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Research Article

Fixed Point Theory in α-Complete Metric Spaces with Applications

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The aim of this paper is to introduce new concepts of α-η-complete metric space and α-η-continuous function and establish fixed point results for modified α-η-ψ-rational contraction mappings in α-η-complete metric spaces. As an application, we derive some Suzuki type fixed point theorems and new fixed point theorems for ψ-graphic-rational contractions. Moreover, some examples and an application to integral equations are given here to illustrate the usability of the obtained results.

This paper is dedicated to Professor Miodrag Mateljević on the occasion of his 65th birthday

1. Preliminaries

We know by the Banach contraction principle [1], which is a classical and powerful tool in nonlinear analysis, that a self-mapping f on a complete metric space (X, d) such that $d(fx, fy) \leq c\, d(x, y)$ for all $x, y \in X$, where $c \in [0, 1)$, has a unique fixed point. Since then, the Banach contraction principle has been generalized in several directions (see [2–26] and references cited therein).

In 2008, Suzuki [21] proved the following result that is an interesting generalization of the Banach contraction principle which also characterizes the metric completeness.

Theorem 1. *Let (X, d) be a complete metric space and let T be a self-mapping on X. Define a nonincreasing function $\theta : [0, 1) \to (1/2, 1]$ by*

$$\theta(r) = \begin{cases} 1, & \text{if } 0 \leq r \leq \dfrac{\left(\sqrt{5}-1\right)}{2}, \\ (1-r)\, r^{-2}, & \text{if } \dfrac{\left(\sqrt{5}-1\right)}{2} < r < 2^{-1/2}, \\ (1+r)^{-1}, & \text{if } 2^{-1/2} \leq r < 1. \end{cases} \tag{1}$$

Assume that there exists $r \in [0, 1)$ such that

$$\theta(r)\, d(x, Tx) \leq d(x, y) \quad \text{implies } d(Tx, Ty) \leq r d(x, y) \tag{2}$$

for all $x, y \in X$. Then there exists a unique fixed point z of T. Moreover, $\lim_{n \to +\infty} T^n x = z$ for all $x \in X$.

In 2012, Samet et al. [19] introduced the concepts of α-ψ-contractive and α-admissible mappings and established various fixed point theorems for such mappings defined on complete metric spaces. Afterwards Salimi et al. [16] and Hussain et al. [7] modified the notions of α-ψ-contractive and α-admissible mappings and established fixed point theorems which are proper generalizations of the recent results in [12, 19].

Definition 2 (see [19]). Let T be a self-mapping on X and let $\alpha : X \times X \to [0, +\infty)$ be a function. One says that T is an α-admissible mapping if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1. \tag{3}$$

Definition 3 (see [16]). Let T be a self-mapping on X and let $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. One says that T is an α-admissible mapping with respect to η if

$$x, y \in X, \quad \alpha(x, y) \geq \eta(x, y) \Longrightarrow \alpha(Tx, Ty) \geq \eta(Tx, Ty). \tag{4}$$

Note that if we take $\eta(x, y) = 1$, then this definition reduces to Definition 2. Also, if we take $\alpha(x, y) = 1$, then we say that T is an η-subadmissible mapping.

Here we introduce the notions of α-η-complete metric space and α-η-continuous function and establish fixed point results for modified α-η-ψ-rational contractions in α-η-complete metric spaces which are not necessarily complete. As an application, we derive some Suzuki type fixed point theorems and new fixed point theorems for ψ-graphic-rational contractions. Moreover, some examples and an application to integral equations are given here to illustrate the usability of the obtained results.

2. Main Results

First, we introduce the notions of α-η-complete metric space and α-η-continuous function.

Definition 4. Let (X, d) be a metric space and $\alpha, \eta : X \times X \to [0, +\infty)$. The metric space X is said to be α-η-complete if and only if every Cauchy sequence $\{x_n\}$ with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ converges in X. One says X is an α-complete metric space when $\eta(x, y) = 1$ for all $x, y \in X$ and one says (X, d) is an η-complete metric space when $\alpha(x, y) = 1$ for all $x, y \in X$.

Example 5. Let $X = (0, \infty)$ and $d(x, y) = |x - y|$ be a metric function on X. Let A be a closed subset of X. Define $\alpha, \eta : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} (x + y)^2, & \text{if } x, y \in A, \\ 0, & \text{otherwise}, \end{cases} \qquad \eta(x, y) = 2xy. \tag{5}$$

Clearly, (X, d) is not a complete metric space, but (X, d) is an α-η-complete metric space. Indeed, if $\{x_n\}$ is a Cauchy sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, then $x_n \in A$ for all $n \in \mathbb{N}$. Now, since (A, d) is a complete metric space, then there exists $x^* \in A$ such that $x_n \to x^*$ as $n \to \infty$.

Remark 6. Let $T : X \to X$ be a self-mapping on metric space X and let X be an orbitally T-complete. Define $\alpha, \eta : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 3, & \text{if } x, y \in O(w), \\ 0, & \text{otherwise}, \end{cases} \qquad \eta(x, y) = 1, \tag{6}$$

where $O(w)$ is an orbit of a point $w \in X$. Then (X, d) is an α-η-complete metric space. Indeed, if $\{x_n\}$ be a Cauchy sequence, where $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, then $\{x_n\} \subseteq O(w)$. Now, since X is an orbitally T-complete metric space, then $\{x_n\}$ converges in X. That is, (X, d) is an α-η-complete metric space. Also, suppose that $\alpha(x, y) \geq \eta(x, y)$; then $x, y \in O(w)$. Hence, $Tx, Ty \in O(w)$. That is, $\alpha(Tx, Ty) \geq \eta(Tx, Ty)$. Thus, T is an α-admissible mapping with respect to η.

Definition 7. Let (X, d) be a metric space. Let $\alpha, \eta : X \times X \to [0, \infty)$ and $T : X \to X$. One says T is an α-η-continuous mapping on (X, d), if for given $x \in X$ and sequence $\{x_n\}$ with

$$x_n \longrightarrow x, \quad \text{as } n \longrightarrow \infty, \qquad \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}), \quad \forall n \in \mathbb{N} \Longrightarrow Tx_n \longrightarrow Tx. \tag{7}$$

Example 8. Let $X = [0, \infty)$ and $d(x, y) = |x - y|$ be a metric on X. Assume that $T : X \to X$ and $\alpha, \eta : X \times X \to [0, +\infty)$ be defined by

$$Tx = \begin{cases} x^5, & \text{if } x \in [0, 1], \\ \sin \pi x + 2, & \text{if } (1, \infty), \end{cases} \qquad \alpha(x, y) = \begin{cases} x^2 + y^2 + 1, & \text{if } x, y \in [0, 1], \\ 0, & \text{otherwise}, \end{cases} \qquad \eta(x, y) = x^2. \tag{8}$$

Clearly, T is not continuous, but T is α-η-continuous on (X, d). Indeed, if $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$, then $x_n \in [0, 1]$ and so $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_n^5 = x^5 = Tx$.

