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Research Article

Admissible Solutions of the Schwarzian Type Difference Equation

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This paper is to investigate the Schwarzian type difference equation $\left[(\Delta^3 f/\Delta f)-(3/2)(\Delta^2 f/\Delta f)^2\right]^k = R(z,f) = (P(z,f)/Q(z,f))$, where $R(z,f)$ is a rational function in f with polynomial coefficients, $P(z,f)$, respectively $Q(z,f)$ are two irreducible polynomials in f of degree p, respectively q. Relationship between p and q is studied for some special case. Denote $d=\max\{p,q\}$. Let $f(z)$ be an admissible solution of $(*)$ such that $\rho_2(f)<1$; then for s (≥ 2) distinct complex constants $\alpha_1,\ldots,\alpha_s$, $q+2k\sum_{j=1}^{s}\delta(\alpha_j,f)\leq 8k$. In particular, if $N(r,f)=S(r,f)$, then $d+2k\sum_{j=1}^{s}\delta(\alpha_j,f)\leq 4k$.

1. Introduction and Results

Throughout this paper, a meromorphic function always means being meromorphic in the whole complex plane, and c always means a nonzero constant. For a meromorphic function $f(z)$, we define its shift by $f(z+c)$ and define its difference operators by

$$\Delta_c f(z) = f(z+c) - f(z), \qquad \Delta_c^n f(z) = \Delta_c^{n-1}\left(\Delta_c f(z)\right), \quad n\in\mathbb{N},\ n\geq 2. \tag{1}$$

In particular, $\Delta_c^n f(z) = \Delta^n f(z)$ for the case $c=1$. We use standard notations of the Nevanlinna theory of meromorphic functions such as $T(r,f)$, $m(r,f)$, and $N(r,f)$ and as stated in [1–3]. For a constant a, we define the Nevanlinna deficiency by

$$\begin{aligned}\delta(a,f) &= \liminf_{r\to\infty}\frac{m(r,1/(f-a))}{T(r,f)} \\ &= 1-\limsup_{r\to\infty}\frac{N(r,1/(f-a))}{T(r,f)}.\end{aligned} \tag{2}$$

Recently, numbers of papers (see, e.g., [4–12]) are devoted to considering the complex difference equations and difference analogues of Nevanlinna theory. Due to some idea of [13], we consider the admissible solution of the Schwarzian type difference equation:

$$\widetilde{S}_k(f) := \left[\frac{\Delta^3 f}{\Delta f} - \frac{3}{2}\left(\frac{\Delta^2 f}{\Delta f}\right)^2\right]^k = R(z,f) = \frac{P(z,f)}{Q(z,f)}, \tag{3}$$

where $R(z,f)$ is a rational function in f with polynomial coefficients, $P(z,f)$, respectively $Q(z,f)$, are two irreducible polynomials in f of degree p, respectively, q. Here and in the following, "admissible" always means "transcendental." And we denote $d=\max\{p,q\}$ from now on. For the existence of solutions of (3), we give some examples below.

Examples. (1) $f(z)=\sin \pi z + z$ is an admissible solution of the Schwarzian type difference equation:

$$\frac{\Delta^3 f}{\Delta f} - \frac{3}{2}\left(\frac{\Delta^2 f}{\Delta f}\right)^2 = \frac{-8\left[f^2+(1-2z)f+z(z-1)\right]}{4f^2-4(2z+1)f+(2z+1)^2}. \tag{4}$$

(2) $f(z) = (e^{z\ln 2}/\sin 2\pi z) + z$ is an admissible solution of the Schwarzian type difference equation

$$\frac{\Delta^3 f}{\Delta f} - \frac{3}{2}\left(\frac{\Delta^2 f}{\Delta f}\right)^2 = \frac{-f^2 + 2(z+1)f - z^2 - 2z}{2f^2 - 4(z-1)f + 2(z-1)^2}. \quad (5)$$

(3) Let $f(z) = z^2 + z$, then $f(z)$ solves the Schwarzian type difference equation:

$$\frac{\Delta^3 f}{\Delta f} - \frac{3}{2}\left(\frac{\Delta^2 f}{\Delta f}\right)^2 = -\frac{3}{2\left[f^2 - 2(z^2-1)f + (z^2-1)^2\right]}. \quad (6)$$

This example shows that (3) may admit polynomial solutions.

Considering the relationship between p and q in those examples above, we prove the following result.

Theorem 1. *For the Schwarzian type difference equation* (3) *with polynomial coefficients, note the following.*

(i) *If it admits an admissible solution $f(z)$ such that $\rho_2(f) < 1$, then*

$$pm(r,f) \le qm(r,f) + S(r,f). \quad (7)$$

In particular, if $m(r,f) \ne S(r,f)$, then $p \le q$.

