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Research Article

Solvability of a Model for the Vibration of a Beam with a Damping Tip Body

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We consider a model for the vibration of a beam with a damping tip body that appeared in a previous article. In this paper we derive a variational form for the motion of the beam and use it to prove that the model problem has a unique solution. The proofs are based on existence results for a general linear vibration model problem, in variational form. Finite element approximation of the solution is discussed briefly.

1. Introduction

In article [1], the authors model and analyze the damped vibration of a cantilever beam with an attached hollow tip body that contains a granular material. The Euler-Bernoulli theory for a beam with Kelvin-Voigt damping is used. The beam is clamped at one end and the tip body is attached to the other end. The authors state that "the problem contains more complicated boundary conditions" than problems "treated previously" (and provide references). This is due to the fact that the model is more realistic—as we explain below. It is an interesting model for more than one reason. The dynamics of the rigid body is treated in a realistic way: the fact that the center of mass of the rigid body is not at the endpoint of the beam is taken into account. The damping mechanism is also explained unlike other papers where vibration models with boundary damping are considered. There are other articles with realistic models of a beam with damping, for example [2]. However, [1] provides the only realistic model for the relevant beam-body configuration.

The existence of a unique solution for the model problem is established in [1]. To obtain the result, the problem is written as an abstract differential equation and an abstract existence result from a previous paper [3] is applied.

In this paper, we also prove the existence of a unique solution. Our approach differs from that in [1]: we write the model problem in variational form and use results from [4] where general linear vibration models in variational form are considered. The existence theorems in [4] were applied to the vibration of a complex plate beam system in [5]. It should be noted that the same variational form can be used for finite element approximations (see Section 7). Since our approach differs from that in [1], it is natural to consider differences regarding assumptions and results. There are indeed some differences but these are not substantial (see Section 6).

The mathematical model is considered in Section 2. In Section 3, we write the model problem in variational form and present the weak variational form in Section 4. Auxiliary results are proved in Section 5. The existence theorems are stated and proved in Section 6 where different methods are also compared. In Section 7, natural frequencies and modes are discussed as well as finite element approximation.

2. Model Problem

The Euler-Bernoulli model for the transverse vibration of a beam is derived from the equation of motion for the beam deflection w:

$$\rho A \partial_t^2 w = \partial_x V + f \tag{1}$$

and the relation

$$\partial_x M = -V. \tag{2}$$

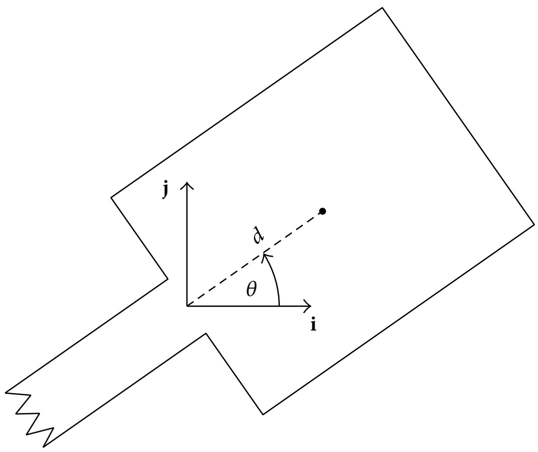

Figure 1: The tip body at the end of the beam.

In these equations, ρ denotes the density, A the area of the cross section, V the shear force, M the bending moment, and f a load (beam models are treated in [6, pages 323-324], [7, pages 337-338], and [8, pages 392–395]).

The usual constitutive equation is $M = EI\partial_x^2 w$, where E is an elastic constant (Young's modulus) and I is the area moment of inertia. Due to Kelvin-Voigt damping, it changes to

$$M = EI\partial_x^2 w + \lambda \partial_t \partial_x^2 w, \tag{3}$$

where λ denotes the damping parameter. The partial differential equation (which we do not use) is

$$\rho A \partial_t^2 w = -EI\partial_x^4 w + \lambda \partial_t \partial_x^4 w + f. \tag{4}$$

The constitutive equation for the moment M and the relation between the moment and shear force V are also used to model the interface conditions.

The left endpoint of the beam is clamped where the boundary conditions are the usual

$$w(0,t) = \partial_x w(0,t) = 0. \tag{5}$$

The interface conditions at the other endpoint are determined by the interaction between the beam and the rigid body. This is explained in [1] in some detail. It is necessary to consider the equations of motion for the rigid body carefully when deriving these conditions.

