Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2014, Article ID 287529, 8 pages
http://dx.doi.org/10.1155/2014/287529

Research Article

Bargmann Type Systems for the Generalization of Toda Lattices

Fang Li[1] and Liping Lu[2]

[1] *College of Science, Henan University of Technology, 100 Lianhua Road, Zhengzhou, Henan 450001, China*
[2] *Department of Information Engineering, Henan College of Finance and Taxation, Zhengkai Road, Zhengzhou, Henan 451464, China*

Correspondence should be addressed to Fang Li; lfseafish@163.com

Received 5 April 2014; Accepted 24 May 2014; Published 9 June 2014

Academic Editor: Senlin Guo

Under a constraint between the potentials and eigenfunctions, the nonlinearization of the Lax pairs associated with the discrete hierarchy of a generalization of the Toda lattice equation is proposed, which leads to a new symplectic map and a class of finite-dimensional Hamiltonian systems. The generating function of the integrals of motion is presented, by which the symplectic map and these finite-dimensional Hamiltonian systems are further proved to be completely integrable in the Liouville sense. Finally, the representation of solutions for a lattice equation in the discrete hierarchy is obtained.

1. Introduction

Differential difference equations have very remarkable applications in modern mathematics and physics; they can model a number of physically interesting phenomena, such as the vibration of particle in lattice [1], the quantum spin chains [2, 3], the Toda lattice [4], the vibration of pulse [5, 6], the nonlinear self-dual network [7], and others. After Toda [8] showed that the Toda lattice was associated with a discretization of the Schrödinger spectral problem, various discrete soliton equations are found, for instance, the discrete nonlinear Schrödinger equation [9], the discrete sine-Gordon equation [10], the discrete KdV equation [11], the discrete mKdV equation [12], and so forth. Recently, the authors have obtained a new discrete hierarchy associated with fourth-order discrete spectral problem, in which a typical member is a generalization of the Toda lattice equation [13].

It has been known that the key to complete integrability of a finite-dimensional Hamiltonian system is the existence of an involutive system of conserved integrals according to the Liouville-Arnold theorem. Many researchers have tried to construct complete integrable Hamiltonian systems. Recently, there are active researches on soliton hierarchies associated with so $(3, \mathbb{R})$ [14]. However, it is a difficult work to search for an involutive system of conserved integrals for a given finite-dimensional Hamiltonian system. An effective method, the nonlinearization of Lax pairs [15, 16], has been developed and applied to various soliton hierarchies associated with 2×2 matrix spectral problems to get finite-dimensional completely integrable systems many years ago, such as the nonlinearization of the AKNS hierarchy [15], the coupled KdV hierarchy [17], the discrete Ablowitz-Ladik hierarchy [18], the Heisenberg hierarchy [19], and the Kac-van Moerbeke hierarchy [20]. Subsequently, this method has been generalized to discuss the nonlinearization of Lax pairs and adjoint Lax pairs of soliton hierarchies [21–23]. Moreover, there are attempts to apply the nonlinearization method to the Lax pairs and adjoint Lax pairs of $(2 + 1)$-dimensional soliton systems, such as the Kadomtsev-Petviashvili equation and the Davey-Stewartson equation, in order to get $(1 + 1)$-dimensional integrable systems [24]. And it is proved that the binary nonlinearization will be more natural to carry out in the case of higher-order matrix spectral problems [25].

Discrete versions of classical integrable systems have become the focus of common concern in recent years because of their importance. However, the known discrete integrable systems are few compared with the continuous case. In the present paper, the nonlinearization approach is developed and applied to the discrete hierarchy associated with a 4×4 discrete eigenvalue problem. Such transformations are adjoint symmetry constraints [26] and a general scheme for doing nonlinearization for lattice soliton hierarchies was presented in [27]. We propose a constraint between the

potentials and eigenfunctions. The nonlinearization of the Lax pairs for the discrete hierarchy leads to a new integrable symplectic map and a class of finite-dimensional integrable Hamiltonian systems.

The outline of this paper is as follows. In Section 2, depending on the spectral problems given in [13], the Bargmann constraint between the potentials and eigenfunctions is introduced, from which a new symplectic map and a class of finite-dimensional Hamiltonian systems are obtained. In Section 3, the generating function approach is used to calculate the involutivity of integrals, by which the symplectic map and these finite-dimensional Hamiltonian systems are further proved to be completely integrable in the Liouville sense. Finally, in Section 4, the representation of solutions for a lattice equation in the discrete hierarchy is obtained.

2. A New Symplectic Map

Consider the discrete 4×4 spectral problem given in [13]

$$E\psi_n = U_n\psi_n, \qquad U_n = \begin{pmatrix} -\frac{a_n}{c_n} & -\frac{1}{c_n} & 0 & \frac{\lambda - b_n}{c_n} \\ 0 & 0 & 1 & a_n \\ 0 & 0 & 0 & c_n \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_n^1 \\ \psi_n^2 \\ \psi_n^3 \\ \psi_n^4 \end{pmatrix}, \tag{1}$$

where a_n, b_n, c_n are three potentials and λ is a constant spectral parameter; E is a translation operator defined by $Ef_n = f_{n+1}$. For the sake of convenience, we usually denote $f_{n+k} = E^k f_n$, $f_{n-k} = E^{-k} f_n$. In order to derive the hierarchy of Lattice equations associated with (1), authors of [13] first solve the stationary discrete zero-curvature equation:

$$V_{n+1}U_n - U_nV_n = 0, \qquad V_n = \left(V_{n,ij}\right)_{4\times 4}, \tag{2}$$

where the entries V_{ij} of the matrix V_n are Laurent expansions of λ. Let ψ_n satisfy the spectral problem (1) and its auxiliary problem:

$$\psi_{n,t} = V_n^{(m)}\psi_n, \qquad V_n^{(m)} = \left(\lambda^m V_n\right)_+; \tag{3}$$

then the zero-curvature equation $U_{n,t} = V_{n+1}^{(m)}U_n - U_nV_n^{(m)}$ yields the discrete hierarchy of a generalization of Toda lattices. The first system of evolution equations in this hierarchy is

$$\begin{aligned} a_{n,t} &= \frac{1}{2}a_n\left(b_n - b_{n+1}\right) + a_{n-1}c_{n-1} - a_{n+1}c_n, \\ b_{n,t} &= a_{n-1}^2 - a_n^2 + c_{n-2}^2 - c_n^2, \\ c_{n,t} &= \frac{1}{2}c_n\left(b_n - b_{n+2}\right), \end{aligned} \tag{4}$$

which is a generalization of Toda lattice equation.