(ii) *If its coefficients are all constants and it admits a polynomial solution $f(z)$ with degree s, then $s \ge 2$ and $qs = ps + 2k$.*

Remark 2. From examples (1) and (2), we conjecture that $p = q$ in Theorem 1(i). However, we cannot prove it currently. From example (3) given before, we see that the restriction on the coefficients in Theorem 1(ii) cannot be omitted.

For the Schwarzian differential equation,

$$S_k(f) = \left[\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right]^k = R(z,f) = \frac{P(z,f)}{Q(z,f)}, \quad (8)$$

where $R(z,f)$, $P(z,f)$, and $Q(z,f)$ are as stated before; Ishizaki [13] proved the following result (see also Theorem 9.3.2 in [2]).

Theorem A (see [2, 13]). *Let $f(z)$ be an admissible solution of* (8) *with polynomial coefficients, and let $\alpha_1, \ldots, \alpha_s$ be s (≥ 2) distinct complex constants. Then*

$$d + 2k\sum_{j=1}^{s}\delta(\alpha_j, f) \le 4k. \quad (9)$$

For the Schwarzian type difference equation (3), we prove the following result.

Theorem 3. *Let $f(z)$ be an admissible solution of* (3) *with polynomial coefficients such that $\rho_2(f) < 1$, and let $\alpha_1, \ldots, \alpha_s$ be s (≥ 2) distinct complex constants. Then*

$$q + 2k\sum_{j=1}^{s}\delta(\alpha_j, f) \le 8k. \quad (10)$$

In particular, if $N(r,f) = S(r,f)$, then

$$d + 2k\sum_{j=1}^{s}\delta(\alpha_j, f) \le 4k. \quad (11)$$

Remark 4. From Theorem 1, under the condition $N(r,f) = S(r,f)$ in Theorem 3, we have $d = q$ in (11). The behavior of the zeros and the poles of $f(z)$ in $\widetilde{S}_k(f)$ is essentially different from that in the $S_k(f)$. We wonder whether the restriction $N(r,f) = S(r,f)$ can be omitted or not.

2. Lemmas

The following lemma plays a very important role in the theory of complex differential equations and difference equations. It can be found in Mohon'ko [14] and Valiron [15] (see also Theorem 2.2.5 in the book of Laine and Yang [2]).

Lemma 5 (see [14, 15]). *Let $f(z)$ be a meromorphic function. Then, for all irreducible rational functions in f,*

$$R(z,f) = \frac{P(z,f)}{Q(z,f)} = \frac{\sum_{i=0}^{p} a_i(z) f^i}{\sum_{j=0}^{q} b_j(z) f^j}, \quad (12)$$

with meromorphic coefficients $a_i(z), b_j(z)$ such that

$$\begin{aligned} T(r,a_i) &= S(r,f), \quad i = 0, \ldots, p, \\ T(r,b_j) &= S(r,f), \quad j = 0, \ldots, q, \end{aligned} \quad (13)$$

and the characteristic function of $R(z,f)$ satisfies

$$T(r, R(z,f)) = dT(r,f) + S(r,f), \quad (14)$$

where $d = \max\{p, q\}$.

The following two results can be found in [10]. In fact, Lemma 6 is a special case of Lemma 8.3 in [10].

Lemma 6 (see [10]). *Let $f(z)$ be a meromorphic function of hyper order $\rho_2(f) = \varsigma < 1$, $c \in \mathbb{C}$ and $\varepsilon > 0$. Then*

$$T(r, f(z+c)) = T(r,f) + S(r,f), \quad (15)$$

possibly outside of a set of r with finite logarithmic measure.

Lemma 7 (see [10]). *Let $f(z)$ be a meromorphic function of hyper order $\rho_2(f) = \varsigma < 1$, $c \in \mathbb{C}$ and $\varepsilon > 0$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{1-\varsigma-\varepsilon}}\right) = S(r,f), \quad (16)$$

possibly outside of a set of r with finite logarithmic measure.

From Lemma 7, we can easily get the following conclusion.

Lemma 8. *Let $f(z)$ be a meromorphic function of hyper order $\rho_2(f)=\varsigma<1$, $c\in\mathbb{C}$ and $\varepsilon>0$. Then*

$$m\left(r,\frac{\Delta_c^n f(z)}{f(z)}\right)=S(r,f),$$
$$m\left(r,\frac{\Delta_c^k f(z)}{\Delta_c^j f(z)}\right)=S(r,f),\quad k>j, \tag{17}$$

possibly outside of a set of r with finite logarithmic measure.

