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Research Article

The Semidirect Sum of Lie Algebras and Its Applications to C-KdV Hierarchy

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By use of the loop algebra $\overline{\overline{G}}$, integrable coupling of C-KdV hierarchy and its bi-Hamiltonian structures are obtained by Tu scheme and the quadratic-form identity. The method can be used to produce the integrable coupling and its Hamiltonian structures to the other integrable systems.

1. Introduction

Integrable coupling is a new topic of the Soliton theory; especially, looking for the new Hamiltonian structure of integrable coupling is more important. The integrable coupling of some known integrable hierarchies is obtained. But their Hamiltonian structure has not been presented because there exists a limitation in trace identity till the quadratic-form identity [1] and the variational identity [2] are proposed. In this paper, a higher-dimensional Lie algebra $\overline{G}$ and the loop algebra $\overline{\overline{G}}$ are constructed [3, 4]. With the help of Tu scheme [5] and the quadratic-form identity, the integrable coupling of C-KdV hierarchy as well as its bi-Hamiltonian structures is produced.

2. Basic Principle of the Semidirect Sum of Lie Algebras

Let $\mathbf{G}$ be a linear space over real or complex number field F together with multiplication, for any $x, y, z \in \mathbf{G}, c \in F$, if $\mathbf{G}$ satisfy

(1) distributive law

$$(x+y)z = xz + yz, \qquad z(x+y) = zx + zy; \tag{1}$$

(2) multiplication commutativity

$$c(xy) = (cx)y = x(cy). \tag{2}$$

Then, $\mathbf{G}$ is called algebra.

Lie algebra $\mathbf{G}$ is an algebra over number field F, if its multiplication satisfies the following:

(1) bilinearity

$$\begin{aligned} [cx + c'y, z] &= c[x,z] + c'[y,z]; \\ [z, cx + c'y] &= c[z,x] + c'[z,y]; \end{aligned} \tag{3}$$

(2) anticommutative

$$[x,y] = -[y,x]; \tag{4}$$

(3) the Jacobi identity

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = 0, \tag{5}$$

where $[\cdot,\cdot]$ denote the multiplication of $\mathbf{G}$, $x, y, z \in \mathbf{G}$, $c, c' \in K$. The multiplication of Lie algebra is called Lie product. One kind of the most important Lie algebras on integrable systems is $A_{n-1} = \{M_{n\times n} \mid \operatorname{tr} M_{n\times n} = 0\}$, where $M_{n\times n}$ denote matrix order n over number field K.

I satisfies $[I, A] \subseteq I$ for arbitrary Lie algebra A; then I is called Lie ideal.

Lie algebra A is called simple Lie algebra if A has A and 0 as Lie ideal and without other Lie ideal. Semisimple Lie algebra R can be written as

$$R = \bigoplus_i A_i, \tag{6}$$

where A_i is simple Lie algebra. We have already known that $A_n, B_n, C_n, D_n, E_{6,7,8}, F_4$, and G_2 are all semisimple Lie algebras which has been studied by Cartan long ago [5]. We also know that Lie algebra R can be written as

$$R = R_1 \uplus R_2, \tag{7}$$

where R_1 is semisimple Lie algebras and R_2 is solvable Lie algebras [3, 6, 7] and $\uplus$ denote the semidirect sum. So we can apply the above basic principle to integrable coupling systems.

3. C-KdV Hierarchy

Firstly, let us recall the construction of the C-KdV hierarchy [8, 9]. Consider the basis of G

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{8}$$

The loop algebra $\widetilde{G}$ is presented as $e_i(n) = e_i\lambda^n$.