Let $\lambda_1, \ldots, \lambda_N$ be N distinct nonzero eigenvalues of (1), and the associated eigenfunctions are denoted by

$$\begin{aligned} q_j^1 &= \psi_n^3\left(\lambda_j\right), \qquad q_j^2 = \psi_n^2\left(\lambda_j\right), \\ p_j^1 &= \psi_n^1\left(\lambda_j\right), \qquad p_j^2 = \psi_n^4\left(\lambda_j\right), \end{aligned} \tag{5}$$

where we denote $q_j^k = q_j^k(n)$ and $p_j^k = p_j^k(n)$ $(k = 1, 2)$ for convenience. Then the system associated with (1) can be written in the form

$$\begin{aligned} Eq_j^1 &= c_n p_j^2, \qquad Eq_j^2 = q_j^1 + a_n p_j^2, \\ Ep_j^1 &= -\frac{1}{c_n}\left(a_n p_j^1 + q_j^2 - \lambda_j p_j^2 + b_n p_j^2\right), \qquad Ep_j^2 = p_j^1. \end{aligned} \tag{6}$$

Now we consider the Bargmann constraint

$$\sum_{j=1}^{N} \nabla\lambda_j = G_n^{(0)}, \tag{7}$$

where $G_n^{(0)} = (2a_n, b_n, 2c_n)^{\mathrm{T}}$ and $\nabla\lambda_j$ is the functional gradient of the eigenvalue λ_j with regard to the potentials a_n, b_n, and c_n; that is,

$$\begin{aligned} \nabla\lambda_j &= \begin{pmatrix} \frac{\delta\lambda_j}{\delta a_n} \\ \frac{\delta\lambda_j}{\delta b_n} \\ \frac{\delta\lambda_j}{\delta c_n} \end{pmatrix} \\ &= \begin{pmatrix} 2p_j^1p_j^2 \\ p_j^2p_j^2 \\ -\frac{2}{c_n}\left[a_n p_j^1p_j^2 + q_j^2p_j^2 + \left(\lambda_j - b_n\right)p_j^2p_j^2\right] \end{pmatrix}. \end{aligned} \tag{8}$$

Combining (7) and (8), it is easy to see that

$$\begin{aligned} a_n &= \langle p^1, p^2\rangle, \qquad b_n = \langle p^2, p^2\rangle, \\ c_n &= \left(-\langle p^1, p^2\rangle^2 - \langle q^2, p^2\rangle + \langle \Lambda p^2, p^2\rangle - \left\langle p^2, p^2\right\rangle^2\right)^{1/2}, \end{aligned} \tag{9}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ and $\langle\cdot,\cdot\rangle$ is the standard inner-product in $\mathbb{R}^N$, $q^i = (q_1^i, \ldots, q_N^i)^{\mathrm{T}}$ and $p^i = (p_1^i, \ldots, p_N^i)^{\mathrm{T}}$. Substituting (9) into (6), we can get the following system:

$$\begin{aligned} Eq_j^1 &= \left(-\left\langle p^1, p^2\right\rangle^2 - \left\langle q^2, p^2\right\rangle + \left\langle \Lambda p^2, p^2\right\rangle - \left\langle p^2, p^2\right\rangle^2\right)^{1/2} p_j^2, \\ Eq_j^2 &= q_j^1 + \left\langle p^1, p^2\right\rangle p_j^2, \\ Ep_j^1 &= -\left(-\left\langle p^1, p^2\right\rangle^2 - \left\langle q^2, p^2\right\rangle + \left\langle \Lambda p^2, p^2\right\rangle - \left\langle p^2, p^2\right\rangle^2\right)^{-1/2} \\ &\quad \times \left(\left\langle p^1, p^2\right\rangle p_j^1 + q_j^2 - \lambda_j p_j^2 + \left\langle p^2, p^2\right\rangle p_j^2\right), \\ Ep_j^2 &= p_j^1. \end{aligned} \tag{10}$$

Through tedious calculations one infers

$$\sum_{j=1}^{N}\sum_{i=1}^{2} d\left(Eq_j^i\right)\wedge d\left(Eq_j^i\right) = \sum_{j=1}^{N}\sum_{i=1}^{2} dq_j^i\wedge dq_j^i. \tag{11}$$

Therefore, (10) determines a symplectic map H of the Bargmann type:

$$\left(Eq^1, Eq^2, Ep^1, Ep^2\right) = H\left(q^1, q^2, p^1, p^2\right). \tag{12}$$

3. Liouville Integrability

Introducing a matrix $\mathscr{V}_\lambda$,

$$\mathscr{V}_\lambda = \left(\mathscr{V}_\lambda^{ij}\right)_{4\times4} = \begin{pmatrix} -Q_\lambda\left(q^1,p^1\right)-\frac{\lambda}{2} & Q_\lambda\left(p^1,p^2\right) & Q_\lambda\left(p^1,p^1\right)+1 & -Q_\lambda\left(q^2,p^1\right) \\ -Q_\lambda\left(q^1,q^2\right)-\left\langle q^1,p^2\right\rangle & Q_\lambda\left(q^2,p^2\right)+\frac{\lambda}{2} & Q_\lambda\left(q^2,p^1\right) & -Q_\lambda\left(q^2,q^2\right)-\left\langle q^2,p^2\right\rangle \\ -Q_\lambda\left(q^1,q^1\right)-\left\langle q^1,p^1\right\rangle & Q_\lambda\left(q^1,p^2\right) & Q_\lambda\left(q^1,p^1\right)+\frac{\lambda}{2} & -Q_\lambda\left(q^1,q^2\right)-\left\langle q^1,p^2\right\rangle \\ -Q_\lambda\left(q^1,p^2\right) & Q_\lambda\left(p^2,p^2\right)+1 & Q_\lambda\left(p^1,p^2\right) & -Q_\lambda\left(q^2,p^2\right)-\frac{\lambda}{2} \end{pmatrix}, \tag{13}$$

where

$$Q_\lambda\left(q^i,p^j\right) = \sum_{k=1}^{N}\frac{q_k^i p_k^j}{\lambda-\lambda_k}, \qquad Q_\lambda\left(p^i,p^j\right) = \sum_{k=1}^{N}\frac{p_k^i p_k^j}{\lambda-\lambda_k},$$
$$Q_\lambda\left(q^i,q^j\right) = \sum_{k=1}^{N}\frac{q_k^i q_k^j}{\lambda-\lambda_k}. \tag{14}$$