Lemma 9. *Let f be an admissible solution of (3) with coefficients. Then, using the notation $Q(z):=Q(z,f(z))$,*

$$qT(r,f)+S(r,f)\le N\left(r,\frac{1}{Q}\right). \tag{18}$$

In particular, if $N(r,f)=S(r,f)$, then

$$dT(r,f)+S(r,f)\le N\left(r,\frac{1}{Q}\right). \tag{19}$$

Proof. We use the idea by Ishizaki [13] (see also [2]) to prove Lemma 9. It follows from Lemma 8 that

$$\begin{aligned}m(r,R)&=m\left(r,\left[\frac{\Delta^3 f}{\Delta f}-\frac{3}{2}\left(\frac{\Delta^2 f}{\Delta f}\right)^2\right]^k\right)\\&\le km\left(r,\frac{\Delta^3 f}{\Delta f}\right)+2km\left(r,\frac{\Delta^2 f}{\Delta f}\right)\\&\quad+S(r,f)=S(r,f).\end{aligned} \tag{20}$$

From this and Lemma 5, we get

$$dT(r,f)+S(r,f)=T(r,R)=N(r,R)+S(r,f), \tag{21}$$

and hence

$$dT(r,f)=N(r,R)+S(r,f). \tag{22}$$

If $d=p>q$, since all coefficients of $P(z,f)$ and $Q(z,f)$ are polynomials, there are at the most finitely many poles of $R(z,f)$, neither the poles of $f(z)$ nor the zeros of $Q(z,f)$. Therefore, we see that

$$\begin{aligned}N(r,R)&\le(p-q)N(r,f)+N\left(r,\frac{1}{Q}\right)+S(r,f)\\&\le(p-q)T(r,f)+N\left(r,\frac{1}{Q}\right)+S(r,f).\end{aligned} \tag{23}$$

We obtain (18) from this and (22) immediately.

If $d=q\ge p$, there are at most finitely many poles of $R(z,f)$, not the zeros of $Q(z,f)$, then

$$N(r,R)\le N\left(r,\frac{1}{Q}\right)+S(r,f). \tag{24}$$

Now (18) follows from (22) and (24).

Notice that if $N(r,f)=S(r,f)$, then (24) always holds. This finishes the proof of Lemma 9. □

3. Proof of Theorem 1

Case 1. Equation (3) admits an admissible solution $f(z)$ such that $\rho_2(f)<1$. Since all coefficients of $P(z,f)$ and $Q(z,f)$ are polynomials, there are at the most finitely many poles of $f(z)$ that are not the poles of $P(z,f)$ and $Q(z,f)$. This implies that

$$\begin{aligned}N(r,P)&=pN(r,f)+S(r,f),\\N(r,Q)&=qN(r,f)+S(r,f).\end{aligned} \tag{25}$$

From Lemma 5, we get

$$\begin{aligned}T(r,P)&=pT(r,f)+S(r,f),\\T(r,Q)&=qT(r,f)+S(r,f).\end{aligned} \tag{26}$$

We can deduce from (3), (25), (26), and Lemma 8 that

$$\begin{aligned}pT(r,f)+S(r,f)&=T(r,P)\\&=m(r,P)+N(r,P)\\&\le pN(r,f)+m\left(r,\widetilde{S}_k(f)Q\right)\\&\quad+S(r,f)\\&\le pN(r,f)+m\left(r,\widetilde{S}_k(f)\right)\\&\quad+m(r,Q)+S(r,f)\\&=pN(r,f)+T(r,Q)-N(r,Q)\\&\quad+S(r,f)\\&=pN(r,f)+qT(r,f)-qN(r,f)\\&\quad+S(r,f)\\&=pN(r,f)+qm(r,f)+S(r,f).\end{aligned} \tag{27}$$

It follows from this that

$$pm(r,f)\le qm(r,f)+S(r,f). \tag{28}$$

What is more is that if $m(r,f)\neq S(r,f)$, then we obtain from (28) that $p\le q$

Case 2. The coefficients of (3) are all constants and it admits a polynomial solution $f(z)$ with degree s. Set

$$f(z)=a_sz^s+a_{s-1}z^{s-1}+\cdots+a_1z+a_0, \tag{29}$$

then

$$f(z+1)=a_sz^s+b_{s-1}z^{s-1}+\cdots+b_1z+b_0, \tag{30}$$

where

$$b_{s-j}=a_sC_s^j+a_{s-1}C_{s-1}^{j-1}+\cdots+a_{s-j+1}C_{s-j+1}^1+a_{s-j}. \tag{31}$$

From (29) and (30), we obtain that

$$\begin{aligned}\Delta f&=sa_sz^{s-1}+(b_{s-2}-a_{s-2})z^{s-2}\\&\quad+\cdots+(b_1-a_1)z+(b_0-a_0).\end{aligned} \tag{32}$$

If $s = 1$, then $\Delta^2 f = \Delta^3 f \equiv 0$, which yields that $P(z, f) \equiv 0$. That is a contradiction to our assumption. Thus, $s \geq 2$.

