_j f \quad (27)$$

is a parabolic operator. We have the following maximal principle from Hopf Lemma [15] or Stahl's corresponding result in [4].

Theorem 6. *Suppose $f : M^n \times [0, T) \rightarrow \mathbb{R}$ satisfies*

$$\frac{df}{dt} - a^{ij}\bar{\nabla}_i\bar{\nabla}_j f(y, t) \leq 0 \quad (28)$$

$$\forall (y, t) \in M^n \times [0, T) \text{ s.t. } \bar{\nabla}f(y) = 0,$$

$$\langle \bar{\nabla}f, \mu \rangle \leq 0 \quad \forall (y, t) \in \partial M^n \times [0, T),$$

and then $f(y, t) \leq \sup_{y \in M^n} f(y, 0)$ for all $(y, t) \in M^n \times [0, T)$.

Now we apply the above theorem to $|F|^2$. Combining (i) in Lemma 3 and (i) in Lemma 5, we immediately have the following estimates.

Corollary 7. *Let $F(y, t)$ be a solution to system (4); then, one has*

$$\min_{y \in M_0^n} |F_0(y)| \leq |F(y, t)| \leq \max_{y \in M_0^n} |F_0(y)| \quad (29)$$

for any $(y, t) \in M^n \times [0, T)$.

This result means that the evolving hypersurface always stays between two spheres with radii $R_1 = \max_{y \in M_0^n} |F_0(y)|$ and $R_2 = \min_{y \in M_0^n} |F_0(y)|$.

In order to obtain the gradient estimate for system (12), we only need to estimate the lower bound for $u = \rho/\omega = \rho/\sqrt{1 + |\nabla r|^2}$. From (iii) in Lemma 3,

$$\begin{aligned} \left(\frac{d}{dt} - u\bar{\Delta}\right)u &= n - 2Hu + |A|^2u^2 + H\langle\bar{\nabla}u, F\rangle \\ &\geq (n - Hu) - \frac{Hu}{n}(n - Hu) + H\langle\bar{\nabla}u, F\rangle \\ &= \frac{1}{n}(n - Hu)^2 + H\langle\bar{\nabla}u, F\rangle. \end{aligned} \tag{30}$$

On the other hand, noticing $\langle\mu, \nu\rangle = 0$ by assumption, ν must be in the tangent space of the boundary cone; that is, $\nu \in T\Sigma$. Combining (ii) in Lemma 5, Proposition 2, and the convexity of the boundary cone, we have

$$\langle\bar{\nabla}u, \mu\rangle = uA^\Sigma(\nu, \nu) \geq 0. \tag{31}$$

Equivalently,

$$\left(\frac{d}{dt} - u\bar{\Delta}\right)(-u) \leq -\frac{1}{n}(n - Hu)^2 \leq 0$$

$$\forall (y, t) \in M^n \times [0, T] \text{ s.t. } \bar{\nabla}u(y, t) = 0,$$

$$\langle\bar{\nabla}(-u), \mu\rangle = -uA^\Sigma(\nu, \nu) \leq 0 \quad \forall (y, t) \in \partial M^n \times [0, T]. \tag{32}$$

By the maximal principle we have the following.

Corollary 8. *For all $(y, t) \in M^n \times [0, T]$, the support function $u(y, t)$ satisfies*

$$u(y, t) \geq \min_{y \in M^n} u(y, 0). \tag{33}$$

Combining Corollaries 7 and 8, we obtain long time existence for the system (12) by the standard argument for divergence PDE (cf. [13]), and then, the first part of Theorem 1 follows by the equivalence of (4) and (12).

Let V be the domain enclosed by the interior of the cone and the evolving hypersurface. The volume element for V is denoted by dV and dV_∂ for its boundary. We will also denote the area elements for the evolving hypersurface and its boundary by dv and dv_∂ , respectively. Flow (4) has the following interesting property.

Proposition 9. *Along flow (4), the volume of V , denoted by $|V|$, is a constant.*

Proof. For the function $\Phi(y, t) = (1/2)|F(y, t)|^2$ defined on the evolving hypersurface M_t , we have

$$\bar{\Delta}\Phi = n - Hu \tag{34}$$

from the proof of Lemma 3(i). Integrating the above equation on M_t and taking into consideration Lemma 5(i) yield

$$\begin{aligned} \int_{M_t} (n - Hu) dv &= \int_{M_t} \bar{\Delta}\Phi dv = \int_{\partial M_t} \langle\bar{\nabla}\Phi, \mu\rangle dv_\partial \\ &= \frac{1}{2} \int_{\partial M_t} \langle\bar{\nabla}|F|^2, \mu\rangle dv_\partial = 0. \end{aligned} \tag{35}$$