We can find that $\mathscr{V}_\lambda$ and $\mu I - \mathscr{V}_\lambda$ are two solutions of the stationary discrete zero-curvature equation (2) under the Bargmann constraint (7), where μ is a parameter and I is a 4×4 unit matrix. Then we assert that $\det\mathscr{V}_\lambda$ and $\det(\mu I-\mathscr{V}_\lambda)$ are independent constants of the discrete variable n. On the other hand,

$$\det\left(\mu I-\mathscr{V}_\lambda\right) = \mu^4 + \mathscr{F}_\lambda^{(1)}\mu^2 + \mathscr{F}_\lambda^{(2)}, \tag{15}$$

where

$$\mathscr{F}_\lambda^{(1)} = \sum_{1\le i<j\le4}\begin{vmatrix}\mathscr{V}_\lambda^{ii} & \mathscr{V}_\lambda^{ij}\\ \mathscr{V}_\lambda^{ji} & \mathscr{V}_\lambda^{jj}\end{vmatrix}, \qquad \mathscr{F}_\lambda^{(2)} = \det\mathscr{V}_\lambda. \tag{16}$$

Substituting the Laurent expansion of $Q_\lambda(q^i,p^j)$, $Q_\lambda(p^i,p^j)$, $Q_\lambda(q^i,q^j)$ into (16) we have

$$\mathscr{F}_\lambda^{(1)} = -\frac{1}{2}\lambda^2 + \sum_{m\ge1}F_m^{(1)}\lambda^{-m}, \qquad \widehat{\mathscr{F}}_\lambda^{(2)} = \sum_{m\ge1}F_m^{(2)}\lambda^{-m}, \tag{17}$$

where

$$\widehat{\mathscr{F}}_\lambda^{(2)} = \mathscr{F}_\lambda^{(2)} + \frac{1}{4}\mathscr{F}_\lambda^{(1)}\lambda^2 + \frac{1}{16}\lambda^4,$$

$$\begin{aligned} F_m^{(1)} &= \sum_{\substack{l+k=m-2\\ l,k\ge0}}\Big(-\left\langle\Lambda^j q^1,p^1\right\rangle\left\langle\Lambda^k q^1,p^1\right\rangle - \left\langle\Lambda^j q^2,p^2\right\rangle\left\langle\Lambda^k q^2,p^2\right\rangle \\ &\quad + \left\langle\Lambda^j p^1,p^1\right\rangle\left\langle\Lambda^k p^1,p^1\right\rangle \\ &\quad + \left\langle\Lambda^j p^2,p^1\right\rangle\left\langle\Lambda^k q^2,q^2\right\rangle + 2\left\langle\Lambda^j p^1,p^2\right\rangle \\ &\quad \times\left\langle\Lambda^k q^1,q^2\right\rangle - 2\left\langle\Lambda^j q^1,p^2\right\rangle\left\langle\Lambda^k q^2,p^1\right\rangle\Big) \\ &\quad + \left\langle q^1,p^1\right\rangle\left\langle\Lambda^{m-1}p^1,p^1\right\rangle + \left\langle q^2,p^2\right\rangle\left\langle\Lambda^{m-1}p^2,p^2\right\rangle \\ &\quad + 2\left\langle q^1,p^2\right\rangle\left\langle\Lambda^{m-1}p^1,p^2\right\rangle + \left\langle\Lambda^{m-1}q^1,q^1\right\rangle \\ &\quad + \left\langle\Lambda^{m-1}q^2,q^2\right\rangle - \left\langle\Lambda^m q^1,p^2\right\rangle - \left\langle\Lambda^m q^2,p^2\right\rangle, \end{aligned}$$