The position of the center of mass of the tip body relative to the endpoint of the beam is

$$d\cos\theta \mathbf{i} + d\sin\theta \mathbf{j}, \tag{6}$$

where θ is the angle of the neutral plane with the horizontal (see Figure 1). Therefore, the velocity $\mathbf{v}_C$ and acceleration $\mathbf{a}_C$ of the center of mass are given by

$$\begin{aligned}\mathbf{v}_C &= \partial_t w(\ell,t)\mathbf{j} - d\dot{\theta}\sin\theta\mathbf{i} + d\dot{\theta}\cos\theta\mathbf{j}, \\ \mathbf{a}_C &= \partial_t^2 w(\ell,t)\mathbf{j} - d\ddot{\theta}\sin\theta\mathbf{i} + d\ddot{\theta}\cos\theta\mathbf{j} \\ &\quad - d\dot{\theta}^2\cos\theta\mathbf{i} - d\dot{\theta}^2\sin\theta\mathbf{j},\end{aligned} \tag{7}$$

where ℓ denotes the length of the beam. For the linear approximation, it is assumed that the term $\dot{\theta}^2\sin\theta\mathbf{j}$ may be neglected, $\dot{\theta} \approx \partial_t\partial_x w(\ell,t)$, $\ddot{\theta} \approx \partial_t^2\partial_x w(\ell,t)$, and $\cos\theta \approx 1$. Using these approximations, we have the following expressions for the vertical components of the velocity and acceleration:

$$\partial_t w(\ell,t) + d\partial_t\partial_x w(\ell,t), \qquad \partial_t^2 w(\ell,t) + d\partial_t^2\partial_x w(\ell,t). \tag{8}$$

In [1], the term $d\partial_t\partial_x w(\ell,t)$ in the expressions for the vertical component of the velocity is neglected. In our opinion, this should not be done and we motivate our point of view in the next section where we discuss the decay of energy for the system.

In the equations below, γ and γ^* denote damping parameters, m the mass, and J the moment of inertia of the rigid body. Using Newton's second law for the motion of the center of mass, we have

$$\begin{aligned}&m\partial_t^2 w(\ell,t) + md\partial_t^2\partial_x w(\ell,t) \\ &\quad = -V(\ell,t) + f_B(t) - \gamma\partial_t w(\ell,t) - \gamma d\partial_t\partial_x w(\ell,t),\end{aligned} \tag{9}$$

where $f_B(t)$ is an external force that may act on the rigid body, for example, gravity. Taking moments about the center of mass, we have

$$J\partial_t^2\partial_x w(\ell,t) = -M(\ell,t) + dV(\ell,t) - d\gamma^*\partial_t\partial_x w(\ell,t). \tag{10}$$

Following [1], we combine (9) and (10) and find that

$$\begin{aligned}&md\partial_t^2 w(\ell,t) + \left(J + md^2\right)\partial_t^2\partial_x w(\ell,t) \\ &\quad = -M(\ell,t) - \gamma d\partial_t w(\ell,t) \\ &\qquad - \left(\gamma d + \gamma^*\right) d\partial_t\partial_x w(\ell,t) + df_B(t).\end{aligned} \tag{11}$$

It is convenient to rewrite (9) and (11) as follows:

$$\begin{aligned}V(\ell,t) = &- m\partial_t^2 w(\ell,t) - md\partial_t^2\partial_x w(\ell,t) \\ &- \gamma\partial_t w(\ell,t) - \gamma d\partial_t\partial_x w(\ell,t) + f_B(t), \\ M(\ell,t) = &- md\partial_t^2 w(\ell,t) - \left(J + md^2\right)\partial_t^2\partial_x w(\ell,t) \\ &- \gamma d\partial_t w(\ell,t) - \left(\gamma d + \gamma^*\right) d\partial_t\partial_x w(\ell,t) + df_B(t).\end{aligned} \tag{12}$$

Model Problem. The mathematical model consists of equations of motion (1) and (2) and constitutive equation (3) for the beam, boundary conditions (5), and interface conditions (12). Initial conditions $w(\cdot,0) = w_0$ and $\partial_t w(\cdot,0) = w_1$ need to be specified.

3. Variational Form

Multiply (1) by an arbitrary smooth function v and integrate. Using integration by parts and (2) yields

$$\begin{aligned}
\int_0^\ell \rho A\left(\partial_t^2 w(\cdot,t)\right) v
&= -\int_0^\ell V(\cdot,t)\, v' + [V(x,t)\, v(x)]_0^\ell + \int_0^\ell f(\cdot,t)\, v \\
&= -\int_0^\ell M(\cdot,t)\, v'' + [V(x,t)\, v(x)]_0^\ell + \left[M(x,t)\, v'(x)\right]_0^\ell \\
&\quad + \int_0^\ell f(\cdot,t)\, v.
\end{aligned} \tag{13}$$

Test Functions. A function v is a test function if $v \in C^1[0,\ell]$, v'' is integrable, and $v(0) = v'(0) = 0$. The space of test functions is denoted by $T[0,\ell]$.