Remark 9. Define (X, d) and $\alpha, \eta : X \times X \to [0, +\infty)$ as in Remark 6. Let $T : X \to X$ be a an orbitally continuous map on (X, d). Then T is α-η-continuous on (X, d). Indeed if $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, so $x_n \in O(w)$ for all $n \in \mathbb{N}$, then there exists sequence $(k_i)_{i\in\mathbb{N}}$ of positive integer such that $x_n = T^{k_i} w \to x$ as $i \to \infty$. Now since T is an orbitally continuous map on (X, d), then $Tx_n = T(T^{k_i} w) \to Tx$ as $i \to \infty$ as required.

A function $\psi : [0, \infty) \to [0, \infty)$ is called Bianchini-Grandolfi gauge function [13, 14, 27] if the following conditions hold:

(i) ψ is nondecreasing;

(ii) there exist $k_0 \in \mathbb{N}$ and $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that

$$\psi^{k+1}(t) \leq a\psi^k(t) + v_k, \tag{9}$$

for $k \geq k_0$ and any $t \in \mathbb{R}^+$.

In some sources, Bianchini-Grandolfi gauge function is known as (c)—comparison function (see e.g., [2]). We denote by Ψ the family of Bianchini-Grandolfi gauge functions. The following lemma illustrates the properties of these functions.

Lemma 10 (see [2]). *If $\psi \in \Psi$, then the following hold:*

(i) $(\psi^n(t))_{n\in\mathbb{N}}$ *converges to* 0 *as* $n \to \infty$ *for all* $t \in \mathbb{R}^+$;

(ii) $\psi(t) < t$, *for any* $t \in (0,\infty)$;

(iii) ψ *is continuous at* 0;

(iv) *the series* $\sum_{k=1}^{\infty} \psi^k(t)$ *converges for any* $t \in \mathbb{R}^+$.

Definition 11. Let (X,d) be a metric space and let T be a self-mapping on X. Let

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,Tx)}{1+d(x,Tx)}, \frac{d(y,Ty)}{1+d(y,Ty)}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}. \tag{10}$$

Then,

(a) we say T is a modified α-η-ψ-rational contraction mapping if

$$x, y \in X, \quad \eta(x,Tx) \le \alpha(x,y) \Longrightarrow d(Tx,Ty) \le \psi(M(x,y)), \tag{11}$$

where $\psi \in \Psi$;

(b) we say T is a modified α-ψ-rational contraction mapping if

$$x, y \in X, \quad \alpha(x,y) \ge 1 \Longrightarrow d(Tx,Ty) \le \psi(M(x,y)), \tag{12}$$

where $\psi \in \Psi$.

The following is our first main result of this section.

Theorem 12. *Let (X,d) be a metric space and let T be a self-mapping on X. Also, suppose that $\alpha, \eta : X \times X \to [0,\infty)$ are two functions and $\psi \in \Psi$. Assume that the following assertions hold true:*

(i) *(X,d) is an α-η-complete metric space;*

(ii) *T is an α-admissible mapping with respect to η;*

(iii) *T is modified α-η-ψ-rational contraction mapping on X;*

(iv) *T is an α-η-continuous mapping on X;*

(v) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge \eta(x_0,Tx_0)$.*

Then T has a fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0,Tx_0) \ge \eta(x_0,Tx_0)$. Define a sequence $\{x_n\}$ in X by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $x = x_n$ is a fixed point for T and the result is proved. Hence, we suppose that $x_{n+1} \ne x_n$ for all $n \in \mathbb{N}$. Since T is α-admissible mapping with respect to η and $\alpha(x_0,Tx_0) \ge \eta(x_0,Tx_0)$, we deduce that $\alpha(x_1,x_2) = \alpha(Tx_0,T^2x_0) \ge \eta(Tx_0,T^2x_0) = \eta(x_1,x_2)$. Continuing this process, we get

$$\alpha(x_n,x_{n+1}) \ge \eta(x_n,x_{n+1}) = \eta(x_n,Tx_n) \tag{13}$$

for all $n \in \mathbb{N} \cup \{0\}$. Now, by (a) we get

$$d(x_n,x_{n+1}) = d(Tx_{n-1},Tx_n) \le \psi(M(x_{n-1},x_n)), \tag{14}$$

where

$$\begin{aligned} M(x_{n-1},x_n) &= \max\left\{d(x_{n-1},x_n), \frac{d(x_{n-1},Tx_{n-1})}{1+d(x_{n-1},Tx_{n-1})}, \frac{d(x_n,Tx_n)}{1+d(x_n,Tx_n)}, \frac{d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})}{2}\right\} \\ &= \max\left\{d(x_{n-1},x_n), \frac{d(x_{n-1},x_n)}{1+d(x_{n-1},x_n)}, \frac{d(x_n,x_{n+1})}{1+d(x_n,x_{n+1})}, \frac{d(x_{n-1},x_{n+1})}{2}\right\} \\ &\le \max\left\{d(x_{n-1},x_n), d(x_n,x_{n+1}), \frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}\right\} \\ &= \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\} \end{aligned} \tag{15}$$

and so, $M(x_{n-1},x_n) \le \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}$. Now since ψ is nondecreasing, so from (14), we have

$$d(x_n,x_{n+1}) \le \psi(\max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}). \tag{16}$$

