Research Article

On the Growth of Solutions of a Class of Higher Order Linear Differential Equations with Extremal Coefficients

Jianren Long,[1,2] Chunhui Qiu,[1] and Pengcheng Wu[2]

[1] School of Mathematical Sciences, Xiamen University, Xiamen 361005, China
[2] School of Mathematics and Computer Science, Guizhou Normal University, Guiyang 550001, China

Correspondence should be addressed to Jianren Long; longjianren2004@163.com

We consider that the linear differential equations $f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0$, where A_j $(j = 0, 1, \ldots, k-1)$, are entire functions. Assume that there exists $l \in \{1, 2, \ldots, k-1\}$, such that A_l is extremal for *Yang's* inequality; then we will give some conditions on other coefficients which can guarantee that every solution $f(\not\equiv 0)$ of the equation is of infinite order. More specifically, we estimate the lower bound of hyperorder of f if every solution $f(\not\equiv 0)$ of the equation is of infinite order.

1. Introduction and Main Results

We will assume that the reader is familiar with the fundamental results and the standard notations of Nevanlinna theory of meromorphic functions (see [1, 2] or [3]). In addition, for a meromorphic function f in the complex plane $\mathbb{C}$, we will use the notations $\rho(f)$ and $\mu(f)$ to denote its order and lower order, respectively.

In order to estimate the rate of growth of meromorphic function of infinite order more precisely, we recall the following definition.

Definition 1 (see [4]). Let f be a meromorphic function in the complex plane $\mathbb{C}$. Then one defines the hyperorder $\rho_2(f)$ of f by

$$\rho_2(f) = \limsup_{r \to \infty} \frac{\log^+ \log^+ T(r, f)}{\log r}. \tag{1}$$

Consider the second order linear differential equation

$$f'' + A(z)f' + B(z)f = 0, \tag{2}$$

where A and $B(\not\equiv 0)$ are entire functions. It is well known that if A is an entire function, $B(\not\equiv 0)$ is a transcendental entire function, and f_1, f_2 are two linearly independent solutions of (2), then at least one of f_1, f_2 must have infinite order. On the

other hand, there are some equations of form (2) that possess a solution $f(\not\equiv 0)$ of finite order; for example, $f(z) = e^z$ satisfies $f'' + e^{-z}f' - (e^{-z} + 1)f = 0$. Therefore, one may ask, what assumptions on $A(z)$ and $B(z)$ will guarantee that every solution $f \not\equiv 0$ of (2) is of infinite order? From the works of Gundersen (see [5]) and Hellerstein et al. (see [6]), we know that if $A(z)$ and $B(z)$ are entire functions with $\rho(A) < \rho(B)$, or $A(z)$ is a polynomial, and $B(z)$ is transcendental, or $\rho(B) < \rho(A) \le 1/2$, then every solution $f(\not\equiv 0)$ of (2) is of infinite order. More results can be found in [7–12]. For entire solutions of infinite order more precise estimates for their rate of growth would be an important achievement. There are many authors investigating the hyperorder $\rho_2(f)$ of solutions of (2), such as Chen and Yang (see [8, 13]) and Kwon (see [14, 15]).

In this paper, we will introduce the deficient value and Borel direction into the studies of the complex differential equations. In order to give the definition of the Borel direction, we need the following notation. Let $\alpha < \beta$ such that $\beta - \alpha < 2\pi$ and $r > 0$; set $\Omega(\alpha, \beta) = \{z : \alpha < \arg z < \beta\}$, $\Omega(\alpha, \beta, r) = \{z : \alpha < \arg z < \beta\} \bigcap \{z : |z| < r\}$.