$$\begin{aligned} F_m^{(2)} &= \sum_{\substack{k+j+s+i=m-4\\ j,k,s,i\ge0}}\begin{vmatrix}\left\langle\Lambda^k q^1,p^1\right\rangle & \left\langle\Lambda^k p^1,p^2\right\rangle & \left\langle\Lambda^k p^1,p^1\right\rangle & \left\langle\Lambda^k q^2,p^1\right\rangle\\ \left\langle\Lambda^j q^1,q^2\right\rangle & \left\langle\Lambda^j q^2,p^2\right\rangle & \left\langle\Lambda^j q^2,p^1\right\rangle & \left\langle\Lambda^j q^2,q^2\right\rangle\\ \left\langle\Lambda^i q^1,q^1\right\rangle & \left\langle\Lambda^i q^1,p^2\right\rangle & \left\langle\Lambda^i q^1,p^1\right\rangle & \left\langle\Lambda^i q^1,q^2\right\rangle\\ \left\langle\Lambda^s q^1,p^2\right\rangle & \left\langle\Lambda^s p^2,p^2\right\rangle & \left\langle\Lambda^s p^1,p^2\right\rangle & \left\langle\Lambda^s q^2,p^2\right\rangle\end{vmatrix} \\ &\quad + \sum_{\substack{k+j+s=m-2\\ j,k,s\ge0}}\Bigg(\begin{vmatrix}\left\langle\Lambda^j q^2,p^2\right\rangle & \left\langle\Lambda^j q^2,p^1\right\rangle & \left\langle\Lambda^j q^2,q^2\right\rangle\\ \left\langle\Lambda^k q^1,p^2\right\rangle & \left\langle\Lambda^k q^1,p^1\right\rangle & \left\langle\Lambda^k q^1,q^2\right\rangle\\ \left\langle\Lambda^s p^2,p^2\right\rangle & \left\langle\Lambda^s p^1,p^2\right\rangle & \left\langle\Lambda^s q^2,p^2\right\rangle\end{vmatrix} \\ &\qquad + \begin{vmatrix}\left\langle\Lambda^k q^1,p^1\right\rangle & \left\langle\Lambda^k p^1,p^1\right\rangle & \left\langle\Lambda^k q^2,p^1\right\rangle\\ \left\langle\Lambda^j q^1,q^1\right\rangle & \left\langle\Lambda^j q^1,p^1\right\rangle & \left\langle\Lambda^j q^1,q^2\right\rangle\\ \left\langle\Lambda^s q^1,p^2\right\rangle & \left\langle\Lambda^s p^1,p^2\right\rangle & \left\langle\Lambda^s q^2,p^2\right\rangle\end{vmatrix}\Bigg) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{\substack{k+j+s=m-3\\ j,k,s\geq 0}}\Bigg(\begin{vmatrix}\langle\Lambda^j q^1,q^2\rangle & \langle\Lambda^j q^2,p^2\rangle & \langle\Lambda^j q^2,q^2\rangle\\ \langle\Lambda^k q^1,q^1\rangle & \langle\Lambda^k q^1,p^2\rangle & \langle\Lambda^k q^1,q^2\rangle\\ \langle\Lambda^s q^1,p^2\rangle & \langle\Lambda^s p^2,p^2\rangle & \langle\Lambda^s q^2,p^2\rangle\end{vmatrix}\\
&\quad+\begin{vmatrix}\langle\Lambda^k q^1,p^1\rangle & \langle\Lambda^k p^1,p^1\rangle & \langle\Lambda^k q^2,p^1\rangle\\ \langle\Lambda^j q^1,q^2\rangle & \langle\Lambda^j q^2,p^1\rangle & \langle\Lambda^j q^2,q^2\rangle\\ \langle\Lambda^s q^1,q^1\rangle & \langle\Lambda^s q^1,p^1\rangle & \langle\Lambda^s q^1,q^2\rangle\end{vmatrix}\\
&\quad+\langle q^2,p^2\rangle\\
&\quad\times\begin{vmatrix}\langle\Lambda^k q^1,p^1\rangle & \langle\Lambda^k p^1,p^2\rangle & \langle\Lambda^k p^1,p^1\rangle\\ \langle\Lambda^j q^1,q^1\rangle & \langle\Lambda^j q^1,p^2\rangle & \langle\Lambda^j q^1,p^1\rangle\\ \langle\Lambda^s q^1,p^2\rangle & \langle\Lambda^s p^2,p^2\rangle & \langle\Lambda^s p^1,p^2\rangle\end{vmatrix}\\
&\quad+\langle q^1,p^1\rangle\\
&\quad\times\begin{vmatrix}\langle\Lambda^k p^1,p^2\rangle & \langle\Lambda^k p^1,p^1\rangle & \langle\Lambda^k q^2,p^1\rangle\\ \langle\Lambda^j q^2,p^2\rangle & \langle\Lambda^j q^2,p^1\rangle & \langle\Lambda^j q^2,q^2\rangle\\ \langle\Lambda^s p^2,p^2\rangle & \langle\Lambda^s p^1,p^2\rangle & \langle\Lambda^s q^2,p^2\rangle\end{vmatrix}\\
&\quad-2\langle q^1,p^2\rangle\\
&\quad\times\begin{vmatrix}\langle\Lambda^k p^1,p^2\rangle & \langle\Lambda^k p^1,p^1\rangle & \langle\Lambda^k q^2,p^1\rangle\\ \langle\Lambda^j q^1,p^2\rangle & \langle\Lambda^j q^1,p^1\rangle & \langle\Lambda^j q^1,q^2\rangle\\ \langle\Lambda^s p^2,p^2\rangle & \langle\Lambda^s p^1,p^2\rangle & \langle\Lambda^s q^2,p^2\rangle\end{vmatrix}\Bigg)\\
&+\sum_{\substack{k+s=m-2\\ k,s\geq 0}}\Big[\langle\Lambda^k q^1,q^1\rangle\langle\Lambda^s q^1,q^1\rangle\\
&\quad-\langle\Lambda^k q^1,q^2\rangle\langle\Lambda^s q^1,q^2\rangle+2\langle q^1,p^2\rangle\\
&\quad\times\big(\langle\Lambda^k q^1,p^1\rangle\langle\Lambda^s q^2,p^1\rangle-\langle\Lambda^k p^1,p^1\rangle\\
&\qquad\times\langle\Lambda^s q^1,q^2\rangle+\langle\Lambda^k q^2,p^2\rangle\langle\Lambda^s q^1,p^2\rangle\\
&\qquad-\langle\Lambda^k q^1,q^2\rangle\langle\Lambda^s p^2,p^2\rangle\big)-\langle q^1,p^1\rangle\\
&\quad\times\big(\langle\Lambda^k q^2,p^1\rangle\langle\Lambda^s q^2,p^1\rangle-\langle\Lambda^k p^1,p^1\rangle\\
&\qquad\times\langle\Lambda^s q^2,q^2\rangle+\langle\Lambda^k q^2,p^2\rangle\langle\Lambda^s q^2,p^2\rangle\\
&\qquad-\langle\Lambda^k q^2,q^2\rangle\langle\Lambda^s p^2,p^2\rangle\big)-\langle q^2,p^2\rangle\\
&\quad\times\big(\langle\Lambda^k q^1,p^1\rangle\langle\Lambda^s q^1,p^1\rangle-\langle\Lambda^k p^1,p^1\rangle\\
&\qquad\times\langle\Lambda^s q^1,q^1\rangle+\langle\Lambda^k q^1,p^2\rangle\langle\Lambda^s q^1,p^2\rangle\\
&\qquad-\langle\Lambda^k q^1,q^1\rangle\langle\Lambda^s p^2,p^2\rangle\big)\\
&\quad+\big(\langle q^1,p^2\rangle^2-\langle q^1,p^1\rangle\langle q^2,q^2\rangle\big)\\
&\quad\times\big(\langle\Lambda^k p^1,p^2\rangle\langle\Lambda^s p^1,p^2\rangle\\
&\qquad-\langle\Lambda^k p^1,p^1\rangle\langle\Lambda^s p^2,p^2\rangle\big)\Big]\\
&+\sum_{\substack{k+s=m-1\\ k,s\geq 0}}\Big[\langle\Lambda^k q^1,q^2\rangle\langle\Lambda^s q^2,p^1\rangle\\
&\qquad-\langle\Lambda^k q^1,p^1\rangle\langle\Lambda^s q^2,q^2\rangle\\
&\qquad+\langle\Lambda^k q^1,q^2\rangle\langle\Lambda^s q^1,p^2\rangle\\
&\qquad-\langle\Lambda^k q^2,p^2\rangle\langle\Lambda^s q^1,q^1\rangle+\langle q^1,p^2\rangle\\
&\qquad\times\big(\langle\Lambda^k q^1,p^2\rangle\langle\Lambda^s p^1,p^1\rangle\\
&\qquad\quad-\langle\Lambda^k q^1,p^1\rangle\langle\Lambda^s p^1,p^2\rangle\\
&\qquad\quad+\langle\Lambda^k q^2,p^1\rangle\langle\Lambda^s p^2,p^2\rangle\\
&\qquad\quad-\langle\Lambda^k p^1,p^2\rangle\langle\Lambda^s p^1,p^2\rangle\big)\\
&\qquad-\langle q^1,p^1\rangle\big(\langle\Lambda^k q^2,p^2\rangle\langle\Lambda^s p^1,p^1\rangle\\
&\qquad\quad-\langle\Lambda^k p^1,p^2\rangle\langle\Lambda^s q^2,p^1\rangle\big)\\
&\qquad+\langle q^2,p^2\rangle\big(\langle\Lambda^k p^1,p^2\rangle\langle\Lambda^s q^1,p^2\rangle\\
&\qquad\quad-\langle\Lambda^k q^1,p^1\rangle\langle\Lambda^s p^2,p^2\rangle\big)\Big]\\
&+\sum_{\substack{k+s=m\\ k,s\geq 0}}\big(\langle\Lambda^k q^1,p^1\rangle\langle\Lambda^s q^2,p^2\rangle\\
&\qquad-\langle\Lambda^k q^2,p^1\rangle\langle\Lambda^s q^1,p^2\rangle\big)\\
&+\langle q^1,p^2\rangle\big(\langle\Lambda^m q^1,p^2\rangle-\langle\Lambda^m q^2,p^1\rangle\big)\\
&-\langle q^1,p^1\rangle\langle\Lambda^m q^2,p^2\rangle-\langle q^2,p^2\rangle\langle\Lambda^m q^1,p^1\rangle\\
&-2\langle q^1,p^2\rangle\langle\Lambda^{m-1}q^1,p^2\rangle-\langle\Lambda^m q^1,q^2\rangle\\
&+\langle q^1,p^1\rangle\langle\Lambda^{m-1}q^2,q^2\rangle+\langle q^2,p^2\rangle\langle\Lambda^{m-1}q^1,q^1\rangle\\
&-\big(\langle q^1,p^2\rangle^2-\langle q^1,p^1\rangle\langle q^2,q^2\rangle\big)\\
&\times\big(\langle\Lambda^{m-1}p^1,p^1\rangle+\langle\Lambda^{m-1}p^2,p^2\rangle\big),\quad m\geq 1.
\end{aligned}
\tag{18}
$$