It follows that

$$\begin{aligned}
\int_0^\ell \rho A\left(\partial_t^2 w(\cdot,t)\right) v
&= -\int_0^\ell M(\cdot,t)\, v'' + V(\ell,t)\, v(\ell) + M(\ell,t)\, v'(\ell) \\
&\quad + \int_0^\ell f(\cdot,t)\, v
\end{aligned} \tag{14}$$

for each $v \in T[0,\ell]$.

We now use the constitutive equation $M = EI\partial_x^2 w + \lambda\partial_t\partial_x^2 w$ and the interface conditions (12) to derive the variational form of the model problem.

It is convenient to introduce the following bilinear forms:

$$\begin{aligned}
\bar{b}(u,v) &= \int_0^\ell EIu''v'', \\
\bar{c}(u,v) &= \int_0^\ell \rho Auv + mu(\ell)\, v(\ell) + mdu'(\ell)\, v(\ell) \\
&\quad + mdu(\ell)\, v'(\ell) + \left(J + md^2\right) u'(\ell)\, v'(\ell), \\
\bar{a}(u,v) &= \int_0^\ell \lambda u''v'' + \gamma u(\ell)\, v(\ell) + \gamma du'(\ell)\, v(\ell) \\
&\quad + \gamma du(\ell)\, v'(\ell) + \left(\gamma d + \gamma^*\right) du'(\ell)\, v'(\ell).
\end{aligned} \tag{15}$$

We now have the variational form of the model problem.

Problem PV. Find w such that, for each $t > 0$, $w(\cdot,t) \in T(0,\ell)$ and

$$\begin{aligned}
&\bar{c}\left(\partial_t^2 w(\cdot,t), v\right) + \bar{a}\left(\partial_t w(\cdot,t), v\right) + \bar{b}\left(w(\cdot,t), v\right) \\
&\quad = \left(f(\cdot,t), v\right) + f_B(t)\, v(\ell) + df_B(t)\, v'(\ell),
\end{aligned} \tag{16}$$

for each $v \in T(0,\ell)$, with $w(\cdot,0) = w_0$ and $\partial_t w(\cdot,0) = w_1$.

Remark 1. Note that the bilinear forms $\bar{a}$, $\bar{b}$, and $\bar{c}$ are symmetric. The additional term $\gamma du'(\ell)v(\ell)$ in the definition of $\bar{a}$ is necessary for symmetry. Problem PV may be used to compute finite element approximations.

Mechanical Energy. The mechanical energy (kinetic energy plus elastic potential energy) of the system is

$$E(t) = \frac{1}{2}\bar{c}\left(\partial_t w(\cdot,t), \partial_t w(\cdot,t)\right) + \frac{1}{2}\bar{b}\left(w(\cdot,t), w(\cdot,t)\right). \tag{17}$$

Using the symmetry of $\bar{b}$ and $\bar{c}$ and assuming that w is sufficiently smooth, we have

$$E'(t) = \bar{c}\left(\partial_t^2 w, \partial_t w\right) + \bar{b}\left(w, \partial_t w\right) = -\bar{a}\left(\partial_t w, \partial_t w\right), \tag{18}$$

for the homogeneous case. It is obvious that $\bar{b}(u,u)$ is nonnegative and not difficult to show that $\bar{a}(u,u)$ and $\bar{c}(u,u)$ are nonnegative:

$$\begin{aligned}
\bar{c}(u,u) &= \int_0^\ell \rho Au^2 + m[u(\ell)]^2 \\
&\quad + 2mdu'(\ell)\, u(\ell) + \left(J + md^2\right)\left[u'(\ell)\right]^2 \\
&= \int_0^\ell \rho Au^2 + m\left[u(\ell) + du'(\ell)\right]^2 + J\left[u'(\ell)\right]^2 \geq 0, \\
\bar{a}(u,u) &= \int_0^\ell \lambda\left(u''\right)^2 + \gamma[u(\ell)]^2 + 2\gamma du(\ell)\, u'(\ell) \\
&\quad + \left(\gamma d + \gamma^*\right) d\left[u'(\ell)\right]^2 \\
&= \int_0^\ell \lambda\left(u''\right)^2 + \gamma\left[u(\ell) + du'(\ell)\right]^2 \\
&\quad + \gamma^* d\left[u'(\ell)\right]^2 \geq 0.
\end{aligned} \tag{19}$$

As a result, $E'(t) \leq 0$. This result is to be expected from Physics. The fact that $\bar{a}$ is symmetric and $\bar{a}(u,u)$ is nonnegative is due to the additional term.