The C-KdV spectral problem reads as

$$\psi_x = U\psi, \qquad \lambda_t = 0$$
$$U = \begin{pmatrix} \dfrac{q-\lambda}{2} & -r \\ 1 & \dfrac{-q+\lambda}{2} \end{pmatrix}, \qquad u = \begin{pmatrix} q \\ r \end{pmatrix}. \tag{9}$$

Upon setting $V = \Sigma_{m\geq 0}(a_m e_1(-m) + b_m e_2(-m) + c_m e_3(-m))$, solving the stationary zero curvature equation,

$$V_x = [U, V], \tag{10}$$

engenders

$$\begin{aligned} a_{nx} &= -rc_m - b_m, \\ b_{mx} &= -b_{m+1} + qb_m + 2ra_m, \\ c_{mx} &= c_{m+1} - qc_m + 2a_m, \\ a_0 &= \frac{1}{2}, \qquad b_0 = c_0 = 0, \\ a_1 &= 0, \qquad b_1 = r, \qquad c_1 = -1. \end{aligned} \tag{11}$$

The compatibility conditions of the spectral problems

$$\begin{aligned} &\psi_x = U\psi, \qquad \psi_t = V^{(n)}\psi; \\ &V^{(n)} = (\lambda^n V)_+ + \frac{1}{2}c_{n+1}e_1(0), \quad n \geq 0, \end{aligned} \tag{12}$$

determine the C-KdV hierarchy of Soliton equations

$$u_t = \begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} 0 & -\partial \\ -\partial & 0 \end{pmatrix}\begin{pmatrix} a_{n+1} \\ -c_{n+1} \end{pmatrix} = J\frac{\delta H_n}{\delta u}, \tag{13}$$

where

$$\begin{aligned} &H_n = \frac{a_{n+1}}{n}, \quad (n \geq 1), \\ &H_1 = r, \qquad H_2 = \frac{1}{2}(-r_x + 2qr). \end{aligned} \tag{14}$$

4. A New Integrable Coupling of the C-KdV Hierarchy

In what follows, we expand Lie algebra G into a bigger one as the following Lie algebra $\overline{G}$:

$$\begin{aligned} &h_1 = \begin{pmatrix} e_1 & 0 \\ 0 & e_1 \end{pmatrix}, \qquad h_2 = \begin{pmatrix} e_2 & 0 \\ 0 & e_2 \end{pmatrix}, \qquad h_3 = \begin{pmatrix} e_3 & 0 \\ 0 & e_3 \end{pmatrix}, \\ &h_4 = \begin{pmatrix} 0 & e_1 \\ 0 & 0 \end{pmatrix}, \qquad h_5 = \begin{pmatrix} 0 & e_2 \\ 0 & 0 \end{pmatrix}, \qquad h_6 = \begin{pmatrix} 0 & e_3 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{15}$$

We do this along with the following commutative relations:

$$\begin{aligned} &[h_1, h_2] = 2h_2, \qquad [h_1, h_3] = -2h_3, \qquad [h_1, h_5] = 2h_5, \\ &[h_1, h_6] = -2h_6, \qquad [h_2, h_3] = h_1, \qquad [h_2, h_4] = -2h_5, \\ &[h_2, h_6] = h_4, \qquad [h_3, h_4] = 2h_6, \qquad [h_3, h_5] = -h_4, \\ &[h_1, h_4] = [h_2, h_5] = [h_3, h_6] = [h_4, h_5] \\ &\quad = [h_4, h_6] = [h_5, h_6] = 0. \end{aligned} \tag{16}$$

Taking $\overline{G}_1 = \operatorname{span}\{h_1, h_2, h_3\}$ and $\overline{G}_2 = \operatorname{span}\{h_4, h_5, h_6\}$, it is easy to verify that

$$\overline{G} = \overline{G}_1 \uplus \overline{G}_2, \quad G \cong \overline{G}_1, \quad \left[\overline{G}_1, \overline{G}_2\right] \subseteq \overline{G}_2, \tag{17}$$

where $\overline{G}_1$ is semisimple Lie algebras and $\overline{G}_2$ is solvable Lie algebras [3, 6, 7].