Now, if $\max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\} = d(x_n,x_{n+1})$ for some $n \in \mathbb{N}$, then

$$\begin{aligned} d(x_n,x_{n+1}) &\le \psi(\max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}) \\ &= \psi(d(x_n,x_{n+1})) < d(x_n,x_{n+1}) \end{aligned} \tag{17}$$

which is a contradiction. Hence, for all $n \in \mathbb{N}$ we have

$$d(x_n,x_{n+1}) \le \psi(d(x_{n-1},x_n)). \tag{18}$$

By induction, we have

$$d(x_n,x_{n+1}) \le \psi^n(d(x_0,x_1)). \tag{19}$$

Fix $\epsilon > 0$; there exists $N \in \mathbb{N}$ such that

$$\sum_{n\ge N} \psi^n(d(x_0,x_1)) < \epsilon. \tag{20}$$

Let $m, n \in \mathbb{N}$ with $m > n \ge N$. Then by triangular inequality we get

$$d(x_n,x_m) \le \sum_{k=n}^{m-1} d(x_k,x_{k+1}) \le \sum_{n\ge N} \psi^n(d(x_0,x_1)) < \epsilon. \tag{21}$$

Consequently $\lim_{m,n,\to+\infty} d(x_n, x_m) = 0$. Hence $\{x_n\}$ is a Cauchy sequence. On the other hand from (13) we know that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Now since X is an α-η-complete metric space, there is $z \in X$ such that $x_n \to z$ as $n \to \infty$. Also, since T is an α-η-continuous mapping, so $x_{n+1} = Tx_n \to Tz$ as $n \to \infty$. That is, $z = Tz$ as required. □

Example 13. Let $X = (-\infty, -2) \cup [-1, 1] \cup (2, +\infty)$. We endow X with the metric

$$d(x, y) = \begin{cases} \max\{|x|, |y|\}, & \text{if } x \neq y, \\ 0, & x = y. \end{cases} \tag{22}$$

Define $T : X \to X$, $\alpha, \eta : X \times X \to [0, \infty)$, and $\psi : [0, \infty) \to [0, \infty)$ by

$$Tx = \begin{cases} \sqrt{2x^2 - 1}, & \text{if } x \in (-\infty, -3], \\ x^3 - 1, & \text{if } x \in (-3, -2), \\ \frac{1}{4}x^2, & \text{if } x \in [-1, 0], \\ \frac{1}{4}x, & \text{if } x \in (0, 1], \\ 5 + \sin \pi x, & \text{if } x \in (2, 4), \\ 3x^3 + \ln x + 1, & \text{if } x \in [4, \infty), \end{cases} \tag{23}$$

$$\alpha(x, y) = \begin{cases} x^2 + y^2 + 1, & \text{if } x, y \in [-1, 1], \\ x^2, & \text{otherwise}, \end{cases}$$

$$\eta(x, y) = x^2 + y^2,$$

$$\psi(t) = \frac{1}{2}t.$$

Clearly, (X, d) is not a complete metric space. However, it is an α-η-complete metric space. In fact, if $\{x_n\}$ is a Cauchy sequence such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, then $\{x_n\} \subseteq [-1, 1]$ for all $n \in \mathbb{N}$. Now, since $([-1, 1], d)$ is a complete metric space, then the sequence $\{x_n\}$ converges in $[-1, 1] \subseteq X$. Let $\alpha(x, y) \geq \eta(x, y)$; then $x, y \in [-1, 1]$. On the other hand, $Tw \in [-1, 1]$ for all $w \in [-1, 1]$. Then, $\alpha(Tx, Ty) \geq \eta(Tx, Ty)$. That is, T is an α-admissible mapping with respect to η. Let $\{x_n\}$ be a sequence, such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_{n+1}, x_n) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Then, $\{x_n\} \subseteq [-1, 1]$ for all $n \in \mathbb{N}$. So, $\{Tx_n\} \subseteq [-1, 1]$ (since $Tw \in [-1, 1]$ for all $w \in [-1, 1]$). Now, since T is continuous on $[-1, 1]$. Then, $Tx_n \to Tx$ as $n \to \infty$. That is, T is an α-η-continuous mapping. Clearly, $\alpha(0, T0) \geq \eta(0, T0)$. Let $\alpha(x, y) \geq \eta(x, Tx)$. Now, if $x \notin [-1, 1]$ or $y \notin [-1, 1]$, then $x^2 \geq x^2 + y^2 + 1$ which implies $y^2 + 1 \leq 0$ which is a contradiction. Then, $x, y \in [-1, 1]$. Now we consider the following cases:

(i) let $x, y \in [-1, 0)$ with $x \neq y$; then,

$$\begin{aligned} d(Tx, Ty) &= \frac{1}{4}\max\{x^2, y^2\} \\ &\leq \frac{1}{2}\max\{|x|, |y|\} = \psi(d(x, y)) \leq \psi(M(x, y)); \end{aligned} \tag{24}$$

(ii) let $x, y \in (0, 1]$ with $x \neq y$; then

$$\begin{aligned} d(Tx, Ty) &= \frac{1}{4}\max\{|x|, |y|\} \\ &\leq \frac{1}{2}\max\{|x|, |y|\} = \psi(d(x, y)) \leq \psi(M(x, y)); \end{aligned} \tag{25}$$

(iii) let $x \in (-1, 0)$ and $y \in (0, 1)$; then

$$\begin{aligned} d(Tx, Ty) &= \frac{1}{4}\max\{x^2, y\} \\ &\leq \frac{1}{2}\max\{|x|, |y|\} = \psi(d(x, y)) \leq \psi(M(x, y)) \end{aligned} \tag{26}$$

(iv) let $x = y \in [-1, 0)$, $x = y \in (0, 1]$ or let $x = -1$, $y = 1$; then, $Tx = Ty$. That is,

$$d(Tx, Ty) = 0 \leq \psi(M(x, y)). \tag{27}$$

Thus T is a modified α-η-ψ-rational contraction mapping. Hence all conditions of Theorem 12 are satisfied and T has a fixed point. Here, $x = 0$ is fixed point of T.

By taking $\eta(x, y) = 1$ for all $x, y \in X$ in Theorem 12, we obtain the following corollary.

Corollary 14. *Let (X, d) be a metric space and let T be a self-mapping on X. Also, suppose that $\alpha : X \times X \to [0, \infty)$ is a function and $\psi \in \Psi$. Assume that the following assertions hold true:*

(i) *(X, d) is an α-complete metric space;*

(ii) *T is an α-admissible mapping;*

(iii) *T is a modified α-ψ-rational contraction on X;*

(iv) *T is an α-continuous mapping on X;*

(v) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.*

Then T has a fixed point.