4. Weak Variational Form

Let $H^m(0,\ell)$ denote the Sobolev space with weak derivatives up to order m in $\mathscr{L}^2(0,\ell)$. The inner product for $H^m(0,\ell)$ is denoted by $(f,g)_m$, with $(f,g)_0 = (f,g)$, the inner product for $\mathscr{L}^2(0,\ell)$. The corresponding norms are $\|f\|_m$ and $\|f\|_0 = \|f\|$.

Consider Problem PV. We start off as usual by considering the closure of the space of test functions. Let $V(0,\ell)$ be the closure of $T[0,\ell]$ in $H^2(0,\ell)$; then $V(0,\ell)$ is a Hilbert space (being a closed subspace of a Hilbert space).

We require the so-called *trace operator* which we denote by Γ. For $u \in C^1[0,\ell]$, $\Gamma u = u(\ell)$ but (as is well known) Γ can be extended to $H^1(0,\ell)$; see, for example, [9]. Clearly $\Gamma u'$ is defined for $u \in H^2(0,\ell)$.

The following product spaces are necessary for the abstract problem:

$$\begin{aligned} X &= \mathscr{L}^2(0,\ell) \times R \times R \\ H^m &= H^m(0,\ell) \times R \times R \\ V_P &= V(0,\ell) \times R \times R \\ V &= \{v \in V_P \mid v_2 = \Gamma v_1, v_3 = \Gamma v_1'\}. \end{aligned} \tag{20}$$

An element $y \in X$ is written as $y = \langle y_1, y_2, y_3 \rangle$. An obvious inner product for X is

$$(u,v)_X = \int_0^\ell u_1 v_1 + u_2 v_2 + u_3 v_3, \tag{21}$$

and we denote the corresponding norm by $\|\cdot\|_X$.

Definition 2 (bilinear forms). For u and v in X,

$$\begin{aligned} c(u,v) = &\int_0^\ell \rho A u_1 v_1 + m u_2 v_2 + md(u_3 v_2 + u_2 v_3) \\ &+ (J + md^2) u_3 v_3, \end{aligned} \tag{22}$$

and for u and v in H^2,

$$\begin{aligned} b(u,v) &= \int_0^\ell EI u_1'' v_1'', \\ a(u,v) &= \int_0^\ell \lambda u_1'' v_1'' + \gamma u_2 v_2 + \gamma d(u_3 v_2 + u_2 v_3) \\ &\quad + (\gamma d + \gamma^*) d u_3 v_3. \end{aligned} \tag{23}$$

Remarks. (1) The bilinear forms a, b, and c are symmetric. (2) For u and v in V, $a(u,v) = \bar{a}(u_1,v_1)$, $b(u,v) = \bar{b}(u_1,v_1)$, and $c(u,v) = \bar{c}(u_1,v_1)$. (3) It is essential to use product spaces since the bilinear form $\bar{c}$ is not defined on $\mathscr{L}^2(0,\ell)$.

For the weak variational form of the model problem, we need to show that the bilinear forms c and b are inner products for X and V, respectively. We use the well-known Poincare type inequalities given in the proposition below. The boundedness of Γ is also required.

Proposition 3. *For each u in $V(0;\ell)$,*

$$\begin{aligned} \|u\| &\le \ell \|u'\| \le \ell^2 \|u''\|, \\ |\Gamma u| &\le \sqrt{\ell} \|u'\|. \end{aligned} \tag{24}$$

Proof. Using the fundamental theorem of calculus, the inequalities are easy to prove for the space of test functions $T[0,\ell]$. Since $V(0,\ell)$ is the closure of $T[0,\ell]$ with respect to the norm of $H^2(0,\ell)$, the result follows. □

Proposition 4. *There exists a constant K such that*

$$c(u,u) \ge K(u,u)_X, \quad \text{for each } u \in X. \tag{25}$$

Proof. It is sufficient to show that there exists a constant K_1 such that $m(u_2 + du_3)^2 + Ju_3^2 \ge K_1(u_2^2 + u_3^2)$ for each $(u_2,u_3) \in \mathbf{R}_2$. From a well-known inequality,

$$m(u_2 + du_3)^2 \ge m(1-\epsilon) u_2^2 + m\left(1 - \frac{1}{\epsilon}\right) d^2 u_3^2. \tag{26}$$