In the above equations, the Poisson bracket of two functions is defined as

$$
\begin{aligned}
\{f,g\}&=\sum_{j=1}^{N}\sum_{i=1}^{2}\left(\frac{\partial f}{\partial q_j^i}\frac{\partial g}{\partial p_j^i}-\frac{\partial g}{\partial q_j^i}\frac{\partial f}{\partial p_j^i}\right)\\
&=\sum_{i=1}^{2}\left(\left\langle\frac{\partial f}{\partial q^i},\frac{\partial g}{\partial p^i}\right\rangle-\left\langle\frac{\partial g}{\partial q^i},\frac{\partial f}{\partial p^i}\right\rangle\right).
\end{aligned}
\tag{19}
$$

Then we can prove the following assertions.

Proposition 1. *The functions* $\{F_m^{(i)} \mid i = 1, 2, m \geq 1\}$ *are in involution in pairs; that is,*

$$\left\{F_m^{(i)}, F_l^{(j)}\right\} = 0, \quad \forall m, l \geq 1,\ 1 \leq i,\ j \leq 2. \tag{20}$$

Proof. Through tedious calculation we can obtain

$$\left\{\mathscr{F}_\lambda^{(1)}, \mathscr{F}_\mu^{(1)}\right\} = \left\{\mathscr{F}_\lambda^{(1)}, \mathscr{F}_\mu^{(2)}\right\} = \left\{\mathscr{F}_\lambda^{(2)}, \mathscr{F}_\mu^{(2)}\right\} = 0, \quad \forall \lambda, \mu \in \mathbb{C}. \tag{21}$$

Then we have

$$\left\{\mathscr{F}_\lambda^{(1)}, \mathscr{F}_\mu^{(1)}\right\} = \left\{\mathscr{F}_\lambda^{(1)}, \widehat{\mathscr{F}}_\mu^{(2)}\right\} = \left\{\widehat{\mathscr{F}}_\lambda^{(2)}, \widehat{\mathscr{F}}_\mu^{(2)}\right\} = 0, \quad \forall \lambda, \mu \in \mathbb{C}. \tag{22}$$

Then relation (20) follows by comparison of power of λ^N in (22) with (17) taken into account. □

Proposition 2. *The* $2N$ *1-forms* $dF_j^{(i)}$ $(1 \leq j \leq N, i = 1, 2)$ *are linearly independent.*

Proof. Assuming that there exist $2N$ constants $b_l^{(i)}$, so that

$$\sum_{j=1}^{N} \left(b_j^{(1)} \frac{\partial F_j^{(1)}}{\partial q^i} + b_j^{(2)} \frac{\partial F_j^{(2)}}{\partial q^i} \right) = 0, \quad i = 1, 2. \tag{23}$$

It is easy to obtain

$$\left.\frac{\partial F_j^{(1)}}{\partial q^1}\right|_{(p^1,p^2,q^2)=0} = 2\Lambda^{j-1} q^1, \qquad \left.\frac{\partial F_j^{(2)}}{\partial q^1}\right|_{(p^1,p^2,q^2)=0} = 0, \quad 1 \leq j \leq N. \tag{24}$$

Then we have

$$\sum_{j=1}^{N} b_j^{(1)} \lambda_k^{j-1} = 0, \quad 1 \leq k \leq N, \tag{25}$$

which gives rise to $b_j^{(1)} = 0$, $1 \leq j \leq N$, by utilizing the fact that the Vandermonde determinant is not zero. Therefore, (23) is reduced to

$$\sum_{j=1}^{N} b_j^{(2)} \frac{\partial F_j^{(2)}}{\partial q^i} = 0, \quad i = 1, 2. \tag{26}$$