Definition 2. Let f be a meromorphic function in the complex plane $\mathbb{C}$ with $0 < \mu(f) < \infty$. Let $\mu(f) \le \lambda \le \rho(f)$ be a finite constant. A ray $\arg z = \theta (0 \le \theta < 2\pi)$ from the origin

is called a Borel direction of order $\geq \lambda$ of f, if for any positive number $\varepsilon > 0$ and for any complex number $a \in \mathbb{C} \bigcup \{\infty\}$, possibly with two exceptions, the following inequality holds:

$$\limsup_{r \to \infty} \frac{\log n \left(\Omega \left(\theta - \varepsilon, \theta + \varepsilon, r \right), f = a \right)}{\log r} \geq \lambda, \qquad (3)$$

where $n(\Omega(\theta - \varepsilon, \theta + \varepsilon, r), f = a)$ denotes the number of zeros, counting the multiplicities, of $f - a$ in the region $\Omega(\theta - \varepsilon, \theta + \varepsilon, r)$.

The fundamental result in angular distribution, due to Valiron, says that a meromorphic function of order $\rho > 0$ must have at least one Borel direction of order ρ; for example, see [3].

It is well known that deficient values and Borel directions are very important concepts in Nevanlinna theory of meromorphic functions. These two concepts are extensively studied. There is a striking relationship between them which was found by Yang and Zhang and says that, for a meromorphic function f of order ρ, the number of deficient values is less or equal to the number of Borel directions of order ρ of f. In 1988, Yang extended the above Y-Z inequality to the case of entire function of finite lower order. In order to use *Yang's* result to study the complex differential equations, we will use the following Theorem which can be easily derived from [16].

Theorem 3 (see [16]). *Suppose that f is an entire function of finite lower order $\mu > 0$. Let q ($< \infty$) denote the number of Borel directions of order $\geq \mu$ and p denote the number of finite deficient values of f; then $p \leq q/2$.*

Note that Theorem 3 is explicitly stated in [17]. To see the valid of the conclusion of the theorem, we note that, in [17, Corollary 1], Wu has proved that if $f(z)$ is of finite lower order μ and the number of Borel directions of order $\geq \mu$ is finite, then the order ρ of $f(z)$ is also finite. As each Borel direction of order ρ is also a Borel direction of order $\geq \mu$, this implies that, for $f(z)$, the number of the Borel directions of order ρ is fewer or equal to the number of the Borel directions of order $\geq \mu$. Therefore Theorem 3 follows from Theorem 6.7 in [3].

In the sequel, we will say that an entire function f is extremal for *Yang's* inequality if f satisfies the assumptions of Theorem 3 with $p = q/2$.

The simplest entire function extremal for *Yang's* inequality is e^z. A little bit complicated example is $A(z) = \int_0^z e^{-t^n} dt$, ($n \geq 2$). We know that (see [3]) $A(z)$ has n deficient values

$$a_l = e^{i(2\pi l/n)} \int_0^\infty e^{-t^n} dt, \quad (l = 1, 2, \dots, n) \qquad (4)$$

and $q = 2n$ Borel direction $\arg z = ((2k - 1)/2n)\pi(k = 1, 2, \dots, 2n)$. So $p = q/2$.

Furthermore, we state the following result due to present authors (see, [18]).

Theorem 4 (see [18]). *Let A be an entire function extremal for Yang's inequality, and let B be a transcendental entire function such that $\rho(A) \neq \rho(B)$. Then every solution $f (\not\equiv 0)$ of (2) is of infinite order.*

In this paper, we will consider the higher order linear differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \qquad (5)$$

where A_j ($j = 0, 1, \dots, k - 1$) are entire functions. Many authors have also investigated the growth of solutions of (5) and obtained lots of results on order and hyperorder of the solutions of (5) (see [19–23]). We will introduce the deficient value and Borel directions into the studies of (5). The main result in the paper is as follows.

Theorem 5. *Let A_j ($j = 0, 1, \dots, k - 1$) be entire functions. Suppose that there exists an integer $l \in \{1, 2, \dots, k - 1\}$, such that $A_l(z)$ is extremal for Yang's inequality. Suppose that A_0 is a transcendental entire function with $\rho(A_0) \neq \rho(A_l)$ and $\rho(A_i) < \rho(A_0)$ for $i \neq l$ ($1 \leq i \leq k - 1$). Then every solution $f (\not\equiv 0)$ of (5) satisfies $\rho(f) = \infty$ and $\rho_2(f) \geq \rho(A_0)$.*

The paper is organized as follows. In Section 2, we will give some lemmas. In Section 3, we will prove Theorem 5. In Section 4, we will discuss some further results related to the two entire coefficients in (5) which are extremal for *Yang's* inequality.

