Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2014, Article ID 276585, 9 pages
http://dx.doi.org/10.1155/2014/276585

Research Article

Numerical Solution of High Order Bernoulli Boundary Value Problems

F. Costabile and A. Napoli

Department of Mathematics, University of Calabria, 87036 Rende, Italy

Correspondence should be addressed to A. Napoli; anna.napoli@unical.it

Received 31 October 2013; Revised 28 May 2014; Accepted 22 June 2014; Published 10 July 2014

Academic Editor: Saeid Abbasbandy

For the numerical solution of high order boundary value problems with special boundary conditions a general procedure to determine collocation methods is derived and studied. Computation of the integrals which appear in the coefficients is generated by a recurrence formula and no integrals are involved in the calculation. Several numerical examples are presented to demonstrate the practical usefulness of the proposed method.

1. Introduction

Higher order differential equations arise in a variety of different areas of science, engineering, and technology (see [1, 2]) since they model a wide spectrum of phenomena.

Particularly, the solutions of fifth-order BVPs model viscoelastic flows [3] and the seventh-order BVPs model induction motors with two rotor circuits [4, 5]. Ordinary differential equations of sixth and eighth order arise in modeling instability when an infinite horizontal layer of fluid is heated from below and is subject to the action of rotation [6]. Moreover, high order boundary value problems arise in hydrodynamic, hydromagnetic stability [7], and other branches of applied sciences.

In [8] the authors presented a class of collocation methods for the numerical solution of high order boundary value problems:

$$y^{(n)}(x) = f(x, \mathbf{y}(x)), \quad x \in I = [a, b], \tag{1}$$

$$B[x, y] = g, \quad x \in \partial I, \tag{2}$$

where $n > 1$, $\mathbf{y}(x) = (y(x), y'(x), \dots, y^{(q)}(x))$, $0 \le q < n$, and B is a linear operator on the boundary ∂I, $g \in \mathbb{R}^n$.

The idea in [8] is the following: the differential problem (1)-(2) is written in the following equivalent integral form:

$$y(x) = P_{n-1}[y, x] + \int_a^b G_{n-1}(x, t) f(t, \mathbf{y}(t))\, dt, \tag{3}$$

where $P_{n-1}[y, x]$ is the unique polynomial which satisfies the boundary conditions

$$B[x, P_{n-1}] = g \tag{4}$$

and $G_{n-1}(x, t)$ is a kernel (Green) function. $G_{n-1}(x, t)$ is such that $B[x, G_{n-1}] = 0$ and it is differentiable under the integral sign such that (3) satisfies (1).

Thus, from (3) and (4) we obtain a collocation polynomial which approximates the solution of problem (1)-(2).

In the present work the authors use this technique to derive collocation methods for the numerical solution of (1) with the particular boundary conditions

$$y(a) = \beta_0, \qquad y^{(k)}(b) - y^{(k)}(a) = \beta_{k+1}, \quad k = 0, \dots, n-2 \tag{5}$$

with β_k, $k = 0, \dots, n-1$, being real constants.

Conditions (5) are called the *Bernoulli boundary conditions*, since they are related to the Bernoulli interpolation problem [9]. They have physical and engineering interpretation [10], but to the authors' knowledge, they have not been considered previously in the literature.

In [9, 10] the BVP (1)–(5) is considered: in [10] a nonconstructive proof of the existence and uniqueness of solution is given, and in [9] Picard's method is applied in connection with Newton's method for the numerical solution of the problem.

The present paper is organized as follows: in Section 2 we summarize some theoretical results on the existence and uniqueness of solution for problem (1)–(5). Then, in Section 3, we present the method for the numerical solution of this type of problems, which produces smooth, global approximations in the form of polynomial functions. In Section 4 we give an a priori estimation of error and, in Section 5, we present some particular cases. In Section 6 we propose an algorithm to compute the numerical solution of (1)–(5) in the nodal points and then, in Section 7, we present some numerical examples of both linear and nonlinear BVPs which confirm the theoretical results.

2. Preliminaries

Let $B_k(x)$ be the Bernoulli polynomial of degree k [11] and let us set

$$S_k(t) = B_k(t) - B_k(0). \tag{6}$$

Moreover, let

$$h = b - a, \qquad \Delta f_a^{(k)} = f^{(k)}(b) - f^{(k)}(a). \tag{7}$$

The following theorems hold.