Take $P_0 \in \mathbb{R}^{4N}$ with the coordinates $q^2 = p^1 = 0$, $q^1 = O(\varepsilon)$, and $p^2 = O(\varepsilon)$, where ε is a small real number. Then, at P_0,

$$\begin{aligned} \left.\frac{\partial F_j^{(1)}}{\partial q^1}\right|_{P_0} &= \langle \Lambda^j q^1, p^2 \rangle p^2 + \langle q^1, p^2 \rangle \Lambda^j p^2 \\ &\quad - 2 \left\langle q^1, p^2 \right\rangle \left\langle \Lambda^{j-1} p^2, p^2 \right\rangle p^2 \\ &= \left\langle \Lambda^j q^1, p^2 \right\rangle p^2 + \left\langle q^1, p^2 \right\rangle \Lambda^j p^2 + O\left(\varepsilon^5\right), \end{aligned} \tag{27}$$

and the determinant of the coefficients of the linear system of equations

$$\sum_{j=1}^{N} b_j^{(2)} \frac{\partial F_j^{(2)}}{\partial q_k^1} = 0, \quad 1 \leq k \leq N, \tag{28}$$

is

$$A = \begin{vmatrix} \left\langle \Lambda q^1, p^2 \right\rangle p_1^2 + \left\langle q^1, p^2 \right\rangle \lambda_1 p_1^2 & \left\langle \Lambda^2 q^1, p^2 \right\rangle p_1^2 + \left\langle q^1, p^2 \right\rangle \lambda_1^2 p_1^2 & \cdots & \left\langle \Lambda^N q^1, p^2 \right\rangle p_1^2 + \left\langle q^1, p^2 \right\rangle \lambda_1^N p_1^2 \\ \left\langle \Lambda q^1, p^2 \right\rangle p_2^2 + \left\langle q^1, p^2 \right\rangle \lambda_2 p_2^2 & \left\langle \Lambda^2 q^1, p^2 \right\rangle p_2^2 + \left\langle q^1, p^2 \right\rangle \lambda_2^2 p_2^2 & \cdots & \left\langle \Lambda^N q^1, p^2 \right\rangle p_2^2 + \left\langle q^1, p^2 \right\rangle \lambda_2^N p_2^2 \\ \vdots & \vdots & \ddots & \vdots \\ \left\langle \Lambda q^1, p^2 \right\rangle p_N^2 + \left\langle q^1, p^2 \right\rangle \lambda_N p_N^2 & \left\langle \Lambda^2 q^1, p^2 \right\rangle p_N^2 + \left\langle q^1, p^2 \right\rangle \lambda_N^2 p_N^2 & \cdots & \left\langle \Lambda^N q^1, p^2 \right\rangle p_N^2 + \left\langle q^1, p^2 \right\rangle \lambda_N^N p_N^2 \end{vmatrix} + O\left(\varepsilon^{5N}\right)$$

$$= \prod_{j=1}^{N} p_j^2 \begin{vmatrix} \left\langle \Lambda q^1, p^2 \right\rangle + \left\langle q^1, p^2 \right\rangle \lambda_1 & \left\langle \Lambda^2 q^1, p^2 \right\rangle + \left\langle q^1, p^2 \right\rangle \lambda_1^2 & \cdots & \left\langle \Lambda^N q^1, p^2 \right\rangle + \left\langle q^1, p^2 \right\rangle \lambda_1^N \\ \left\langle q^1, p^2 \right\rangle (\lambda_2 - \lambda_1) & \left\langle q^1, p^2 \right\rangle (\lambda_2^2 - \lambda_1^2) & \cdots & \left\langle q^1, p^2 \right\rangle (\lambda_2^N - \lambda_1^N) \\ \vdots & \vdots & \ddots & \vdots \\ \left\langle q^1, p^2 \right\rangle (\lambda_N - \lambda_1) & \left\langle q^1, p^2 \right\rangle (\lambda_N^2 - \lambda_1^2) & \cdots & \left\langle q^1, p^2 \right\rangle (\lambda_N^N - \lambda_1^N) \end{vmatrix} + O\left(\varepsilon^{5N}\right)$$

$$= \prod_{j=1}^{N} p_j^2 \left(\left\langle q^1, p^2 \right\rangle^{N-1} \begin{vmatrix} \left\langle \Lambda q^1, p^2 \right\rangle & \left\langle \Lambda^2 q^1, p^2 \right\rangle & \cdots & \left\langle \Lambda^N q^1, p^2 \right\rangle \\ \lambda_2 - \lambda_1 & \lambda_2^2 - \lambda_1^2 & \cdots & \lambda_2^N - \lambda_1^N \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_N - \lambda_1 & \lambda_N^2 - \lambda_1^2 & \cdots & \lambda_N^N - \lambda_1^N \end{vmatrix} + \left\langle q^1, p^2 \right\rangle^N \begin{vmatrix} \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^N \\ \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^N \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_N & \lambda_N^2 & \cdots & \lambda_N^N \end{vmatrix} \right) + O\left(\varepsilon^{5N}\right), \tag{29}$$

where

$$\begin{vmatrix} \left\langle \Lambda q^1, p^2 \right\rangle & \left\langle \Lambda^2 q^1, p^2 \right\rangle & \cdots & \left\langle \Lambda^N q^1, p^2 \right\rangle \\ \lambda_2 - \lambda_1 & \lambda_2^2 - \lambda_1^2 & \cdots & \lambda_2^N - \lambda_1^N \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_N - \lambda_1 & \lambda_N^2 - \lambda_1^2 & \cdots & \lambda_N^N - \lambda_1^N \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^N \\ 0 & \left\langle \Lambda q^1, p^2 \right\rangle & \left\langle \Lambda^2 q^1, p^2 \right\rangle & \cdots & \left\langle \Lambda^N q^1, p^2 \right\rangle \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_N & \lambda_N^2 & \cdots & \lambda_N^N \end{vmatrix} \tag{30}$$

$$= \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^N \\ -\left\langle q^1, p^2 \right\rangle & 0 & 0 & \cdots & 0 \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_N & \lambda_N^2 & \cdots & \lambda_N^N \end{vmatrix}$$

$$= \left\langle q^1, p^2 \right\rangle \prod_{\substack{i,j=1 \\ i>j}}^{N} \left(\lambda_i - \lambda_j\right).$$

Therefore,

$$A = 2\left\langle q^1, p^2 \right\rangle^N \left(\prod_{j=1}^{N} p_j^2 \right) \left(\prod_{\substack{i,j=1 \\ i>j}}^{N} \left(\lambda_i - \lambda_j\right) \right) + O\left(\varepsilon^{5N}\right) \neq 0. \tag{31}$$

Then we obtain $b_j^{(2)} = 0$, $1 \leq j \leq N$. The proof is complete. □

Combining Propositions 1 and 2, we have immediately the following conclusions.