2. Lemmas

In this section, we need some auxiliary results. The following lemma is by Gundersen.

Lemma 6 (see [24]). *Let (f, Γ) denote a pair that consists of a transcendental meromorphic function f and a finite set*

$$\Gamma = \left\{ (k_1, j_1), (k_2, j_2), \dots, (k_q, j_q) \right\} \qquad (6)$$

of distinct pairs of integers that satisfy $k_i > j_i \geq 0$ for $i = 1, \dots, q$. Let $\alpha > 1$ and $\varepsilon > 0$ be given real constants. Then the following three statements hold.

(i) There exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero, and there exists a constant $c > 0$ that depends only on α and Γ, such that if $\psi_0 \in [0, 2\pi) - E_1$, then there is a constant $R_0 = R_0(\psi_0) > 1$ such that, for all z satisfying $\arg z = \psi_0$ and $|z| = r \geq R_0$ and for all $(k, j) \in \Gamma$, we have

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq c \left(\frac{T(\alpha r, f)}{r} \log^\alpha r \log T(\alpha r, f) \right)^{k-j}. \qquad (7)$$

In particular, if f has finite order $\rho(f)$, then (7) is replaced by

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho(f)-1+\varepsilon)}. \qquad (8)$$

(ii) There exists a set $E_2 \subset (1, \infty)$ that has finite logarithmic measure, and there exists a constant $c > 0$ that depends only on α and Γ, such that, for all z satisfying $|z| = r \notin E_2 \bigcup [0, 1]$ and for all $(k, j) \in \Gamma$, inequality (7) holds.

In particular, if f has finite order $\rho(f)$, then inequality (8) holds.

(iii) There exists a set $E_3 \subset [0, \infty)$ that has finite linear measure, and there exists a constant $c > 0$ that depends only on α and Γ, such that, for all z satisfying $|z| = r \notin E_3$ and for all $(k, j) \in \Gamma$, we have

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \le c \left(T(\alpha r, f) r^\varepsilon \log T(\alpha r, f) \right)^{k-j}. \tag{9}$$

In particular, if f has finite order $\rho(f)$, then (9) is replaced by

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \le |z|^{(k-j)(\rho(f)+\varepsilon)}. \tag{10}$$

Let f be an entire function extremal for *Yang's* inequality $p = q/2$. Suppose that the rays $\arg z = \theta_k$ $(k = 1, 2, \ldots, q)$ $(0 \le \theta_1 < \theta_2 < \cdots < \theta_q < \theta_{q+1} = \theta_1 + 2\pi)$ are the q distinct Borel directions of order $\ge \mu$ of f. In [17], Wu studied the entire functions which are extremal for *Yang's* inequality systematically. The following results play an important role in the proof of our results.

Lemma 7 (see [17]). *Suppose that f is extremal for Yang's inequality. Then $\mu(f) = \rho(f)$. Moreover, for every deficient value a_i $(i = 1, 2, \ldots, p)$ there exists a corresponding angular domain $\Omega(\theta_{k_i}, \theta_{k_i+1}) = \{z : \theta_{k_i} < \arg z < \theta_{k_i+1}\}$ such that for every $\varepsilon > 0$ inequality*

$$\log \frac{1}{|f(z) - a_i|} > C\left(\theta_{k_i}, \theta_{k_i+1}, \varepsilon, \delta(a_i, f) \right) T(|z|, f) \tag{11}$$

holds for $z \in \Omega(\theta_{k_i}+\varepsilon, \theta_{k_i+1}-\varepsilon, r, +\infty) = \{z : \theta_{k_i}+\varepsilon < \arg z < \theta_{k_i+1} - \varepsilon\} \bigcap \{z : r < |z| < \infty\}$, where $C(\theta_{k_i}, \theta_{k_i+1}, \varepsilon, \delta(a_i, f))$ is a positive constant depending only on $\theta_{k_i}, \theta_{k_i+1}, \varepsilon$ and $\delta(a_i, f)$.