Theorem 15. *Let (X, d) be a metric space and let T be a self-mapping on X. Also, suppose that $\alpha, \eta : X \times X \to [0, \infty)$ are two functions and $\psi \in \Psi$. Assume that the following assertions hold true:*

(i) *(X, d) is an α-η-complete metric space;*

(ii) *T is an α-admissible mapping with respect to η;*

(iii) *T is a modified α-η-ψ-rational contraction on X;*

(iv) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;

(v) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ *with* $x_n \to x$ *as* $n \to \infty$, *then either*

$$\begin{aligned} &\eta\left(Tx_n, T^2x_n\right) \leq \alpha\left(Tx_n, x\right) \\ &\text{or } \eta\left(T^2x_n, T^3x_n\right) \leq \alpha\left(T^2x_n, x\right) \end{aligned} \tag{28}$$

holds for all $n \in \mathbb{N}$.

Then T *has a fixed point.*

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. Define a sequence $\{x_n\}$ in X by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. Now as in the proof of Theorem 12 we have $\alpha(x_{n+1}, x_n) \geq \eta(x_{n+1}, x_n)$ for all $n \in \mathbb{N}$ and there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Let $d(z, Tz) \neq 0$. From (v) either

$$\begin{aligned} &\eta\left(Tx_{n-1}, T^2x_{n-1}\right) \leq \alpha\left(Tx_{n-1}, z\right) \\ &\text{or } \eta\left(T^2x_{n-1}, T^3x_{n-1}\right) \leq \alpha\left(T^2x_{n-1}, z\right) \end{aligned} \tag{29}$$

holds for all $n \in \mathbb{N}$. Then,

$$\begin{aligned} &\eta\left(x_n, x_{n+1}\right) \leq \alpha\left(x_n, z\right) \\ &\text{or} \quad \eta\left(x_{n+1}, x_{n+2}\right) \leq \alpha\left(x_{n+1}, z\right) \end{aligned} \tag{30}$$

holds for all $n \in \mathbb{N}$. Let $\eta(x_n, x_{n+1}) \leq \alpha(x_n, z)$ hold for all $n \in \mathbb{N}$. Now from (a) we get

$$\begin{aligned} &d\left(x_{n_k+1}, Tz\right) \\ &= d\left(Tx_{n_k}, Tz\right) \\ &\leq \psi\left(\max\left\{d\left(x_{n_k}, z\right), \frac{d\left(x_{n_k}, Tx_{n_k}\right)}{1 + d\left(x_{n_k}, Tx_{n_k}\right)}, \right.\right. \\ &\qquad \left.\left. \frac{d(z, Tz)}{1 + d(z, Tz)}, \frac{d\left(x_{n_k}, Tz\right) + d\left(z, Tx_{n_k}\right)}{2}\right\}\right) \\ &= \psi\left(\max\left\{d\left(x_{n_k}, z\right), \frac{d\left(x_{n_k}, x_{n_k+1}\right)}{1 + d\left(x_{n_k}, x_{n_k+1}\right)}, \right.\right. \\ &\qquad \left.\left. \frac{d(z, Tz)}{1 + d(z, Tz)}, \frac{d\left(x_{n_k}, Tz\right) + d\left(z, x_{n_k+1}\right)}{2}\right\}\right) \\ &< \max\left\{d\left(x_{n_k}, z\right), \frac{d\left(x_{n_k}, x_{n_k+1}\right)}{1 + d\left(x_{n_k}, x_{n_k+1}\right)}, \right. \\ &\qquad \left. \frac{d(z, Tz)}{1 + d(z, Tz)}, \frac{d\left(x_{n_k}, Tz\right) + d\left(z, x_{n_k+1}\right)}{2}\right\}. \end{aligned} \tag{31}$$

By taking limit as $k \to \infty$ in the above inequality we get

$$d(z, Tz) \leq \max\left\{\frac{d(z, Tz)}{1 + d(z, Tz)}, \frac{d(z, Tz)}{2}\right\} < d(z, Tz) \tag{32}$$

which is a contradiction. Hence, $d(z, Tz) = 0$ implies $z = Tz$. By the similar method we can show that $z = Tz$ if $\eta(x_{n+1}, x_{n+2}) \leq \alpha(x_{n+1}, z)$ holds for all $n \in \mathbb{N}$. □

Example 16. Let $X = (0, +\infty)$. We endow X with usual metric. Define $T : X \to X$, $\alpha, \eta : X \times X \to [0, \infty)$, and $\psi : [0, \infty) \to [0, \infty)$ by

$$\begin{aligned} Tx &= \begin{cases} \dfrac{\sqrt{x^2+1}}{\sin x + \cos x + 3}, & \text{if } x \in (0, 1), \\ \dfrac{1}{16}x^2 + 1, & \text{if } x \in [1, 2], \\ \dfrac{x^3+1}{\sqrt{x^2+1}}, & \text{if } x \in (2, \infty), \end{cases} \\ \alpha(x, y) &= \begin{cases} \dfrac{1}{2}, & \text{if } x, y \in [1, 2], \\ 0, & \text{otherwise}, \end{cases} \\ \eta(x, y) &= \frac{1}{4}, \qquad \psi(t) = \frac{1}{4}t. \end{aligned} \tag{33}$$

Note that (X, d) is not a complete metric space. But it is an α-η-complete metric space. Indeed, if $\{x_n\}$ is a Cauchy sequence such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, then $\{x_n\} \subseteq [1, 2]$ for all $n \in \mathbb{N}$. Now, since $([1, 2], d)$ is a complete metric space, then the sequence $\{x_n\}$ converges in $[1, 2] \subseteq X$. Let $\alpha(x, y) \geq \eta(x, y)$; then $x, y \in [1, 2]$. On the other hand, $Tw \in [1, 2]$ for all $w \in [1, 2]$. Then, $\alpha(Tx, Ty) \geq \eta(Tx, Ty)$. That is, T is an α-admissible mapping with respect to η. If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to \infty$. Then, $Tx_n, T^2x_n, T^3x_n \in [1, 2]$ for all $n \in \mathbb{N}$. That is,

$$\begin{aligned} &\eta\left(Tx_n, T^2x_n\right) \leq \alpha\left(Tx_n, x\right), \\ &\eta\left(T^2x_n, T^3x_n\right) \leq \alpha\left(T^2x_n, x\right), \end{aligned} \tag{34}$$

holds for all $n \in \mathbb{N}$. Clearly, $\alpha(0, T0) \geq \eta(0, T0)$. Let, $\alpha(x, y) \geq \eta(x, Tx)$. Now, if $x \notin [1, 2]$ or $y \notin [1, 2]$, then $0 \geq 1/4$, which is a contradiction. So, $x, y \in [1, 2]$. Therefore,

$$\begin{aligned} d(Tx, Ty) &= \frac{1}{16}\left|x^2 - y^2\right| \\ &= \frac{1}{16}|x - y||x + y| \leq \frac{1}{4}|x - y| \\ &= \frac{1}{4}d(x, y) \leq \frac{1}{4}M(x, y) = \psi(M(x, y)). \end{aligned} \tag{35}$$

Therefore T is a modified α-η-ψ-rational contraction mapping. Hence all conditions of Theorem 15 hold and T has a fixed point. Here, $x = 8 - 2\sqrt{14}$ is a fixed point of T.