Proposition 3. *The symplectic map of the Bargmann type defined by* (10) *is completely integrable in the Liouville sense.*

Proposition 4. *The systems defined as follows are completely integrable in the Liouville sense:*

$$\frac{\partial q^i}{\partial t} = \frac{\partial F_m^{(1)}}{\partial p^i}, \quad \frac{\partial p^i}{\partial t} = -\frac{\partial F_m^{(1)}}{\partial q^i}, \quad m \geq 1, \ i = 1, 2. \tag{32}$$

4. The Representation of Solutions

Consider the following initial value problem:

$$\begin{aligned} q_t^i &= \frac{\partial H_1}{\partial p^i}, \qquad p_t^i = -\frac{\partial H_1}{\partial q^i}, \\ \left(q^i, p^i\right)\Big|_{t=0} &= \left(q^i(0), p^i(0)\right), \\ & \qquad i = 1, 2, \end{aligned} \tag{33}$$

where $H_1 = -(1/2)F_1^{(1)}$. In fact, the first two equations in (33) are

$$\begin{aligned} q_t^1 &= \frac{1}{2}\left(\Lambda - \left\langle p^1, p^1 \right\rangle\right) q^1 - \left\langle q^1, p^1 \right\rangle p^1 - \left\langle q^1, p^2 \right\rangle p^2, \\ q_t^2 &= -\left\langle p^1, p^2 \right\rangle q^1 + \frac{1}{2}\left(\Lambda - \left\langle p^2, p^2 \right\rangle\right) q^2 \\ &\quad - \left\langle q^1, p^2 \right\rangle p^1 - \left\langle q^2, p^2 \right\rangle p^2, \\ p_t^1 &= q^1 + \frac{1}{2}\left(\left\langle p^1, p^1 \right\rangle - \Lambda\right) p^1 + \left\langle p^1, p^2 \right\rangle p^2, \\ p_t^2 &= q^2 + \frac{1}{2}\left(\left\langle p^2, p^2 \right\rangle - \Lambda\right) p^2. \end{aligned} \tag{34}$$

Then we can obtain the presentation of solutions for the lattice equation (4).

Proposition 5. *Let $q^i(t)$ and $p^i(t)$ $(1 \leq i \leq 3)$ be a solution of* (33)*; define*

$$\begin{aligned} &\left(q^1(n,t), q^2(n,t), p^1(n,t), p^2(n,t)\right) \\ &\quad = H^n\left(q^1(t), q^2(t), p^1(t), p^2(t)\right). \end{aligned} \tag{35}$$

Then

$$a_n = \left\langle p^1, p^2 \right\rangle, \qquad b_n = \left\langle p^2, p^2 \right\rangle,$$

$$c_n = \left(-\left\langle p^1, p^2 \right\rangle^2 - \left\langle q^2, p^2 \right\rangle + \left\langle \Lambda p^2, p^2 \right\rangle - \left\langle p^2, p^2 \right\rangle^2\right)^{1/2}, \tag{36}$$

and solve the lattice equation (4).

Proof. It is easy to see that (35) is equivalent to (12), that is, (10) with $(q^i(0,t), p^i(0,t)) = (q^i(t), p^i(t))$. Using (33), (36), and (10), a direct calculation shows that

$$\begin{aligned}
a_{n,t} &= \frac{1}{2}\left\langle p^1, p^2\right\rangle\left(\left\langle p^1, p^1\right\rangle + \left\langle p^2, p^2\right\rangle\right) + \left\langle p^2, p^2\right\rangle\left\langle p^1, p^2\right\rangle \\
&\quad + \left\langle q^1, p^2\right\rangle + \left\langle q^2, p^1\right\rangle - \left\langle \Lambda p^1, p^2\right\rangle \\
&= \frac{1}{2}\left(E^{-1}c_n - c_n E\right)\left(2\langle p^1, p^2\rangle\right) + \frac{1}{2}a_n(1-E)\left\langle p^2, p^2\right\rangle, \\
b_{n,t} &= \left\langle p^2, p^2\right\rangle\left\langle p^2, p^2\right\rangle + 2\left\langle \Lambda q^2, p^2\right\rangle - \left\langle p^2, p^2\right\rangle \\
&= \frac{1}{2}\left(E^{-1}-1\right)a_n\left(2\left\langle p^1, p^2\right\rangle\right) + \left(E^{-2}-1\right) \\
&\quad \times\left(-a_n\left\langle p^1, p^2\right\rangle - \left\langle q^2, p^2\right\rangle + \left\langle \Lambda p^2, p^2\right\rangle - b_n\left\langle p^2, p^2\right\rangle\right), \\
c_{n,t} &= \frac{1}{2}\left(-\left\langle p^1, p^2\right\rangle^2 - \left\langle q^2, p^2\right\rangle + \left\langle \Lambda p^2, p^2\right\rangle - \left\langle p^2, p^2\right\rangle^2\right)^{-1/2} \\
&\quad \times\Big[-\left\langle p^2, p^2\right\rangle\left(\left\langle p^1, p^2\right\rangle^2 + \left\langle q^2, p^2\right\rangle - \left\langle \Lambda p^2, p^2\right\rangle\right. \\
&\qquad \left. + 2\left\langle p^2, p^2\right\rangle^2\right) - \left\langle p^1, p^2\right\rangle \\
&\qquad \times\left(\left\langle p^1, p^2\right\rangle\left\langle p^1, p^1\right\rangle + 2\left\langle p^2, p^2\right\rangle\left\langle p^1, p^2\right\rangle\right. \\
&\qquad \left. - 2\left\langle \Lambda p^1, p^2\right\rangle + 2\left\langle q^2, p^1\right\rangle\right) - 2\left\langle p^2, p^2\right\rangle \\
&\qquad \times\left(\left\langle \Lambda q^2, p^2\right\rangle - \left\langle \Lambda p^2, p^2\right\rangle\right) \\
&\qquad - \left\langle q^2, q^2\right\rangle + 2\left\langle \Lambda q^2, p^2\right\rangle - \left\langle \Lambda^2 p^2, p^2\right\rangle\Big] \\
&= \frac{1}{2}c_n\left(1-E^2\right)\left\langle p^2, p^2\right\rangle.
\end{aligned} \tag{37}$$

Therefore, we have

$$\frac{\partial}{\partial t}(a_n, b_n, c_n)^{\mathrm{T}} = J_n\sum_{j=1}^{N}\nabla\lambda_j = J_n G_n^{(0)}, \tag{38}$$

where

$$J_n = \begin{pmatrix} \frac{1}{2}\left(E^{-1}c_n - c_n E\right) & \frac{1}{2}a_n(1-E) & 0 \\ \frac{1}{2}\left(E^{-1}-1\right)a_n & 0 & \frac{1}{2}\left(E^{-2}-1\right)c_n \\ 0 & \frac{1}{2}c_n\left(1-E^2\right) & 0 \end{pmatrix}. \tag{39}$$

Then (38) is equivalent to the generalization of Toda lattice equation (4). This proves Proposition 5. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant no. 11301487), a Foundation for the Author of National Excellent Doctoral Dissertation of China (no. 201313), the Research Foundation of Henan University of Technology (Grant no. 2009BS041), and Soft Science Research Project of the Science and Technology Department of Henan Province (no. 142400410274).