In the sequel, we will say that f decays exponentially to a_i in $\Omega(\theta_{k_i}, \theta_{k_i+1})$, if (11) holds in $\Omega(\theta_{k_i}, \theta_{k_i+1})$. Note that if f is extremal for *Yang's* inequality, then $\mu(f) = \rho(f)$. Thus, for these functions, we need only to consider the Borel direction of order $\rho(f)$.

Lemma 8 (see [18]). *Let f be extremal for Yang's inequality. Suppose that there exists $\theta \in \Omega(\theta_j, \theta_{j+1})$ $(1 \le j \le q)$ such that*

$$\limsup_{r \to \infty} \frac{\log^+ \log^+ \left| f\left(re^{i\theta} \right) \right|}{\log r} = \rho(f), \tag{12}$$

where $\arg z = \theta_j$ $(j = 1, 2, \ldots, q)$ are Borel directions of f. Then $\theta_{j+1} - \theta_j = \pi/\rho(f)$.

Before stating the following lemmas, for $E \subset [0, \infty)$, we define the Lebesgue measure of E by $\mathrm{mes}(E)$ and the logarithmic measure of $E \subset [1, \infty)$ by $m_l(E) = \int_E (dt/t)$ and define the upper and lower logarithmic density of $E \subset [1, \infty)$, respectively, by

$$\overline{\log \mathrm{dens}} E = \overline{\lim_{r \to \infty}} \frac{m_l(E \bigcap [1, r])}{\log r},$$
$$\underline{\log \mathrm{dens}} E = \underline{\lim_{r \to \infty}} \frac{m_l(E \bigcap [1, r])}{\log r}. \tag{13}$$

Lemma 9 (see [25]). *Let f be an entire function with $\rho(f) = \rho < 1/2$ and suppose that $m(r)$ is defined as*

$$m(r) = \inf_{|z|=r} \log |f(z)|. \tag{14}$$

If $\sigma < \rho$, then the set $\{r : m(r) > r^\sigma\}$ has a positive upper logarithmic density.

Lemma 10 (see [26]). *Let $g(r)$ and $h(r)$ be monotone nondecreasing functions on $(0, \infty)$ such that $g(r) \le h(r)$ for all r outside some set of finite logarithmic measure. Let $\lambda > 1$ be a given real constant. Then there exists a constant $r_0 > 0$ such that $g(r) \le h(\lambda r)$ for all $r \ge r_0$.*

Lemma 11. *Let $f(z)$ be an entire function with order $\rho(0 < \rho < \infty)$, let and $\Omega(\phi_1, \phi_2) = \{z : \phi_1 < \arg z < \phi_2\}$ be a sector with $\phi_2 - \phi_1 < \pi/\rho$. If there is Borel direction of $f(z)$ in $\Omega(\phi_1, \phi_2)$, then there exists at least one of the two rays $L_j : \arg z = \phi_j$ $(j = 1, 2)$, without lose of generality, says, L_2, such that*

$$\limsup_{r \to \infty} \frac{\log^+ \log^+ \left| f\left(re^{i\phi_2} \right) \right|}{\log r} = \rho. \tag{15}$$

Lemma 11 can be founded in [27, Lemma 1], which can be proved by using a result in [28, Page 119-120].

3. Proof of Theorem 5

Now we prove our main result.

Since $\rho(A_i) < \rho(A_0)$ $(i \ne l, 1 \le i \le k - 1)$, we know that for any given constant η with $\max\{\rho(A_i), i \ne l, 1 \le i \le k-1\} < \eta < \rho(A_0)$, there exists a constant $R_1 > 0$ such that

$$|A_i(z)| \le \exp(r^\eta) \tag{16}$$

holds for all $|z| = r > R_1$.

We consider the following two cases.

Case 1. We suppose that $\rho(A_l) < \rho(A_0)$. Now to the contrary assume that there is a solution $f(\not\equiv 0)$ of (5) with $\rho(f) < +\infty$. We will seek a contradiction. By Lemma 6(ii), there exists a set $E_1 \subset [1, +\infty)$ that has finite logarithmic measure, such that the following inequality

$$\left| \frac{f^{(m)}(z)}{f(z)} \right| \le |z|^{k\rho(f)}, \quad m = 1, 2, \ldots, k \tag{17}$$

holds for all z with $|z| = r \notin E_1 \bigcup [0, 1]$.