By the definition of the moment of inertia of a rigid body, $J \ge \beta m d^2$ for some $\beta > 0$. Now choose $0 < \epsilon < 1$ and $\epsilon < (1+\beta)^{-1}$ and the desired result follows. □

Corollary 5. *The bilinear form c is an inner product for the space X.*

Definition 6 (inertia space). The norm $\|\cdot\|_W$ is defined by $\|u\|_W = \sqrt{c(u,u)}$. We refer to the vector space X equipped with this norm as the inertia space and denote it by W.

Proposition 7. *There exists a constant K such that*

$$b(u,u) \ge K \|u_1\|_2^2, \quad \text{for each } u \in V. \tag{27}$$

Proof. We use Proposition 3 and the definition of the bilinear form b:

$$\|u_1\|_2^2 \le \frac{\ell^2 + \ell + 1}{EI} b(u,u). \tag{28}$$

□

Corollary 8. *The bilinear form b is an inner product for V.*

Proof. Clearly $b(u,u) = 0$ implies that $u_1 = 0$ and therefore $u_2 = \Gamma u_1 = 0$ and $u_3 = \Gamma u_1' = 0$. □

Definition 9 (energy space). The space V equipped with the inner product b is referred to as the energy space. The norm $\|\cdot\|_V$ is defined by $\|u\|_V = \sqrt{b(u,u)}$.

We proceed to determine a weak variational form of the model problem. Let $\tilde{f}(t) = \langle f(\cdot,t), f_B(t), df_B(t) \rangle$, $u_0 = \langle w_0, \tilde{u}_{0,2}, \tilde{u}_{0,3} \rangle$, and $u_1 = \langle w_1, \tilde{u}_{1,2}, \tilde{u}_{1,3} \rangle$, with $\tilde{u}_{0,2}, \tilde{u}_{0,3}, \tilde{u}_{1,2}$, and $\tilde{u}_{1,3}$ arbitrary.

Problem PW. Find u such that, for each $t > 0$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$, and

$$\begin{aligned} c(u''(t), v) + a(u'(t), v) + b(u(t), v) = (\tilde{f}(t), v)_X, \\ \text{for each } v \in V, \end{aligned} \tag{29}$$

with $u(0) = u_0$ and $u'(0) = u_1$.

Remark 10. It is natural to think that $\tilde{u}_{0,2} = \Gamma w_0$ and so forth are the correct initial conditions. This is discussed in Section 6.

5. Auxiliary Results

We need the results of this section to apply Theorems 15 and 16 in Section 6.

Proposition 11. *Space V is a dense subset of X.*

Proof. Consider any $y \in W$. Since $C_0^\infty(0,\ell)$ is dense in $\mathscr{L}^2(0,\ell)$, there exists a sequence $\{\phi_n\} \subset C_0^\infty(0,\ell)$ such that $\|\phi_n - y_1\| \to 0$.

It is not difficult to construct sequences $\{\eta_n\}$ and $\{\zeta_n\}$ in $H^2(0,\ell)$ with the following properties:

$$\begin{gathered} \Gamma\eta_n = 1, \quad \Gamma\eta_n' = 0, \quad \|\eta_n\| \longrightarrow 0, \\ \Gamma\zeta_n = 0, \quad \Gamma\zeta_n' = 1, \quad \|\zeta_n\| \longrightarrow 0. \end{gathered} \tag{30}$$

Now, let $v_n = \phi_n + y_2\eta_n + y_3\zeta_n$; then $v_n \in V(0,\ell)$, $\Gamma v_n = y_2$, and $\Gamma v_n' = y_3$.

Consequently, $u_n = \langle v_n, v_n(\ell), v_n'(\ell)\rangle \in V$ and $\|u_n - y\|_X \to 0$. □

Proposition 12. *There exists a constant K such that*

$$b(u,u) \geq Kc(u,u), \quad \textit{for each } u \in V. \tag{31}$$

Proof. We use Proposition 3.

Consider

$$\begin{aligned} c(u,u) &= \int_0^\ell \rho A u_1^2 + m(u_2 + du_3)^2 + Ju_3^2 \\ &\leq \rho A\ell^4 \|u_1''\|^2 + 2m(\Gamma u_1)^2 + 2md^2(\Gamma u_1')^2 + J(\Gamma u_1')^2 \\ &\leq \|u_1\|_2^2 \left(\rho A\ell^4 + 2m\ell^3 + 2md^2\ell + J\ell\right). \end{aligned} \tag{32}$$

Now apply Proposition 7. □

Proposition 13. *There exists a constant K such that*

$$|a(u,v)| \leq K\|u\|_V\|v\|_V, \tag{33}$$

for each u and v in V.