If in Theorem 15 we take $\eta(x, y) = 1$ for all $x, y \in X$, then we obtain the following result.

Corollary 17. *Let* (X, d) *be a metric space and let* T *be a self-mapping on* X. *Also, suppose that* $\alpha : X \times X \to [0, \infty)$ *is a function and* $\psi \in \Psi$. *Assume that the following assertions hold true:*

(i) (X, d) *is a* α*-complete metric space;*

(ii) T *is an* α*-admissible mapping;*

(iii) T *is a modified* α-ψ*-rational contraction mapping on* X*;*

(iv) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$*;*

(v) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *with* $x_n \to x$ *as* $n \to \infty$*, then either*

$$\alpha(Tx_n, x) \geq 1 \quad \text{or } \alpha\left(T^2 x_n, x\right) \geq 1 \tag{36}$$

holds for all $n \in \mathbb{N}$.

Then T *has a fixed point.*

Corollary 18. *Let* (X, d) *be a complete metric space and let* T *be a continuous self-mapping on* X*. Assume that* T *is a modified rational contraction mapping, that is,*

$$\forall x, y \in X, \quad d(Tx, Ty) \leq \psi(M(x, y)), \tag{37}$$

where $\psi \in \Psi$*. Then* T *has a fixed point.*

Corollary 19. *Let* (X, d) *be a complete metric space and let* T *be a continuous self-mapping on* X*. Assume that* T *satisfies the following rational inequality:*

$$\forall x, y \in X, \quad d(Tx, Ty) \leq rM(x, y), \tag{38}$$

where $0 \leq r < 1$ *and*

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx)}{1 + d(x, Tx)}, \frac{d(y, Ty)}{1 + d(y, Ty)}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}. \tag{39}$$

Then T *has a fixed point.*

3. Consequences

3.1. Suzuki Type Fixed Point Results. From Theorem 12 we deduce the following Suzuki type fixed point result.

Theorem 20. *Let* (X, d) *be a complete metric space and let* T *be a continuous self-mapping on* X*. Assume that there exists* $r \in [0, 1)$ *such that*

$$d(x, Tx) \leq d(x, y) \quad \text{implies } d(Tx, Ty) \leq rM(x, y) \tag{40}$$

for all $x, y \in X$*, where*

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx)}{1 + d(x, Tx)}, \frac{d(y, Ty)}{1 + d(y, Ty)}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}. \tag{41}$$

Then T *has a unique fixed point.*

Proof. Define $\alpha, \eta : X \times X \to [0, \infty)$ and $\psi : [0, \infty) \to [0, \infty)$ by

$$\alpha(x, y) = d(x, y), \qquad \eta(x, y) = d(x, y), \tag{42}$$

for all $x, y \in X$ and $\psi(t) = rt$, where $0 \leq r < 1$. Clearly, $\eta(x, y) \leq \alpha(x, y)$ for all $x, y \in X$. That is, conditions (i)–(v) of Theorem 12 hold true. Let $\eta(x, Tx) \leq \alpha(x, y)$. Then, $d(x, Tx) \leq d(x, y)$. Now from (40) we have $d(Tx, Ty) \leq rM(x, y) = \psi(M(x, y))$. That is, T is a modified α-η-ψ-rational contraction mapping on X. Then all conditions of Theorem 12 hold and T has a fixed point. The uniqueness of the fixed point follows easily from (40). □

Corollary 21. *Let* (X, d) *be a complete metric space and let* T *be a continuous self-mapping on* X*. Assume that there exists* $r \in [0, 1)$ *such that*

$$d(x, Tx) \leq d(x, y) \quad \text{implies } d(Tx, Ty) \leq rd(x, y) \tag{43}$$

for all $x, y \in X$*. Then* T *has a unique fixed point.*

Now, we prove the following Suzuki type fixed point theorem without continuity of T.

Theorem 22. *Let* (X, d) *be a complete metric space and let* T *be a self-mapping on* X*. Define a nonincreasing function* $\rho : [0, 1) \to (1/2, 1]$ *by*

$$\rho(r) = \frac{1}{1 + r}. \tag{44}$$

Assume that there exists $r \in [0, 1)$ *such that*

$$\rho(r) d(x, Tx) \leq d(x, y) \quad \text{implies } d(Tx, Ty) \leq rd(x, y) \tag{45}$$

for all $x, y \in X$*. Then* T *has a unique fixed point.*

Proof. Define $\alpha, \eta : X \times X \to [0, \infty)$ and $\psi : [0, \infty) \to [0, \infty)$ by

$$\alpha(x, y) = d(x, y), \qquad \eta(x, y) = \rho(r) d(x, y) \tag{46}$$

for all $x, y \in X$ and $\psi(t) = rt$, where $0 \leq r < 1$. Now, since $\rho(r)d(x, y) \leq d(x, y)$ for all $x, y \in X$, $\eta(x, y) \leq \alpha(x, y)$ for all $x, y \in X$. That is, conditions (i)–(iv) of Theorem 15 hold true. Let $\{x_n\}$ be a sequence with $x_n \to x$ as $n \to \infty$. Since $\rho(r)d(Tx_n, T^2 x_n) \leq d(Tx_n, T^2 x_n)$ for all $n \in \mathbb{N}$, then from (45) we get

$$d\left(T^2 x_n, T^3 x_n\right) \leq rd\left(Tx_n, T^2 x_n\right) \tag{47}$$

for all $n \in \mathbb{N}$.