We deduce from (17) and (5) that

$$\left| A_0(z) \right| \le \left| \frac{f^{(k)}}{f} \right| + \left| A_{k-1}(z) \right| \left| \frac{f^{(k-1)}}{f} \right|$$
$$+ \cdots + \left| A_l(z) \right| \left| \frac{f^{(l)}}{f} \right| + \cdots + \left| A_1(z) \right| \left| \frac{f'}{f} \right| \tag{18}$$
$$\le |z|^{k\rho(f)} \left(1 + \left| A_{k-1}(z) \right| + \cdots + \left| A_l(z) \right| + \cdots + \left| A_1(z) \right| \right)$$

holds for all z with $|z| = r \notin E_1 \bigcup [0, 1]$.

Thus,

$$T(r, A_0) \leq k\rho(f) \log r + (k-1) T(r, A_i) \qquad (19)$$

holds for all z with $|z| = r \notin E_1 \bigcup [0, 1]$, where $T(r, A_i) = \max_{1 \leq h \leq k-1} T(r, A_h)$. By Lemma 10, we have $\rho(A_0) \leq \rho(A_i)$; this is a contradiction. Therefore, every solution $f(\not\equiv 0)$ of (5) is of infinite order.

By using similar methods as [14], we can easily prove that $\rho_2(f) \geq \rho(A_0)$ in this case. We omit the details here.

Case 2. We suppose that $\rho(A_l) > \rho(A_0)$. Now to the contrary assume that there is a solution $f(\not\equiv 0)$ of (5) with $\rho(f) < +\infty$. We will seek a contradiction.

Suppose that a_i $(i = 1, 2, \ldots, p)$ are all the finite deficient values of $A_l(z)$. Thus we have $2p$ angular domains $S_j = \{z \mid \theta_j < \arg z < \theta_{j+1}\}$ $(j = 1, 2, \ldots, 2p)$. For any $\varepsilon > 0$, by using Lemmas 7 and 11, we can easily obtain that $A_l(z)$ has the following properties: in each sector S_j, either there exists some a_i such that

$$\log \frac{1}{|A_l(z) - a_i|} > C(\theta_j, \theta_{j+1}, \varepsilon, \delta(a_i, A_l)) T(|z|, A_l) \qquad (20)$$

holds for $z \in \Omega(\theta_j + \varepsilon, \theta_{j+1} - \varepsilon, r, +\infty)$, where $C(\theta_j, \theta_{j+1}, \varepsilon, \delta(a_i, A_l))$ is a positive constant depending only on $\theta_j, \theta_{j+1}, \varepsilon$ and $\delta(a_i, A_l)$, or there exists $\theta \in S_j$ such that

$$\limsup_{r \to \infty} \frac{\log^+\log^+ |A_l(re^{i\theta})|}{\log r} = \rho(A_l) \qquad (21)$$

holds. For the sake of simplicity, in the sequel we use C to represent $C(\theta_j, \theta_{j+1}, \varepsilon, \delta(a_i, A_l))$. Note that if there exists some a_i such that (20) holds in S_j, then there exists θ such that (21) holds in S_{j-1} and S_{j+1}. And if there exists $\theta \in S_j$ such that (21) holds, then there are a_i $(a_{i'})$ such that (20) holds in S_{j-1} and S_{j+1}, respectively.

Without loss of generality, we assume that there is a ray $\arg z = \theta$ in S_1 such that (21) holds. Therefore, there exists a ray in each sector $S_3, S_5, \ldots, S_{2p-1}$, such that (21) holds. By using Lemma 8, we know that all the sectors have the same magnitude $\pi/\rho(A_l)$.