Proof. We use Proposition 3.

Consider

$$\begin{aligned} |a(u,v)| &\leq \lambda \|u_1''\| \|v_1''\| + \gamma |\Gamma u_1 \Gamma v_1| + \gamma d |\Gamma u_1' \Gamma v_1| \\ &\quad + \gamma d |\Gamma u_1 \Gamma v_1'| + (\gamma d + \gamma^*) d |\Gamma u_1' \Gamma v_1'| \\ &\leq \|u_1\|_2 \|v_1\|_2 \left(\lambda + \gamma\ell^3 + 2\gamma d\ell^2 + (\gamma d + \gamma^*) d\ell\right). \end{aligned} \tag{34}$$

Now use Proposition 7. □

The result above is true for $\lambda \geq 0$. If $\lambda > 0$, the bilinear form a is positive definite on V and this has implications for existence results.

Proposition 14. *Consider*

$$a(u,u) \geq \frac{\lambda}{EI}\|u\|_V^2. \tag{35}$$

Proof. We have that $a(u,u) \geq \lambda\|u_1''\|^2 = (\lambda/EI)\|u\|_V^2$. □

6. Existence

In this section, we apply the existence results from [4]. For convenience, we formulate the general linear vibration problem and present the relevant existence theorems. Let V, W, and X be real Hilbert spaces with $V \subset W \subset X$. Spaces X, W, and V have inner products $(\cdot,\cdot)_X$, c, and b and norms $\|\cdot\|_X$, $\|\cdot\|_W$, and $\|\cdot\|_V$, respectively. Consider also a bilinear form a defined on V.

Problem PG. Find a function u such that, for each $t > 0$, $u(t) \in V$, $u'(t) \in V$, $u''(t) \in W$, and

$$c(u''(t), v) + a(u'(t), v) + b(u(t), v) = (f(t), v)_X, \quad \text{for each } v \in V, \tag{36}$$

with $u(0) = u_0$ and $u'(0) = u_1$.

In Theorems 15 and 16, the following is assumed.

Assumptions. (A1) V is dense in W and W is dense in X. (A2) There exists a constant C_b such that $\|v\|_W \leq C_b\|v\|_V$ for each $v \in V$. (A3) There exists a constant C_c such that $\|v\|_X \leq C_c\|v\|_W$ for each $v \in W$. (A4) The bilinear form a is symmetric, nonnegative, and bounded on V; that is, $|a(u,v)| \leq C\|u\|_V\|v\|_V$ for each u and v in V.

Theorem 15 (see [4, Theorem 1]). *Suppose that assumptions (A1), (A2), (A3), and (A4) are satisfied. If*

(a) $f \in C^1([0,\tau), X)$,

(b) $u_0 \in V$, $u_1 \in V$ *and there exists a* $y \in W$ *such that*

$$b(u_0, v) + a(u_1, v) = c(y, v) \quad \textit{for each } v \in V, \tag{37}$$

then problem PG has a unique solution:

$$\begin{gathered} u \in C([0,\tau), V) \cap C^1([0,\tau), W) \\ \cap C^1((0,\tau), V) \cap C^2((0,\tau), W). \end{gathered} \tag{38}$$

Theorem 16 (see [4, Theorem 3]). *Suppose that assumptions (A1), (A2), (A3), and (A4) are satisfied. If*

(a) *the bilinear form a is positive definite, that is, there exists a positive constant C such that $a(u,u) \geq C\|u\|_V^2$, for each $u \in V$,*

(b) *f is locally Hölder continuous on* $(0,\tau)$,

(c) $u_0 \in V$, $u_1 \in W$,

then problem PG has a unique solution:

$$\begin{gathered} u \in C([0,\tau), V) \cap C^1([0,\tau), W) \\ \cap C^1((0,\tau), V) \cap C^2((0,\tau), W). \end{gathered} \tag{39}$$

If $f = 0$, then $u \in C([0,\infty), V) \cap C^1([0,\infty), W) \cap C^\infty((0,\infty), V)$.

6.1. Applying General Results. Theorem 15 above is applied to the case where $\lambda = 0$ and Theorem 16 to the case where $\lambda > 0$. Note that assumptions (A1), (A2), (A3), and (A4) are satisfied due to Propositions 11, 12, 4, and 13, respectively. In the formulation of the theorems, we denote the function $t \to f(\cdot, t)$ by f_1.