In terms of the Lie algebra $\overline{G}$, we constructed the loop algebra $\widetilde{\overline{G}}$ as follows $h_k(i, n) = h_k\lambda^{2n+i}$ [4, 10], with the following commutative relations:

$$\begin{aligned} &[h_1(i, m), h_2(j, n)] = 2h_2\left(\delta_{ij}, m + n + \rho_{ij}\right) \\ &[h_1(i, m), h_3(j, n)] = -2h_3\left(\delta_{ij}, m + n + \rho_{ij}\right) \\ &[h_1(i, m), h_5(j, n)] = 2h_5\left(\delta_{ij}, m + n + \rho_{ij}\right) \end{aligned}$$

$$\begin{aligned}
&[h_1(i,m), h_6(j,n)] = -2h_6(\delta_{ij}, m+n+\rho_{ij})\\
&[h_2(i,m), h_3(j,n)] = h_1(\delta_{ij}, m+n+\rho_{ij})\\
&[h_2(i,m), h_4(j,n)] = -2h_5(\delta_{ij}, m+n+\rho_{ij})\\
&[h_2(i,m), h_4(j,n)] = -2h_5(\delta_{ij}, m+n+\rho_{ij})\\
&[h_2(i,m), h_4(j,n)] = -2h_5(\delta_{ij}, m+n+\rho_{ij})\\
&[h_2(i,m), h_6(j,n)] = h_4(\delta_{ij}, m+n+\rho_{ij})\\
&[h_3(i,m), h_4(j,n)] = 2h_6(\delta_{ij}, m+n+\rho_{ij})\\
&[h_3(i,m), h_5(j,n)] = -h_4(\delta_{ij}, m+n+\rho_{ij})\\
&[h_1(i,m), h_4(j,n)] = [h_2(i,m), h_5(j,n)] = 0\\
&[h_3(i,m), h_6(j,n)] = [h_4(i,m), h_5(j,n)] = 0\\
&[h_4(i,m), h_6(j,n)] = [h_5(i,m), h_6(j,n)] = 0.
\end{aligned} \tag{18}$$

$\delta_{ij} = i + j$, $\rho_{ij} = 0$, when $i + j < 2$, and $\delta_{ij} = 0$, $\rho_{ij} = 1$, when $i + j = 2$. With the help of above equations, we consider an isospectral problem:

$$\begin{aligned}
&\psi_x = U\psi, \qquad \lambda_t = 0.\\
&U = -\frac{1}{2}h_1(1,0) + \frac{q}{2}h_1(0,0) - rh_2(0,0)\\
&\quad + h_3(0,0) + u_1h_4(0,0) + u_2h_5(0,0)
\end{aligned} \tag{19}$$

Set

$$\begin{aligned}
V = \sum_{m\geq 0}\sum_{i=0}^{1} &\left(a(i,m)h_1(i,-m) + b(i,m)h_2(i,-m)\right.\\
&+ c(i,m)h_3(i,-m) + d(i,m)h_4(i,-m)\\
&\left.+ e(i,m)h_5(i,-m) + f(i,m)h_6(i,-m)\right).
\end{aligned} \tag{20}$$

Solving the stationary zero curvature equation (10) permits that

$$\begin{aligned}
a_x(0,m) &= -rc(0,m) - b(0,m),\\
a_x(1,m) &= -rc(1,m) - b(1,m),\\
b_x(0,m) &= -b(1,m+1) + qb(0,m)\\
&\quad + 2ra(0,m),\\
b_x(1,m) &= -b(0,m) + qb(1,m) + 2ra(1,m),\\
c_x(0,m) &= c(1,m+1) - qc(0,m) + 2a(0,m),\\
c_x(1,m) &= c(0,m) - qc(1,m) + 2a(1,m),\\
d_x(0,m) &= -rf(0,m) - e(0,m) + u_2c(0,m),\\
d_x(1,m) &= -rf(1,m) - e(1,m) + u_2c(1,m),\\
e_x(0,m) &= -e(1,m+1) + qe(0,m)\\
&\quad + 2rd(0,m) + 2u_1b(0,m)\\
&\quad - 2u_2a(0,m),\\
e_x(1,m) &= -e(0,m) + qe(1,m) + 2rd(1,m)\\
&\quad + 2u_1b(1,m) - 2u_2a(1,m),\\
f_x(0,m) &= f(1,m+1) - qf(0,m)\\
&\quad + 2d(0,m) - 2u_1c(0,m),\\
f_x(1,m) &= f(0,m) - qf(1,m)\\
&\quad + 2d(1,m) - 2u_1c(1,m),
\end{aligned} \tag{21}$$