Firstly, suppose that $\rho(A_0) \geq 1/2$. Since $A_0(z)$ must have a Borel direction of order $\rho(A_0)$, by using Lemma 11, we can see that there exists a sector $\Omega(\alpha, \beta)(\alpha < \beta)$ such that $\beta - \alpha \geq \pi/\rho(A_0)$ and such that for all the rays $\arg z = \theta$ $(\alpha < \theta < \beta)$ we have

$$\limsup_{r \to \infty} \frac{\log^+\log^+ |A_0(re^{i\theta})|}{\log r} = \rho(A_0). \qquad (22)$$

Note that $\rho(A_l) > \rho(A_0)$. It is not hard to see that there exists a sector $\Omega(\alpha', \beta')(\alpha < \alpha' < \beta' < \beta)$ such that there is an a_{j_0} such that

$$\log \frac{1}{|A_l(re^{i\theta}) - a_{j_0}|} > CT(r, A_l) \qquad (23)$$

holds for all $\alpha' \leq \theta \leq \beta'$.

By Lemma 6(i), there exist θ_0 $(\alpha' \leq \theta_0 \leq \beta')$ and $R_2 > 0$ such that

$$\left| \frac{f^{(m)}(re^{i\theta_0})}{f(re^{i\theta_0})} \right| \leq r^{k\rho(f)}, \quad m = 1, 2, \ldots, k \qquad (24)$$

holds for all $r > R_2$.

Note that

$$\limsup_{r \to \infty} \frac{\log^+\log^+ |A_0(re^{i\theta_0})|}{\log r} = \rho(A_0). \qquad (25)$$

Thus there is a sequence of $\{r_n\}$ with $\lim_{n \to \infty} r_n = \infty$ such that

$$|A_0(r_n e^{i\theta_0})| \geq \exp(r_n^{\rho(A_0)-\varepsilon}) \qquad (26)$$

holds for every $0 < \varepsilon < (\rho(A_0) - \eta)/2$. Therefore, we deduce from (16), (23), and (24) that

$$
\begin{aligned}
&\left| A_0(r_n e^{i\theta_0}) \right| \\
&\leq r_n^{k\rho(f)} \left(1 + \left| A_{k-1}(r_n e^{i\theta_0}) \right| + \cdots + \left| A_l(r_n e^{i\theta_0}) - a_{j_0} \right| \right. \\
&\qquad \left. + \left| a_{j_0} \right| + \cdots + \left| A_1(r_n e^{i\theta_0}) \right| \right) \\
&\leq r_n^{k\rho(f)} \left(1 + \exp(r_n^\eta) + \cdots + \exp\{-CT(r_n, A_l)\} \right. \\
&\qquad \left. + \left| a_{j_0} \right| + \cdots + \exp(r_n^\eta) \right)
\end{aligned}
\qquad (27)
$$

holds for all sufficiently large n. Therefore, combining (26) with (27), we have that

$$
\begin{aligned}
&\exp\{r_n^{\rho(A_0)-\varepsilon}\} \\
&< r_n^{k\rho(f)} \left(1 + (k-2)\exp(r_n^\eta) + \exp\{-CT(r_n, A_l)\} + \left| a_{j_0} \right| \right)
\end{aligned}
\qquad (28)
$$

holds for all sufficiently large n. This is a contradiction, so every solution $f(\not\equiv 0)$ of (5) is infinite order in this case.

Secondly, suppose that $0 < \rho(A_0) < 1/2$. By Lemma 9, there is a sequence of $\{r_n\}$ with $\lim_{n \to \infty} r_n = \infty$ such that for any $\theta \in [0, 2\pi)$, we must have

$$|A_0(r_n e^{i\theta})| > \exp(r_n^{\rho(A_0)-\varepsilon}). \qquad (29)$$

Thus we can get a contradiction by using similar argument for the proof of case $\rho(A_0) \geq 1/2$. So every solution $f(\not\equiv 0)$ of (5) is infinite order in this case.

Lastly, suppose that $\rho(A_0) = 0$. Note that $A_0(z)$ is a transcendental entire function. By using the results of [29] or [8], for any $\theta \in [0, 2\pi)$, we have

$$\limsup_{r \to \infty} \frac{\log |A_0(re^{i\theta})|}{\log r} = \infty. \qquad (30)$$

Thus we can get a contradiction by using similar argument for the proof of case $\rho(A_0) \geq 1/2$. Therefore, every solution $f(\not\equiv 0)$ of (5) is infinite order in this case.