Assume there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} \eta\left(Tx_{n_0}, T^2 x_{n_0}\right) &> \alpha\left(Tx_{n_0}, x\right), \\ \eta\left(T^2 x_{n_0}, T^3 x_{n_0}\right) &> \alpha\left(T^2 x_{n_0}, x\right); \end{aligned} \tag{48}$$

then,

$$\rho(r)\, d\left(Tx_{n_0}, T^2x_{n_0}\right) > d\left(Tx_{n_0}, x\right),$$
$$\rho(r)\, d\left(T^2x_{n_0}, T^3x_{n_0}\right) > d\left(T^2x_{n_0}, x\right), \tag{49}$$

and so by (47) we have

$$\begin{aligned} &d\left(Tx_{n_0}, T^2x_{n_0}\right) \\ &\quad \le d\left(Tx_{n_0}, x\right) + d\left(T^2x_{n_0}, x\right) \\ &\quad < \rho(r)\, d\left(Tx_{n_0}, T^2x_{n_0}\right) + \rho(r)\, d\left(T^2x_{n_0}, T^3x_{n_0}\right) \\ &\quad \le \rho(r)\, d\left(Tx_{n_0}, T^2x_{n_0}\right) + r\rho(r)\, d\left(Tx_{n_0}, T^2x_{n_0}\right) \\ &\quad = \rho(r)(1+r)\, d\left(Tx_{n_0}, T^2x_{n_0}\right) = d\left(Tx_{n_0}, T^2x_{n_0}\right) \end{aligned} \tag{50}$$

which is a contradiction. Hence, either

$$\begin{aligned} &\eta\left(Tx_n, T^2x_n\right) \le \alpha\left(Tx_n, x\right) \\ &\text{or } \eta\left(T^2x_n, T^3x_n\right) \le \alpha\left(T^2x_n, x\right) \end{aligned} \tag{51}$$

holds for all $n \in \mathbb{N}$. That is condition (v) of Theorem 15 holds.

Let, $\eta(x, Tx) \le \alpha(x, y)$. So, $\rho(r)d(x, Tx) \le d(x, y)$. Then from (45) we get $d(Tx, Ty) \le rd(x, y) \le rM(x, y) = \psi(M(x, y))$. Hence, all conditions of Theorem 15 hold and T has a fixed point. The uniqueness of the fixed point follows easily from (45). □

3.2. Fixed Point Results in Orbitally T-Complete Metric Spaces

Theorem 23. *Let (X, d) be a metric space and let $T : X \to X$ be a self-mapping on X. Suppose the following assertions hold:*

(i) *(X, d) is an orbitally T-complete metric space;*

(ii) *there exists $\psi \in \Psi$ such that*

$$d(Tx, Ty) \le \psi(M(x, y)) \tag{52}$$

holds for all $x, y \in O(w)$ for some $w \in X$, where

$$\begin{aligned} &M(x, y) \\ &\quad = \max\left\{d(x, y), \frac{d(x, Tx)}{1 + d(x, Tx)}, \right. \\ &\qquad \left. \frac{d(y, Ty)}{1 + d(y, Ty)}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}; \end{aligned} \tag{53}$$

(iii) *if $\{x_n\}$ is a sequence such that $\{x_n\} \subseteq O(w)$ with $x_n \to x$ as $n \to \infty$, then $x \in O(w)$.*

Then T has a fixed point.

Proof. Define $\alpha : X \times X \to [0, +\infty)$ as in Remark 6. From Remark 6 we know that (X, d) is an α-complete metric space and T is an α-admissible mapping. Let $\alpha(x, y) \ge 1$; then $x, y \in O(w)$. Then from (ii) we have

$$d(Tx, Ty) \le \psi(M(x, y)). \tag{54}$$

That is, T is a modified α-ψ-rational contraction mapping. Let $\{x_n\}$ be a sequence such that $\alpha(x_n, x_{n+1}) \ge 1$ with $x_n \to x$ as $n \to \infty$. So, $\{x_n\} \subseteq O(w)$. From (iii) we have $x \in O(w)$. That is, $\alpha(x_n, x) \ge 1$. Hence, all conditions of Corollary 17 hold and T has a fixed point. □

Corollary 24. *Let (X, d) be a metric space and let $T : X \to X$ be a self-mapping on X. Suppose the following assertions hold:*

(i) *(X, d) is an orbitally T-complete metric space;*

(ii) *there exists $r \in [0, 1)$ such that*

$$d(Tx, Ty) \le rM(x, y) \tag{55}$$

holds for all $x, y \in O(w)$ for some $w \in X$, where

$$\begin{aligned} M(x, y) = \max\Bigg\{&d(x, y), \frac{d(x, Tx)}{1 + d(x, Tx)}, \\ &\frac{d(y, Ty)}{1 + d(y, Ty)}, \frac{d(x, Ty) + d(y, Tx)}{2}\Bigg\}; \end{aligned} \tag{56}$$

(iii) *if $\{x_n\}$ is a sequence such that $\{x_n\} \subseteq O(w)$ with $x_n \to x$ as $n \to \infty$, then $x \in O(w)$.*

Then T has a fixed point.

3.3. Fixed Point Results for Graphic Contractions. Consistent with Jachymski [11], let (X, d) be a metric space and let Δ denote the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops; that is, $E(G) \supseteq \Delta$. We assume that G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph (see [11]) by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G, then a path in G from x to y of length $N (N \in \mathbb{N})$ is a sequence $\{x_i\}_{i=0}^{N}$ of $N + 1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{n-1}, x_n) \in E(G)$ for $i = 1, \ldots, N$. A graph G is connected if there is a path between any two vertices. G is weakly connected if $\widetilde{G}$ is connected (see for details [3, 6, 10, 11]).

Recently, some results have appeared providing sufficient conditions for a mapping to be a Picard operator if (X, d) is endowed with a graph. The first result in this direction was given by Jachymski [11].

Definition 25 (see [11]). We say that a mapping $T : X \to X$ is a Banach G-contraction or simply G-contraction if T preserves edges of G; that is,

$$\forall x, y \in X \quad ((x, y) \in E(G) \Longrightarrow (T(x), T(y)) \in E(G)) \tag{57}$$

and T decreases weights of edges of G in the following way:

$$\begin{aligned} &\exists\, \alpha \in (0, 1), \quad \forall x, y \in X \\ &((x, y) \in E(G) \Longrightarrow d(T(x), T(y)) \le \alpha d(x, y)). \end{aligned} \tag{58}$$

Definition 26 (see [11]). A mapping $T : X \to X$ is called G-continuous, if given $x \in X$ and sequence $\{x_n\}$

$$\begin{gathered} x_n \longrightarrow x, \quad \text{as } n \longrightarrow \infty, \\ (x_n, x_{n+1}) \in E(G), \quad \forall n \in \mathbb{N} \text{ implying } Tx_n \longrightarrow Tx. \end{gathered} \tag{59}$$

Theorem 27. *Let (X, d) be a metric space endowed with a graph G and let T be a self-mapping on X. Suppose that the following assertions hold:*

(i) *for all $x, y \in X$, $(x, y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G)$;*

(ii) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$;*

(iii) *there exists $\psi \in \Psi$ such that*

$$d(Tx, Ty) \le \psi(M(x, y)) \tag{60}$$

for all $(x, y) \in E(G)$, where

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx)}{1 + d(x, Tx)}, \frac{d(y, Ty)}{1 + d(y, Ty)}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}; \tag{61}$$

(iv) *T is G-continuous;*

(v) *if $\{x_n\}$ is a Cauchy sequence in X with $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then $\{x_n\}$ is convergent in X.*

Then T has a fixed point.