Theorem 17. *Suppose that $\lambda = 0$ and*

(a) $f_1 \in C^1([0,\tau), \mathscr{L}^2(0,\ell))$ *and* $f_B \in C^1([0,\tau), \mathscr{R})$,

(b) $u_0 \in V$, $u_1 \in V$ *and there exists a* $y \in W$ *such that*

$$b(u_0, v) + a(u_1, v) = c(y, v) \quad \textit{for each } v \in V. \tag{40}$$

Then problem PW has a unique solution:

$$\begin{aligned} u \in C([0,\tau), V) \cap C^1([0,\tau), W) \\ \cap C^1((0,\tau), V) \cap C^2((0,\tau), W). \end{aligned} \tag{41}$$

Proof. Clearly $\tilde{f} \in C^1([0,\tau), X)$. The result follows from Theorem 15. □

Theorem 18. *Suppose that $\lambda > 0$ and*

(a) f_1 *is locally Hölder continuous on* $[0,\tau)$ *with respect to the norm of* $\mathscr{L}^2(0,\ell)$ *and* f_B *is locally Hölder continuous on* $[0,\tau)$,

(b) $u_0 \in V$ *and* $u_1 \in W$.

Then problem PW has a unique solution:

$$u \in C([0,\tau), V) \cap C^1([0,\tau), W) \cap C^2((0,\tau), W). \tag{42}$$

If $f_1 = f_B = 0$, then $u \in C([0,\infty); V) \cap C^1([0,\infty); W) \cap C^\infty((0,\infty); V)$.

Proof. Clearly $\tilde{f}$ is locally Hölder continuous on $[0,\tau)$ with respect to the norm $\|\cdot\|_X$ and the bilinear form a is positive definite by Proposition 14. The result follows from Theorem 16. □

6.2. Sufficient Conditions for Existence. The case $\lambda > 0$ is trivial.

If $\lambda = 0$, sufficient conditions on $u_0 = \langle w_0, \tilde{u}_{0,2}, \tilde{u}_{0,3}\rangle$ and $u_1 = \langle w_1, \tilde{u}_{1,2}, \tilde{u}_{1,3}\rangle$ are required to satisfy condition (b) in Theorem 17. It is obviously necessary to assume that u_0 and u_1 are in V which implies that $u_0 = \langle w_0, \Gamma w_0, \Gamma w_0'\rangle$ and $u_1 = \langle w_1, \Gamma w_1, \Gamma w_1'\rangle$. Suppose that $w_0 \in C^4[0,\ell]$ and $w_1 \in C^2[0,\ell]$. From the definition of the bilinear form b, using integration by parts, we obtain

$$b(u,v) = \int_0^\ell EIw_0^{(4)} v_1 - EIw_0'''(\ell) v_1(\ell) + EIw_0''(\ell) v_1'(\ell). \tag{43}$$

From the definition of a, we have

$$\begin{aligned} &b(u,v) + a(u,v) \\ &\quad = \int_0^\ell EIw_0^{(4)} v_1 - EIw_0'''(\ell) v_1(\ell) + EIw_0''(\ell) v_1'(\ell) \\ &\qquad + \gamma w_1(\ell) v_1(\ell) + \gamma d w_1'(\ell) v_1(\ell) \\ &\qquad + \gamma d w_1(\ell) v_1'(\ell) + (\gamma d + \gamma^*) d w_1'(\ell) v_1'(\ell). \end{aligned} \tag{44}$$

Therefore,

$$b(u,v) + a(u,v) = \int_0^\ell EIw_0^{(4)} v_1, \quad \text{for each } v \in V, \tag{45}$$

if and only if

$$\begin{aligned} 0 = &- EIw_0'''(\ell) v_1(\ell) + EIw_0''(\ell) v_1'(\ell) \\ &+ \gamma w_1(\ell) v_1(\ell) + \gamma d w_1'(\ell) v_1(\ell) + \gamma d w_1(\ell) v_1'(\ell) \\ &+ (\gamma d + \gamma^*) d w_1'(\ell) v_1'(\ell), \end{aligned} \tag{46}$$

for each $v \in V$. Since $v_1(\ell)$ and $v_1'(\ell)$ are arbitrary, it follows that

$$\begin{aligned} &- EIw_0'''(\ell) + \gamma w_1(\ell) + \gamma d w_1'(\ell) = 0; \\ &EIw_0''(\ell) + \gamma d w_1(\ell) + (\gamma d + \gamma^*) d w_1'(\ell) = 0. \end{aligned} \tag{47}$$