$$\begin{aligned}
&a(0,0) = v_1, \qquad d(0,0) = v_2,\\
&b(0,0) = c(0,0) = e(0,0) = f(0,0) = 0,\\
&a(1,0) = b(1,0) = c(1,0) = d(1,0)\\
&\quad = e(1,0) = f(1,0) = 0,\\
&a(0,1) = 2v_1r,\\
&b(0,1) = 2v_1qr + 2v_3r - 2v_1r_x,\\
&c(0,1) = -2v_1q - 2v_3,\\
&d(0,1) = 2v_2r - 2v_1u_2,\\
&e(0,1) = 2v_2qr - 2v_1u_2q + 2v_4r + 4v_1ru_1\\
&\quad - 2v_3u_2 - 2v_2r_x + 2v_1u_{2x},\\
&f(0,1) = -2v_2q - 2v_4 - 4v_1u_1, \qquad a(1,1) = v_3,\\
&b(1,1) = 2v_1r, \qquad c(1,1) = -2v_1,\\
&d(1,1) = v_4, \qquad e(1,1) = 2v_2r - 2v_1u_2,\\
&f(1,1) = -2v_2,
\end{aligned} \tag{22}$$

where v_1, v_2, v_3, and v_4 are nonzero constants.

Assume that $V_+^{(n)} = \sum_{m=0}^{n}\sum_{i=0}^{1}(a(i,m)h_1(i,n-m) + b(i,m)h_2(i,n-m) + c(i,m)h_3(i,n-m) + d(i,m)h_4(i,n-m) + e(i,m)h_5(i,n-m) + f(i,m)h_6(i,n-m)) = \lambda^{2n}V - V_-^{(n)}$; then (10) may be written as

$$-V_{+x}^{(n)} + \left[U, V_+^{(n)}\right] = V_{-x}^{(n)} - \left[U, V_-^{(n)}\right]. \tag{23}$$

A direct calculation reads

$$\begin{aligned} & -V_{+x}^{(n)}+\left[U,V_{+}^{(n)}\right] \\ & \quad = b\left(1,n+1\right)h_2\left(0,0\right)-c\left(1,n+1\right)h_3\left(0,0\right) \\ & \qquad + e\left(1,n+1\right)h_5\left(0,0\right)-f\left(1,n+1\right)h_6\left(0,0\right). \end{aligned} \tag{24}$$

Take $V^{(n)} = V_{+}^{(n)}+1/2c(1,n+1)h_1(0,0)+1/2f(1,n+1)h_4(0,0)$; then the zero curvature equation

$$U_t - V_x^{(n)} + \left[U,V^{(n)}\right] = 0 \tag{25}$$

is equivalent to

$$\begin{aligned} u_t &= \begin{pmatrix} q \\ r \\ u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} 0 & 0 & 0 & \partial \\ 0 & 0 & -\frac{1}{2}\partial & 0 \\ 0 & -\frac{1}{2}\partial & 0 & -\frac{1}{2}\partial \\ \partial & 0 & -\frac{1}{2}\partial & 0 \end{pmatrix} \cdot \begin{pmatrix} a(1,n+1)+d(1,n+1) \\ -c(1,n+1)-f(1,n+1) \\ 2a(1,n+1) \\ c(1,n+1) \end{pmatrix} \\ &= J_1 \begin{pmatrix} a(1,n+1)+d(1,n+1) \\ -c(1,n+1)-f(1,n+1) \\ 2a(1,n+1) \\ c(1,n+1) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & -\partial & \partial^2+\partial q \\ 0 & 0 & \frac{1}{2}\left(\partial^2-q\partial\right) & r\partial+\partial r \\ -\partial & -\frac{1}{2}\left(\partial^2+\partial q\right) & \frac{1}{2}\partial & -\frac{1}{2}\left(\partial^2+\partial q\right)+\partial u_1 \\ -\partial^2+q\partial & r\partial+\partial r & \frac{1}{2}\left(\partial^2-q\partial\right)+u_1\partial & r\partial+\partial r+u_2\partial+\partial u_2 \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} a(0,n)+d(0,n) \\ -c(0,n)-f(0,n) \\ 2a(0,n) \\ c(0,n) \end{pmatrix} = J_2 \begin{pmatrix} a(0,n)+d(0,n) \\ -c(0,n)-f(0,n) \\ 2a(0,n) \\ c(0,n) \end{pmatrix}, \end{aligned} \tag{26}$$