References

[1] M. J. Ablowitz and J. F. Ladik, "Nonlinear differential-difference equations," *Journal of Mathematical Physics*, vol. 16, pp. 598–603, 1975.

[2] K. Daikoku, Y. Mizushima, and T. Tamama, "Computer experiments on new lattice solitons propagating in Volterra's system," *Japanese Journal of Applied Physics*, vol. 14, no. 3, pp. 367–376, 1975.

[3] M. Wadati, K. Konno, and Y. H. Ichikawa, "A generalization of inverse scattering method," *Journal of the Physical Society of Japan*, vol. 46, no. 6, pp. 1965–1966, 1979.

[4] M. Toda, "Waves in nonlinear lattice," *Progress of Theoretical Physics*, vol. 45, supplement, pp. 174–200, 1970.

[5] X. Geng and C. Cao, "Quasi-periodic solutions of the 2 + 1 dimensional modified Korteweg-de Vries equation," *Physics Letters A*, vol. 261, no. 5-6, pp. 289–296, 1999.

[6] X. Geng, H. H. Dai, and C. W. Cao, "Algebro-geometric constructions of the discrete Ablowitz-Ladik flows and applications," *Journal of Mathematical Physics*, vol. 44, no. 10, pp. 4573–4588, 2003.

[7] X. Geng and T. Su, "Decomposition and straightening out of the discrete Kaup-Newell flows," *International Journal of Modern Physics B: Condensed Matter Physics, Statistical Physics, Applied Physics*, vol. 25, no. 32, pp. 4513–4531, 2011.

[8] M. Toda, *Theory of Nonlinear Lattices*, Springer, Berlin, Germany, 1982.

[9] M. J. Ablowitz and J. F. Ladik, "A nonlinear difference scheme and inverse scattering," vol. 55, no. 3, pp. 213–229, 1976.

[10] S. J. Orfanidis, "Sine-Gordon equation and nonlinear σ model on a lattice," *Physical Review D: Particles and Fields*, vol. 18, no. 10, pp. 3828–3832, 1978.

[11] S. V. Manakov, "Complete integrability and stochastization of discrete dynamical systems," *Journal of Experimental and Theoretical Physics*, vol. 40, no. 2, pp. 269–274, 1975.

[12] M. J. Ablowitz and J. F. Ladik, "On the solution of a class of nonlinear partial difference equations," vol. 57, no. 1, pp. 1–12, 1977.

[13] X. Geng, F. Li, and B. Xue, "A generalization of Toda lattices and their bi-Hamiltonian structures," *Modern Physics Letters B: Condensed Matter Physics, Statistical Physics, Applied Physics*, vol. 26, no. 13, Article ID 1250078, p. 7, 2012.

[14] W.-X. Ma, "A soliton hierarchy associated with (3, $\mathbb{R}$)," *Applied Mathematics and Computation*, vol. 220, pp. 117–122, 2013.

[15] C. W. Cao, "Nonlinearization of the Lax system for AKNS hierarchy," *Science in China A*, vol. 33, no. 5, pp. 528–536, 1990.

[16] X. G. Geng, "Finite-dimensional discrete systems and integrable systems through nonlinearization of the discrete eigenvalue problem," *Journal of Mathematical Physics*, vol. 34, no. 2, pp. 805–817, 1993.

[17] C. W. Cao and X. G. Geng, "C. Neumann and Bargmann systems associated with the coupled KdV soliton hierarchy," *Journal of Physics A: Mathematical and General*, vol. 23, no. 18, pp. 4117–4125, 1990.

[18] X. Geng and H. H. Dai, "Nonlinearization of the Lax pairs for discrete Ablowitz-Ladik hierarchy," *Journal of Mathematical Analysis and Applications*, vol. 327, no. 2, pp. 829–853, 2007.

[19] D. Du, "Complex form, reduction and Lie-Poisson structure for the nonlinearized spectral problem of the Heisenberg hierarchy," *Physica A: Statistical Mechanics and its Applications*, vol. 303, no. 3-4, pp. 439–456, 2002.

[20] D. Du and C. Cao, "The Lie-Poisson representation of the nonlinearized eigenvalue problem of the Kac-van Moerbeke hierarchy," *Physics Letters A*, vol. 278, no. 4, pp. 209–224, 2001.

[21] W. X. Ma and W. Strampp, "An explicit symmetry constraint for the Lax pairs and the adjoint Lax pairs of AKNS systems," *Physics Letters A*, vol. 185, no. 3, pp. 277–286, 1994.

[22] X. G. Geng, "The Hamiltonian structure and new finite-dimensional integrable system associated with Harry-Dym type equations," *Physics Letters A*, vol. 194, no. 1-2, pp. 44–48, 1994.

[23] Y. Wu and X. Geng, "A new integrable symplectic map associated with lattice soliton equations," *Journal of Mathematical Physics*, vol. 37, no. 5, pp. 2338–2345, 1996.

[24] B. Konopelchenko, J. Sidorenko, and W. Strampp, "(1 + 1)-dimensional integrable systems as symmetry constraints of (2 + 1)-dimensional systems," *Physics Letters A*, vol. 157, no. 1, pp. 17–21, 1991.

[25] W.-X. Ma and Z. Zhou, "Binary symmetry constraints of N-wave interaction equations in 1+1 and 2+1 dimensions," *Journal of Mathematical Physics*, vol. 42, no. 9, pp. 4345–4382, 2001.

[26] W.-X. Ma and R. Zhou, "Adjoint symmetry constraints leading to binary nonlinearization," *Journal of Nonlinear Mathematical Physics*, vol. 9, supplement 1, pp. 106–126, 2002.

[27] W.-X. Ma and X. Geng, "Bäcklund transformations of soliton systems from symmetry constraints," in *CRM Proceedings and Lecture Notes*, vol. 29, pp. 313–323, 2001.