Therefore, a sufficient condition for existence is $w_0 \in C^4[0,\ell]$, $w_1 \in C^2[0,\ell]$, and

$$\begin{aligned} &w_0(0) = w_0'(0) = w_1(0) = w_1'(0) = 0; \\ &V(\ell, 0) = - EIw_0'''(\ell) = -\gamma w_1(\ell) - \gamma d w_1'(\ell); \\ &M(\ell, 0) = EIw_0''(\ell) = -\gamma d w_1(\ell) - (\gamma d + \gamma^*) d w_1'(\ell). \end{aligned} \tag{48}$$

The conditions for the shear force $V(\ell, 0)$ and bending moment $M(\ell, 0)$ have an interesting physical interpretation. Comparing them to (9) and (11), we see that the force and moment at the endpoint must match the force and moment due to damping.

6.3. Discussion. In [1], the model problem is written in the form

$$(By)' + (Ay) = g, \tag{49}$$

equation (3.11). The existence of a unique solution then follows from the theory in [3]. It should also be possible to use [10]. Relevant abstract existence results may also be found in other publications, for example, [11].

Theorems 15 and 16 (existence results in [4]) are convenient for application when the model problem is in weak variational form. This is so because the assumptions are in terms of the bilinear forms a, b, and c and it is not necessary to consider linear operators as in the publications cited above.

The approach in [4] is relatively new and therefore we discuss briefly how it is related to semigroup theory. Problem G is equivalent to a first order differential equation in the product space $H = V \times W$. A linear operator A with domain $\mathscr{D}(A) \subset V \times W$ is constructed in [4] and problem G is equivalent to an initial value problem of the form

$$w' = Aw + g \quad \text{with } w(0) = \langle u_0, u_1 \rangle . \tag{50}$$

The pair $\langle u_0, u_1 \rangle \in \mathscr{D}(A)$ if and only if u_0 and u_1 in V satisfy condition (b) in Theorem 15. The properties of A are determined by the properties of the three bilinear forms c, a, and b. Under the assumptions in Theorem 15 A is the infinitesimal generator of a C_0 semigroup of contractions and under the assumptions in Theorem 16 A is the infinitesimal generator of an analytic semigroup.

7. Application

7.1. Natural Frequencies. In the second half of Section 4 in [1], the sequence of natural frequencies of the undamped system is considered. The conjecture on p1041 concerning the eigenvalues for the undamped system is indeed correct. For each $f \in W$, the problem $b(x, v) = c(f, v)$, for each $v \in V$, has a unique solution $x \in V$. The mapping K, defined by $x = Kf$, is a symmetric linear operator. Since K is bounded as a mapping from W into V and the embedding of V into W is compact, K is compact. Considering b as a bilinear form in W, its eigenvalues are real and if λ is an eigenvalue, then λ^{-1} is an eigenvalue of K. The corresponding eigenvectors are the same. It follows that the sequence of eigenvalues tends to infinity and the sequence of eigenvectors is complete in W (see, e.g., [12, Theorem 4.A, p.232]).

For the model problem with the damping tip body, the situation is different. To the best of our knowledge, there is no general spectral theory for systems with boundary damping. However, results are known for specific model problems. In [13], it is proved that the sequence of eigenfunctions for an Euler-Bernoulli beam with boundary damping has the Riesz basis property, but there is no attached body.

Galerkin Approximation for Problem PW. Let S^h be a finite dimensional subspace of V.

Problem PW^h. Find u_h such that, for each $t > 0$, $u_h(t) \in S^h$ and

$$c\left(u_h''(t), v\right) + a\left(u_h'(t), v\right) + b\left(u_h(t), v\right) = \left(\tilde{f}(t), v\right)_X, \quad \text{for each } v \in S^h, \tag{51}$$

with $u_h(0) = u_0^h$ and $u_h'(0) = u_1^h$. The functions u_0^h and u_1^h are suitable approximations for u_0 and u_1 in S^h.

Problem PW^h is equivalent to a system of ordinary differential equations:

$$M\overline{u}'' + C\overline{u}' + K\overline{u} = F(t). \tag{52}$$

The system can be used to approximate the solution of problem PW. How to construct the relevant matrices is explained in [14].

The quadratic eigenvalue problem

$$\lambda^2 M\overline{u} + \lambda C\,\overline{u} + K\overline{u} = 0 \tag{53}$$

can be used to calculate approximations for the natural frequencies (see [14]).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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