where J_1 and J_2 are Hamiltonian operators.

From (22), a recurrence operator L is obtained, which satisfies

$$\begin{pmatrix} a(1,n+1)+d(1,n+1) \\ -c(1,n+1)-f(1,n+1) \\ 2a(1,n+1) \\ c(1,n+1) \end{pmatrix} = L \begin{pmatrix} a(0,n)+d(0,n) \\ -c(0,n)-f(0,n) \\ 2a(0,n) \\ c(0,n) \end{pmatrix}, \tag{27}$$

where

$$L = \begin{pmatrix} -\partial+\partial^{-1}q\partial & \partial^{-1}r\partial+r & \partial^{-1}u_1\partial & \partial^{-1}u_2\partial+u_2 \\ 2 & q+\partial & 0 & -2u_1 \\ 0 & 0 & -\partial-\partial^{-1}q\partial & -2\partial^{-1}r\partial-2r \\ 0 & 0 & -1 & \partial+q \end{pmatrix}. \tag{28}$$

It is easy to verify that

$$J_1 L = L^* J_1 = J_2. \tag{29}$$

Therefore, the hierarchy (26) is Liouville integrable. Taking $q = r = 0$, $u_1 = q/2$, and $u_2 = -r$, (26) reduces to (13). According to the integrable theory, the hierarchy (26) is the integrable coupling of the C-KdV hierarchy. Furthermore, in the following part we will point out that there exist bi-Hamiltonian structures from constructing of Lie loop algebras.

5. The Bi-Hamiltonian Structures of the Hierarchy (26)

Let

$$a = \sum_{i=1}^{6} a_i h_i, \qquad b = \sum_{i=1}^{6} b_i h_i. \tag{30}$$

We have $[a,b] = (a_2b_3-a_3b_2, 2a_1b_2-2a_2b_1, 2a_3b_1-2a_1b_3, a_2b_6-a_6b_2+a_5b_3-a_3b_5, 2a_1b_5-2a_5b_1+2a_4b_2-2a_2b_4, 2a_3b_4-2a_1b_6+2a_6b_1-2a_4b_3)^T$.

In what follows, from $[a,b]^T = a^T R(b)$, we get

$$R(b) = \begin{pmatrix} 0 & 2b_2 & -2b_3 & 0 & 2b_5 & -2b_6 \\ b_3 & -2b_1 & 0 & b_6-2b_4 & 0 & \\ -b_2 & 0 & 2b_1 & -b_5 & 0 & 2b_4 \\ 0 & 0 & 0 & 0 & 2b_2 & -2b_3 \\ 0 & 0 & 0 & b_3 & -2b_1 & 0 \\ 0 & 0 & 0 & -b_2 & 0 & 2b_1 \end{pmatrix}. \tag{31}$$

Solving the matrix equation $R(b)F = -(R(b)F)^T$ for F gives rise to

$$F = \begin{pmatrix} 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{32}$$

So we have $\{a, b\} = a^T F b = (2a_1 + 2a_4)b_1 + (a_3 + a_6)b_2 + (a_2 + a_5)b_3 + 2a_1 b_4 + a_3 b_5 + a_2 b_6$.