Next we prove that $\rho_2(f) \geq \rho(A_0)$. Firstly, suppose that $\rho(A_0) > 0$. By using similar argument as above, there exists $\theta_0(\alpha' \leq \theta_0 \leq \beta')$ and there is a sequence of $\{r_n\}$ with $\lim_{n \to \infty} r_n = \infty$ such that (23) and (26) (or (29)) hold for all sufficiently large n.

By Lemma 6(i) that there exist θ_0 $(\alpha' \leq \theta_0 \leq \beta')$ and constants $R_3 > 0, d > 0$, such that the following inequality

$$\left| \frac{f^{(m)}(z)}{f(z)} \right| \leq dT(2r, f)^{2k}, \quad m = 1, \ldots, k \qquad (31)$$

holds for $|z| = r > R_3$.

Hence, calculating at the points $z_n = r_n e^{i\theta_0}$ with $r_n \notin [0, R_1] \bigcup [0, R_3]$, from (16), (23), (26) (or (29)), and (31), we get

$$\exp\left(r_n^{\rho(A_0)-\varepsilon} \right) \leq dT(2r_n, f)^{2k} \left(1 + (k-2)\exp\left(r_n^{\eta} \right) \right.$$
$$\left. + \exp\left\{ -CT\left(r_n, A_l \right) \right\} + \left| a_{j_0} \right| \right). \qquad (32)$$

Thus

$$\varlimsup_{r \to \infty} \frac{\log^+\log^+ T(r, f)}{\log r} \geq \rho(A_0). \qquad (33)$$

This gives $\rho_2(f) \geq \rho(A_0)$.

If $\rho(A_0) = 0$, obviously, $\rho_2(f) \geq 0$. The proof of Theorem 5 is completed.

4. Further Results

In this section, we will study (5) with coefficients A_0 and A_l which are both extremal for *Yang's* inequality.

Theorem 12. *Let A_j $(j = 0, 1, \ldots, k-1)$ be entire functions. Suppose that there exists an integer $l \in \{1, 2, \ldots, k-1\}$, such that A_l is extremal for Yang's inequality $p_1 = q_1/2$. Suppose that A_0 is an entire function extremal for Yang's inequality $p_2 = q_2/2$ and $\rho(A_i) < \rho(A_0)$ for $i \neq l$ $(1 \leq i \leq k-1)$. Suppose that one of the following conditions holds:*

(1) $q_1 \neq q_2$,

(2) $q_1 = q_2$, *and the set of Borel directions of A_l is different from that of A_0.*

Then every solution $f(\not\equiv 0)$ of (5) satisfies $\rho(f) = \infty$ and $\rho_2(f) \geq \rho(A_0)$.

Proof. We first treat the case that the entire functions A_l and A_0 satisfy condition (1).

Note that if $\rho(A_l) \neq \rho(A_0)$, then the conclusion of Theorem 12 follows from Theorem 5. Now suppose that $\rho(A_l) = \rho(A_0) = \rho$. We divide the proof into two cases: (a) $q_1 < q_2$ and (b) $q_1 > q_2$.

Now suppose that (a) $q_1 < q_2$ holds. It is easy to see from Lemmas 7 and 8 that there are $q_2/2$ sectors with magnitude π/ρ such that

$$\limsup_{r \to \infty} \frac{\log^+\log^+ \left| A_0\left(re^{i\theta} \right) \right|}{\log r} = \rho, \qquad (34)$$

while there are $q_1/2$ sectors with magnitude π/ρ such that

$$\limsup_{r \to \infty} \frac{\log^+\log^+ \left| A_l\left(re^{i\theta} \right) \right|}{\log r} = \rho. \qquad (35)$$

Note that $q_1 < q_2$. It is easy to see that there exists a sector $\Omega(\alpha, \beta)(0 < \alpha < \beta < 2\pi)$ such that, for every $\theta \in (\alpha, \beta)$, $A_l(z)$ must be bounded in $\Omega(\alpha, \beta)$, while $A_0(z)$ satisfies (34). So, by using the same argument in the proof of Theorem 5, we can easily prove the theorem.