Proof. Define $\alpha : X^2 \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in E(G), \\ 0, & \text{otherwise.} \end{cases} \tag{62}$$

At first we prove that T is an α-admissible mapping. Let $\alpha(x, y) \ge 1$; then $(x, y) \in E(G)$. From (i), we have $(Tx, Ty) \in E(G)$. That is, $\alpha(Tx, Ty) \ge 1$. Thus T is an α-admissible mapping. Let T be G-continuous on (X, d). Then,

$$\begin{gathered} x_n \longrightarrow x, \quad \text{as } n \longrightarrow \infty, \\ (x_n, x_{n+1}) \in E(G), \quad \forall n \in \mathbb{N} \text{ implying } Tx_n \longrightarrow Tx. \end{gathered} \tag{63}$$

That is,

$$\begin{gathered} x_n \longrightarrow x, \quad \text{as } n \longrightarrow \infty, \\ \alpha(x_n, x_{n+1}) \ge 1, \quad \forall n \in \mathbb{N} \text{ implying } Tx_n \longrightarrow Tx \end{gathered} \tag{64}$$

which implies that T is α-continuous on (X, d). From (ii) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$. That is, $\alpha(x_0, Tx_0) \ge 1$. Let $\alpha(x, y) \ge 1$; then $(x, y) \in E(G)$. Now, from (iii) we have $d(Tx, Ty) \le \psi(M(x, y))$. That is,

$$\alpha(x, y) \ge 1 \Longrightarrow d(Tx, Ty) \le \psi(M(x, y)). \tag{65}$$

Condition (v) implies that (X, d) is an α-complete metric space. Hence, all conditions of Corollary 14 are satisfied and T has a fixed point. □

Theorem 28. *Let (X, d) be a complete metric space endowed with a graph G and let T be a self-mapping on X. Suppose that the following assertions hold:*

(i) *for all $x, y \in X$, $(x, y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G)$;*

(ii) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$;*

(iii) *there exists $\psi \in \Psi$ such that*

$$d(Tx, Ty) \le \psi(M(x, y)) \tag{66}$$

for all $(x, y) \in E(G)$, where

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx)}{1 + d(x, Tx)}, \frac{d(y, Ty)}{1 + d(y, Ty)}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}; \tag{67}$$

(iv) *T is G-continuous.*

Then T has a fixed point.

As an application of Corollary 17, we obtain.

Theorem 29. *Let (X, d) be a metric space endowed with a graph G and let T be a self-mapping on X. Suppose that the following assertions hold:*

(i) *for all $x, y \in X$, $(x, y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G)$;*

(ii) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$;*

(iii) *there exists $\psi \in \Psi$ such that*

$$d(Tx, Ty) \le \psi(M(x, y)) \tag{68}$$

for all $(x, y) \in E(G)$, where

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx)}{1 + d(x, Tx)}, \frac{d(y, Ty)}{1 + d(y, Ty)}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}; \tag{69}$$

(iv) *if $\{x_n\}$ is a sequence such that $(x_n, x_{n+1}) \in E(G)$ with $x_n \to x$ as $n \to \infty$, then either*

$$(Tx_n, x) \in E(G) \quad \text{or} \quad (T^2 x_n, x) \in E(G) \tag{70}$$

holds for all $n \in \mathbb{N}$;

(v) *if $\{x_n\}$ is a Cauchy sequence in X with $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then either $\{x_n\}$ is convergent in X or (X, d) is a complete metric space.*

Then T has a fixed point.

Let $(X, d, \preceq)$ be a partially ordered metric space. Define the graph G by

$$E(G) := \{(x, y) \in X \times X : x \preceq y\}. \tag{71}$$

For this graph, condition (i) in Theorem 27 means that T is nondecreasing with respect to this order [5]. From Theorems 27–29 we derive the following important results in partially ordered metric spaces.

Theorem 30. *Let $(X, d, \preceq)$ be a partially ordered metric space and let T be a self-mapping on X. Suppose that the following assertions hold:*

(i) *T is nondecreasing map;*

(ii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(iii) *there exists $\psi \in \Psi$ such that*

$$d(Tx, Ty) \leq \psi(M(x, y)) \tag{72}$$

for all $x \preceq y$, where

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx)}{1 + d(x, Tx)}, \frac{d(y, Ty)}{1 + d(y, Ty)}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}; \tag{73}$$

(iv) *either for a given $x \in X$ and sequence $\{x_n\}$*

$$\begin{gathered} x_n \longrightarrow x, \quad \text{as } n \longrightarrow \infty, \\ x_n \preceq x_{n+1}, \quad \forall n \in \mathbb{N}, \text{ one has } Tx_n \longrightarrow Tx \end{gathered} \tag{74}$$

or T is continuous;

(v) *if $\{x_n\}$ is a Cauchy sequence in X with $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$, then either $\{x_n\}$ is convergent in X or (X, d) is a complete metric space.*

Then T has a fixed point.

Corollary 31 (Ran and Reurings [15]). *Let $(X, d, \preceq)$ be a partially ordered complete metric space and let $T : X \to X$ be a continuous nondecreasing self-mapping such that $x_0 \preceq Tx_0$ for some $x_0 \in X$. Assume that*

$$d(Tx, Ty) \leq rd(x, y) \tag{75}$$

holds for all $x, y \in X$ with $x \preceq y$, where $0 \leq r < 1$. Then T has a fixed point.