Denote by $\widetilde{\text{div}}X$ the divergence of the position vector X for V in \mathbb{R}^{n+1} ; we have

$$\int_V \widetilde{\text{div}}X dV = (n + 1)|V|. \tag{36}$$

Taking derivative with respect to time t and combining divergence theorem yield

$$\begin{aligned} (n + 1) \frac{d}{dt} |V| &= \int_V \widetilde{\text{div}}\left(\frac{dX}{dt}\right) dV = \int_{\partial V} \left\langle\frac{dX}{dt}, \eta\right\rangle dV_\partial \\ &= \int_\Sigma \left\langle\frac{dX}{dt}, \mu\right\rangle dv_\Sigma + \int_{M_t} \left\langle\frac{dX}{dt}, \nu\right\rangle dv \\ &= \int_{M_t} (n - Hu) dv = 0, \end{aligned} \tag{37}$$

where η is the outward unit normal vector to the $(n + 1)$ -dimensional region V , and dv_Σ is the area element on boundary cone Σ . Hence, $|V|$ is a constant and the result follows. \square

5. Convergence

In this section, we use an idea of Guan and Li in [1] to obtain our estimate and prove the exponential convergence. For that purpose, the system (9) or (12) is convenient for us. Now, we choose a local coordinate $\{x_1, \dots, x_n\}$ on $\Omega \subset \mathbb{S}^n$. ∇ and Δ are again the covariant derivative and Laplace operator with respect to the standard metric e_{ij} on \mathbb{S}^n . Let \mathfrak{L} be a parabolic operator defined by

$$\mathfrak{L}f = \frac{df}{dt} - \frac{1}{\rho\omega} \left(e^{ij} - \frac{\nabla^i r \nabla^j r}{\omega^2} \right) f_{ij} \tag{38}$$

on $\Omega \times [0, T]$ for a smooth function $f : \Omega \times [0, T] \rightarrow \mathbb{R}$. From [1], we have the following evolution equation for $|\nabla r|^2/2$ at the critical point

$$\begin{aligned} \mathfrak{L}\left(\frac{|\nabla r|^2}{2}\right) &= -\frac{n}{\rho\omega} |\nabla r|^4 - \frac{n-1}{\rho\omega} |\nabla r|^2 \\ &\quad - \frac{1}{\rho\omega} \Delta r |\nabla r|^2 - \frac{1}{\rho\omega} |\nabla^2 r|^2. \end{aligned} \tag{39}$$

Using Cauchy-Schwartz inequality and similar rearrangement as in [10], we have

$$\begin{aligned} \mathfrak{L}|\nabla r|^2 &\leq -\frac{3n}{2\rho\omega} |\nabla r|^4 - \frac{2(n-1)}{\rho\omega} |\nabla r|^2 \\ &\quad - \frac{2}{n\rho\omega} \left(\Delta r + \frac{n}{2} |\nabla r|^2 \right)^2. \end{aligned} \tag{40}$$

Therefore,

$$\mathfrak{Q}|\nabla r|^2 \leq -\frac{2(n-1)}{\rho\omega}|\nabla r|^2 \leq 0. \quad (41)$$

Recall that $\rho = \omega u$ and $\omega = \sqrt{1 + |\nabla r|^2}$; we have, by Lemma 5 and the Neumann boundary assumption, for any points $(x, t) \in \partial\Omega \times [0, T)$,

$$\begin{aligned} \langle \nabla(|\nabla r|^2), \mu \rangle &= \langle \nabla(\omega^2 - 1), \mu \rangle \\ &= \frac{2\rho}{u^2} \langle \nabla\rho, \mu \rangle \\ &\quad - \frac{2\rho^2}{u^3} \langle \nabla u, \mu \rangle = -\frac{2\rho^2}{u^2} A^\Sigma(\nu, \nu). \end{aligned} \quad (42)$$

With the assumption of convexity on the boundary cone, we obtain, by Theorem 6,

$$|\nabla r|^2(x, t) \leq \max_{x \in \Omega} |\nabla r|^2(x, 0). \quad (43)$$

By the equivalence of (4) and (9), we have

$$\min_{x \in \Omega} \rho(x, 0) \leq \rho(x, t) \leq \max_{x \in \Omega} \rho(x, 0) \quad (44)$$

from Corollary 7.

From the above estimate, there exists a uniform positive constant $\alpha \leq 2(n-1)/\rho\omega$ depending only on the upper bound of ω and ρ such that (41) reads as

$$\mathfrak{Q}|\nabla r|^2 \leq -\alpha|\nabla r|^2. \quad (45)$$

Denote by $g = e^{\alpha t}|\nabla r|^2$

$$\begin{aligned} \mathfrak{Q}g &= \mathfrak{Q}(e^{\alpha t}|\nabla r|^2) = \alpha e^{\alpha t}|\nabla r|^2 + e^{\alpha t}\mathfrak{Q}|\nabla r|^2 \leq 0, \\ \langle \nabla g, \mu \rangle &= e^{\alpha t} \langle \nabla\omega^2, \mu \rangle = -2\frac{\rho^2}{u^2} e^{\alpha t} A^\Sigma(\nu, \nu) \leq 0. \end{aligned} \quad (46)$$