We next suppose that (b) $q_1 > q_2$. It is not hard to see that there must exist a sector $\Omega(\alpha, \beta)$ such that $A_l(z)$ is bounded in $\Omega(\alpha, \beta)$, while for any $\theta \in \Omega(\alpha, \beta) A_0(z)$ satisfies (34). By using similar arguments as we did before, we can prove the theorem under the condition that $A_l(z)$ and $A_0(z)$ satisfy (1).

We turn to the case that $A_l(z)$ and $A_0(z)$ satisfy (2). In this case, it is easy to see that there exists a sector such that in it $A_l(z)$ is bounded, while $A_0(z)$ satisfies (34). By using similar arguments as we did before, we can prove the theorem in this case. We omit the details here. The proof of Theorem 12 is completed. □

Finally, in [16], we note that if an entire function f is extremal for *Yang's* inequality $p = q/2$, then for any positive integer m, $f^{(m)}$ also has some special properties. In the sectors where, for any θ, f satisfies

$$\limsup_{r \to \infty} \frac{\log^+\log^+ \left| f\left(re^{i\theta} \right) \right|}{\log r} = \rho, \qquad (36)$$

for any θ, $f^{(m)}$ satisfies

$$\limsup_{r \to \infty} \frac{\log^+\log^+ \left| f^{(m)}\left(re^{i\theta} \right) \right|}{\log r} = \rho, \qquad (37)$$

f decays to some deficient values exponentially and $f^{(m)}$ decays to 0 exponentially. Therefore, in the same manner as in the proofs of Theorems 5 and 12, we have the following result.

Theorem 13. *Let A_j $(j = 0, 1, \ldots, k-1)$ be entire functions. Suppose that there exists an integer $l \in \{1, 2, \ldots, k-1\}$, such that A_l is extremal for Yang's inequality $p_1 = q_1/2$. Suppose that A_0 is a transcendental entire function with $\rho(A_0) \neq \rho(A_l)$ and $\rho(A_i) < \rho(A_0)$ for $i \neq l$ $(1 \leq i \leq k-1)$. Then every solution $f \not\equiv 0$ of*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_l^{(m)}(z) f^{(l)} + \cdots + A_0(z) f$$
$$= 0 \qquad (38)$$

satisfies $\rho(f) = \infty$ and $\rho_2(f) \geq \rho(A_0)$.

Moreover, suppose that A_0 is an entire function extremal for *Yang's* inequality $p_2 = q_2/2$ and that one of the following assumptions holds:

(1) $q_1 \neq q_2$,

(2) $q_1 = q_2$, and the set of Borel directions of A_l is different from that of A_0.

Then every solution $f(\not\equiv 0)$ of

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_l^{(m)}(z) f^{(l)} + \cdots + A_0^{(n)}(z) f = 0 \tag{39}$$

satisfies $\rho(f) = \infty$ and $\rho_2(f) \geq \rho(A_0)$, where m and n are two nonnegative integers.

Finally we give an example satisfying the conditions of Theorem 12. Let again $A_n(z) = \int_0^z e^{-t^n}dt$, $(n \geq 2)$. So if we let $A_l = A_n(z)$, $A_0 = A_m(z) = \int_0^z e^{-t^m}dt$ with $n \neq m$ and all other coefficients A_i satisfy $\rho(A_i) < \rho(A_0)$ for $i \neq l$ $(1 \leq i \leq k-1)$, then, by Theorem 12(1), every solution $f(\not\equiv 0)$ of (5) satisfies $\rho(f) = \infty$ and $\rho_2(f) \geq \rho(A_0)$. Furthermore, if we let $A_0(z) = A_n(e^{i\theta}z)$ with $\theta \in (0, \pi/2n)$, then, by Theorem 12(2), every solution $f(\not\equiv 0)$ of (5) satisfies $\rho(f) = \infty$ and $\rho_2(f) \geq \rho(A_0)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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