Theorem 32. *Let $(X, d, \preceq)$ be a partially ordered metric space and let T be a self-mapping on X. Suppose that the following assertions hold:*

(i) *T is nondecreasing map;*

(ii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(iii) *there exists $\psi \in \Psi$ such that*

$$d(Tx, Ty) \leq \psi(M(x, y)) \tag{76}$$

for all $x \preceq y$, where

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx)}{1 + d(x, Tx)}, \frac{d(y, Ty)}{1 + d(y, Ty)}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}; \tag{77}$$

(iv) *if $\{x_n\}$ is a sequence such that $x_n \preceq x_{n+1}$ with $x_n \to x$ as $n \to \infty$, then either*

$$Tx_n \preceq x \quad \text{or } T^2 x_n \preceq x \tag{78}$$

holds for all $n \in \mathbb{N}$;

(v) *if $\{x_n\}$ is a Cauchy sequence in X with $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$, then either $\{x_n\}$ is convergent in X or (X, d) is a complete metric space.*

Then T has a fixed point.

4. Application to Existence of Solutions of Integral Equations

Fixed point theorems for monotone operators in ordered metric spaces are widely investigated and have found various applications in differential and integral equations (see [28–30] and references therein). In this section, we apply our result to the existence of a solution of an integral equation. Let $X = C([0, T], \mathbb{R})$ be the set of real continuous functions defined on $[0, T]$ and let $d : X \times X \to \mathbb{R}_+$ be defined by

$$d(x, y) = \|x - y\|_\infty \tag{79}$$

for all $x, y \in X$. Then (X, d) is a complete metric space. Also, assume this metric space endowed with a graph G.

Consider the integral equation as follows:

$$x(t) = p(t) + \int_0^T S(t, s) f(s, x(s))\, ds \tag{80}$$

and let $F : X \to X$ be defined by

$$F(x)(t) = p(t) + \int_0^T S(t, s) f(s, x(s))\, ds. \tag{81}$$

We assume that

(A) $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous;

(B) $p : [0, T] \to \mathbb{R}$ is continuous;

(C) $S : [0, T] \times \mathbb{R} \to [0, +\infty)$ is continuous;

(D) there exists a $\psi \in \Psi$ such that for all $s \in [0,T]$

$$\forall x, y \in X \quad (x,y) \in E(G) \Longrightarrow (F(x), F(y)) \in E(G),$$

$$\begin{aligned}
&\forall x, y \in X \quad (x,y) \in E(G) \Longrightarrow 0 \\
&\le f(s, x(s)) - f(s, y(s)) \\
&\le \psi\left(\max\left\{\frac{|x(s)-y(s)|}{1+|x(s)-y(s)|},\right.\right. \\
&\qquad \frac{|x(s)-F(x(s))|}{1+|x(s)-F(x(s))|}, |y(s)-F(y(s))|, \\
&\qquad \left.\left.\frac{1}{2}\left[|x(s)-F(y(s))|+|y(s)-F(x(s))|\right]\right\}\right);
\end{aligned} \tag{82}$$

(E) there exists $x_0 \in X$ such that $(x_0, F(x_0)) \in E(G)$;

(F) if $\{x_n\}$ is a sequence such that $(x_n, x_{n+1}) \in E(G)$ with $x_n \to x$ as $n \to \infty$, then either

$$(Fx_n, x) \in E(G) \quad or \quad (F^2 x_n, x) \in E(G) \tag{83}$$

holds for all $n \in \mathbb{N}$;

(G) $\int_0^T S(t,s)ds \le 1$ for all t.

Theorem 33. *Under assumptions (A)–(G), the integral equation* (80) *has a solution in* $X = C([0,T], \mathbb{R})$.

Proof. Consider the mapping $F : X \to X$ defined by (81). Let $(x,y) \in E(G)$. Then from (D) we deduce

$$\begin{aligned}
&|F(x)(t) - F(y)(t)| \\
&= \left|\int_0^T S(t,s)\left[f(s,x(s)) - f(s,y(s))\right]ds\right| \\
&\le \int_0^T S(t,s)\left|f(s,x(s)) - f(s,y(s))\right|ds \\
&\le \int_0^T S(t,s)\,\psi \\
&\quad \times\left(\max\left\{|x(s)-y(s)|,\right.\right. \\
&\qquad \frac{|x(s)-F(x(s))|}{1+|x(s)-F(x(s))|}, \frac{|y(s)-F(y(s))|}{1+|y(s)-F(y(s))|}, \\
&\qquad \left.\left.\frac{1}{2}\left[|x(s)-F(y(s))|+|y(s)-F(x(s))|\right]\right\}\right)ds \\
&\le \left(\int_0^T S(t,s)\,ds\right)\psi \\
&\quad \times\left(\max\left\{\|x(s)-y(s)\|,\right.\right. \\
&\qquad \frac{\|x(s)-F(x(s))\|}{1+\|x(s)-F(x(s))\|}, \frac{\|y(s)-F(y(s))\|}{1+\|y(s)-F(y(s))\|}, \\
&\qquad \left.\left.\frac{1}{2}\left[\|x(s)-F(y(s))\|+\|y(s)-F(x(s))\|\right]\right\}\right).
\end{aligned} \tag{84}$$

Then

$$\begin{aligned}
&\|Fx - Fy\|_\infty \\
&\le \psi\left(\max\left\{\|x(s)-y(s)\|,\right.\right. \\
&\qquad \frac{\|x(s)-F(x(s))\|}{1+\|x(s)-F(x(s))\|}, \frac{\|y(s)-F(y(s))\|}{1+\|y(s)-F(y(s))\|}, \\
&\qquad \left.\left.\frac{1}{2}\left[\|x(s)-F(y(s))\|+\|y(s)-F(x(s))\|\right]\right\}\right).
\end{aligned} \tag{85}$$

That is, $(x,y) \in E(G)$ implies

$$\begin{aligned}
&\|Fx - Fy\|_\infty \\
&\le \psi\left(\max\left\{\|x-y\|_\infty,\right.\right. \\
&\qquad \frac{\|x-F(x)\|_\infty}{1+\|x-F(x)\|_\infty}, \frac{\|y-F(y)\|_\infty}{1+\|y-F(y)\|_\infty}, \\
&\qquad \left.\left.\frac{1}{2}\left[\|x-F(y)\|_\infty+\|y-F(x)\|_\infty\right]\right\}\right).
\end{aligned} \tag{86}$$

It easily shows that all the hypotheses of Theorem 29 are satisfied and hence the mapping F has a fixed point that is a solution in $X = C([0,T], \mathbb{R})$ of the integral equation (80). □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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