If $s = 2$, then $\Delta f = 2a_2 z + a_2 + a_1$, $\Delta^2 f = 2a_2$, and $\Delta^3 f \equiv 0$. Now from (3), we get

$$(-3)^k Q(z, f)\left(\Delta^2 f\right)^{2k} = 2^k P(z, f)(\Delta f)^{2k}. \quad (33)$$

Considering degrees of both sides of the equation above, we can see that $q = p + k$.

If $s \geq 3$, we can deduce similarly that

$$\begin{aligned} \Delta^2 f &= s(s-1) a_s z^{s-2} + P_1(z), \\ \Delta^3 f &= s(s-1)(s-2) a_s z^{s-3} + P_2(z), \end{aligned} \quad (34)$$

where $P_1(z), P_2(z)$ are polynomials such that $\deg P_1 \leq s - 3$, $\deg P_2 \leq s - 4$.

Rewrite (3) as follows:

$$Q(z, f)\left[2\Delta^3 f \cdot \Delta f - 3\left(\Delta^2 f\right)^2\right]^k = 2^k P(z, f)(\Delta f)^{2k}. \quad (35)$$

From (34), we find that the leading coefficient of $2\Delta^3 f \cdot \Delta f - 3(\Delta^2 f)^2$ is

$$-a_s^2 s^2 (s-1)(s+1) \neq 0. \quad (36)$$

Considering degrees of both sides of (35), we prove that $qs = ps + 2k$.

4. Proof of Theorem 3

Firstly, we consider the general case. As mentioned in Remark 1 in [13], due to Jank and Volkmann [16], if (3) admits an admissible solution, then there are at most $S(r, f)$ common zeros of $P(z, f)$ and $Q(z, f)$. Since all coefficients of $Q(z, f)$ are polynomials, there are at the most finitely many poles of f that are the zeros of $Q(z, f)$. Therefore, from (3), we have

$$\begin{aligned} \frac{1}{2k} N\left(r, \frac{1}{Q}\right) &\leq N\left(r, \frac{1}{\Delta f}\right) + S(r, f) \leq T(r, \Delta f) + S(r, f) \\ &= T(r, f(z+1) - f(z)) + S(r, f) \\ &\leq 2T(r, f) + S(r, f). \end{aligned} \quad (37)$$

Combining this and Lemma 9, applying the second main theorem, we get

$$\begin{aligned} &\frac{q}{2k} T(r, f) + \sum_{j=1}^{s} m\left(r, \frac{1}{f - \alpha_j}\right) \\ &\quad \leq \frac{q}{2k} T(r, f) + m(r, f) + \sum_{j=1}^{s} m\left(r, \frac{1}{f - \alpha_j}\right) \\ &\quad \leq \frac{1}{2k} N\left(r, \frac{1}{Q}\right) + m(r, f) + \sum_{j=1}^{s} m\left(r, \frac{1}{f - \alpha_j}\right) + S(r, f) \\ &\quad \leq 2T(r, f) + m(r, f) + \sum_{j=1}^{s} m\left(r, \frac{1}{f - \alpha_j}\right) + S(r, f) \\ &\quad \leq 4T(r, f) + S(r, f). \end{aligned} \quad (38)$$

Thus, we prove that (10) holds.

Secondly, we consider the case that $N(r, f) = S(r, f)$. From (3) and Lemma 8, we similarly get that

$$\begin{aligned} \frac{1}{2k} N\left(r, \frac{1}{Q}\right) &\leq N\left(r, \frac{1}{\Delta f}\right) + S(r, f) \leq T(r, \Delta f) + S(r, f) \\ &= m(r, \Delta f) + N(r, \Delta f) + S(r, f) \\ &\leq m\left(r, \frac{\Delta f}{f}\right) + m(r, f) + S(r, f) \\ &\leq m(r, f) + S(r, f). \end{aligned} \quad (39)$$

From this and applying Lemma 9 with (19), as arguing before, we can prove that (11) holds.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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