Then, by Theorem 6 again, $g = e^{\alpha t}|\nabla r|^2$ has uniform upper bound $C_1 = \max_{x \in \Omega} g(x, 0)$; that is

$$|\nabla r|^2 \leq C_1 e^{-\alpha t}, \quad t \in [0, \infty). \quad (47)$$

Or, equivalently, for a different constant $C_2 = \rho_{\max, x \in \Omega}^2(x, 0)C_1$,

$$|\nabla\rho|^2 \leq C_2 e^{-\alpha t}, \quad t \in [0, \infty). \quad (48)$$

This means that M_t converges exponentially fast to some part of a sphere.

Assume the radial function $\rho(x, t_0)$ attains its minimum at a point $p \in \Omega$ for some $t_0 \in [0, \infty)$; that is, $\rho(p, t_0) = \rho_{\min, x \in \Omega}(x, t_0)$. Let $\sigma : [0, s) \rightarrow \Omega$ be a geodesic on the unit sphere \mathbb{S}^n starting from p to any point $x \in \Omega$ with $\sigma(s) = x$. Integrating both sides of the last inequality on $[0, s]$ and taking into account the boundedness of s , we have, for some constant $C_3 = C_3(\Omega, C_2, \alpha)$ and $\beta = \alpha/2$,

$$\int_0^s \frac{d\rho}{ds} ds \leq \int_0^s |\nabla\rho| ds \leq C_3 e^{-\beta t_0}, \quad (49)$$

and then,

$$\rho(x, t_0) - \rho(p, t_0) = \rho(x, t_0) - \rho_{\min, x \in \Omega}(x, t_0) \leq C_3 e^{-\beta t_0}. \quad (50)$$

So we have

$$\rho_{\max, x \in \Omega}(x, t) - \rho_{\min, x \in \Omega}(x, t) \leq C_3 e^{-\beta t}, \quad t \in [0, \infty). \quad (51)$$

On the other hand,

$$\begin{aligned} (n+1)|V| &= \int_V \widetilde{\operatorname{div}} X dV = \int_{\partial V} \langle X, \eta \rangle dV_{\partial} \\ &= \int_{\Sigma} \langle X, \mu \rangle dV_{\Sigma} + \int_{M_t} \langle X, \nu \rangle dV = \int_{M_t} u dV. \end{aligned} \quad (52)$$

Because

$$\begin{aligned} \rho_{\max, x \in \Omega}(x, t) - C_3 e^{-\beta t} &\leq \rho(x, t) \leq \rho_{\min, x \in \Omega}(x, t) + C_3 e^{-\beta t}, \\ 1 \leq \omega &= \sqrt{1 + |\nabla r|^2} \leq \sqrt{1 + C_1 e^{-\alpha t}}, \end{aligned} \quad (53)$$

the support function $u = \rho/\omega$ satisfies

$$\frac{\rho_{\max, x \in \Omega}(x, t) - C_3 e^{-\beta t}}{\sqrt{1 + C_1 e^{-\alpha t}}} \leq u \leq \rho_{\min, x \in \Omega}(x, t) + C_3 e^{-\beta t}. \quad (54)$$

Denote by $|M_t|$ the area of M_t ; combining (52), we have

$$\begin{aligned} |M_t| \frac{\rho_{\max, x \in \Omega}(x, t) - C_3 e^{-\beta t}}{\sqrt{1 + C_1 e^{-\alpha t}}} &\leq (n+1)|V| \\ &\leq |M_t| \left(\rho_{\min, x \in \Omega}(x, t) + C_3 e^{-\beta t} \right), \end{aligned} \quad (55)$$

or, equivalently,

$$\begin{aligned} \frac{(n+1)|V|}{\rho_{\min, x \in \Omega}(x, t) + C_3 e^{-\beta t}} &\leq |M_t| \leq (n+1)|V| \frac{\sqrt{1 + C_1 e^{-\alpha t}}}{\rho_{\max, x \in \Omega}(x, t) - C_3 e^{-\beta t}}. \end{aligned} \quad (56)$$

Let $\lim_{t \rightarrow \infty} \rho(x, t) = \rho(x)$, and from the above inequality, we obtain $\rho(x) = R$, a constant, and therefore,

$$|M_t| \longrightarrow \frac{(n+1)|V|}{R}, \quad t \longrightarrow \infty. \quad (57)$$

That is to say, M_t converges exponentially to the intersection of the interior of the cone Σ and a sphere centered at the vertex of the cone with radius

$$R = \frac{(n+1)|V|}{|M_{\infty}|}, \quad (58)$$

where $|M_{\infty}|$ denotes the area of the limit sphere M_{∞} . This finishes the proof of Theorem 1.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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