A direct calculation reads

$$\begin{aligned} \left\{V, \frac{\partial U}{\partial \lambda}\right\} &= -a(0) - a(1) - d(0) - d(1) \\ \left\{V, \frac{\partial U}{\partial q}\right\} &= a(0) + a(1) + d(0) + d(1) \\ \left\{V, \frac{\partial U}{\partial r}\right\} &= -c(0) - c(1) - f(0) - f(1) \\ \left\{V, \frac{\partial U}{\partial u_1}\right\} &= 2a(0) + 2a(1) \\ \left\{V, \frac{\partial U}{\partial u_2}\right\} &= c(0) + c(1), \end{aligned} \tag{33}$$

where $a(0) = \sum_{m\geq 0} a(0,m)\lambda^{-2m}$ and $a(1) = \sum_{m\geq 0} a(1,m)\lambda^{-2m+1}\cdots$.

Substituting the above formulas into the quadratic-form identity yields

$$\begin{aligned} &\frac{\delta}{\delta u}\left(-a(0) - a(1) - d(0) - d(1)\right) \\ &\quad = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{pmatrix} a(0) + a(1) + d(0) + d(1) \\ -c(0) - c(1) - f(0) - f(1) \\ 2a(0) + 2a(1) \\ c(0) + c(1) \end{pmatrix}. \end{aligned} \tag{34}$$

Comparison of coefficients of λ^{-2n-2} of both sides of the above equations leads to

$$\begin{aligned} &\frac{\delta}{\delta u}\left(-a(0,n+1) - d(0,n+1)\right) \\ &\quad = (-2n-1+\gamma)\begin{pmatrix} a(1,n+1) + d(1,n+1) \\ -c(1,n+1) - f(1,n+1) \\ 2a(1,n+1) \\ c(1,n+1) \end{pmatrix}. \end{aligned} \tag{35}$$

To fix the γ we take $n = 0$ into the above equation and find $\gamma = 1$.

So

$$\begin{gathered} \begin{pmatrix} a(1,n+1) + d(1,n+1) \\ -c(1,n+1) - f(1,n+1) \\ 2a(1,n+1) \\ c(1,n+1) \end{pmatrix} = \frac{\delta H(1,n)}{\delta u}, \\ H(1,n) = \frac{a(0,n+1) + d(0,n+1)}{2n}. \end{gathered} \tag{36}$$

Comparison of coefficients of λ^{-2n-1} of both sides of the above equations gives

$$\begin{aligned} &\frac{\delta}{\delta u}\left(-a(1,n+1) - d(1,n+1)\right) \\ &\quad = (-2n+\gamma)\begin{pmatrix} a(0,n) + d(0,n) \\ -c(0,n) - f(0,n) \\ 2a(0,n) \\ c(0,n) \end{pmatrix}. \end{aligned} \tag{37}$$

In this situation, we have $\gamma = 0$.

So

$$\begin{pmatrix} a(0,n) + d(0,n) \\ -c(0,n) - f(0,n) \\ 2a(0,n) \\ c(0,n) \end{pmatrix} = \frac{\delta H(2,n)}{\delta u}, \tag{38}$$

$$H(2,n) = \frac{a(1,n+1) + d(1,n+1)}{2n}. \tag{39}$$

Thus the bi-Hamiltonian structures of the system (26) are given by

$$u_t = J_1 \frac{\delta H(1,n)}{\delta u} = J_2 \frac{\delta H(2,n)}{\delta u}. \tag{40}$$

From the system (26), we easily give the following equations:

$$\begin{aligned} q_t &= -2v_1 q_{xx} - 4v_1 q q_x - 2v_3 q_x - 4v_1 r_x, \\ r_t &= -4v_1 (qr)_{xx} - 2v_3 r_x - 2v_1 r_{xx}. \end{aligned} \tag{41}$$

6. Conclusion

On the one hand, we obtain a new integrable coupling of C-KdV hierarchy by expanding a bigger Lie algebra. On the other hand, the bi-Hamiltonian structures of the integrable coupling of C-KdV hierarchy are observed by use of the quadratic-form identity.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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