Theorem 1 (see [12]). *Let $f \in C^{(n+1)}[a,b]$. Then*

$$f(x) = f(a) + \sum_{k=1}^{n} S_k\left(\frac{x-a}{h}\right)\frac{h^{k-1}}{k!}\Delta f_a^{(k-1)} + R_n[f,x], \tag{8}$$

where $R_n[f,x]$ is the remainder term

$$R_n[f,x] = \int_a^b G_n(x,t)\, f^{(n+1)}(t)\,dt \tag{9}$$

with $G_n(x,t)$ being Peano's kernel:

$$G_n(x,t) = \frac{1}{n!}\left[(x-t)_+^n - \sum_{k=1}^{n} S_k\left(\frac{x-a}{h}\right)\frac{h^{k-1}}{k} \times \binom{n}{k-1}(b-t)^{n-k+1}\right]. \tag{10}$$

Theorem 2 (see [12]). *If $f \in C^{(n-1)}[a,b]$, then the polynomial*

$$P_n[f,x] = f(a) + \sum_{k=1}^{n} S_k\left(\frac{x-a}{h}\right)\frac{h^{k-1}}{k!}\Delta f_a^{(k-1)} \tag{11}$$

satisfies the Bernoulli interpolation problem

$$\begin{gathered} P_n[f,a] = f(a), \\ \Delta P_n^{(k)} = P_n^{(k)}[f,b] - P_n^{(k)}[f,a] = f^{(k)}(b) - f^{(k)}(a) = \Delta f_a^{(k)}, \\ k = 0,\ldots,n-1. \end{gathered} \tag{12}$$

The proof of the existence and uniqueness of solution of (1)–(5) is based on (3) [8], under the hypothesis that the function $f(x,\mathbf{y})$ satisfies the Lipschitz condition

$$\left|f(x,\mathbf{y}_1(x)) - f(x,\mathbf{y}_2(x))\right| \le \sum_{k=0}^{q} L_k \left|y_1^{(k)}(x) - y_2^{(k)}(x)\right| \tag{13}$$

in a certain domain interval of $[a,b]\times\mathbb{R}^{q+1}$.

3. The Collocation Method

Let $y(x)$ be the solution of (1)–(5). If x_i, $i = 1,\ldots,m$, are m distinct points in $[a,b]$ and $y(x) \in C^{(n+m)}[a,b]$, using Lagrange interpolation, we get

$$y^{(n)}(x) = \sum_{i=1}^{m} l_i(x)\, y^{(n)}(x_i) + \overline{R}_m(y,x), \tag{14}$$

where

$$\overline{R}_m(y,x) = \frac{(x-x_1)\cdots(x-x_m)}{m!}\, y^{(n+m)}(\xi_x), \quad \xi_x \in (a,b) \tag{15}$$

and $l_i(t)$ are the fundamental Lagrange polynomials on the m points x_i.

Inserting (14) into (3), in view of (1), we obtain

$$\begin{aligned} y(x) = {} & P_{n-1}[y,x] \\ & + \sum_{i=1}^{m} f(x_i,\mathbf{y}(x_i)) \int_a^b G_{n-1}(x,t)\, l_i(t)\,dt \\ & + \int_a^b G_{n-1}(x,t)\,\overline{R}_m(y,t)\,dt. \end{aligned} \tag{16}$$

Hence the following identity holds:

$$y(x) = P_{n-1}[y,x] + \sum_{i=1}^{m} p_{n,i}(x)\, f(x_i,\mathbf{y}(x_i)) + T_{n,m}(y,x), \tag{17}$$

where

$$p_{n,i}(x) = \int_a^b G_{n-1}(x,t)\, l_i(t)\,dt, \quad i = 1,\ldots,m, \tag{18}$$

$$T_{n,m}(y,x) = \int_a^b G_{n-1}(x,t)\,\overline{R}_m(y,t)\,dt. \tag{19}$$

This suggests defining the polynomials

$$y_{n,m}(x) = P_{n-1}[y,x] + \sum_{i=1}^{m} p_{n,i}(x)\, f(x_i,\mathbf{y}_{n,m}(x_i)), \tag{20}$$

where $\mathbf{y}_{n,m}(x) = (y_{n,m}(x), y'_{n,m}(x),\ldots,y_{n,m}^{(q)}(x))$, $0 \le q \le n-1$.

Theorem 3. *The polynomial of degree $n+m$ implicitly defined by* (20) *satisfies the relations*

$$y_{n,m}(a) = y(a), \tag{21}$$

$$y_{n,m}^{(k)}(b) - y_{n,m}^{(k)}(a) = \beta_{k+1}, \quad k = 0,\dots,n-2, \tag{22}$$

$$y_{n,m}^{(n)}(x_i) = f(x_i, \mathbf{y}_{n,m}(x_i)), \quad i = 1,\dots,m; \tag{23}$$

that is, $y_{n,m}(x)$ is a collocation polynomial for (1)–(5) *on the nodes x_i, $i = 1,\dots,m$.*

Proof. From (18), $p_{n,i}(a) = p_{n,i}(b) = 0$, $i = 0,\dots,m$, and thus relations (21) follow from direct computation. To prove (22) we derive $G_{n-1}(x,t)$ k times, $k = 1,\dots,n-2$, with respect to x, and using the well-known relation [11] $B_s'(x) = sB_{s-1}(x)$, $s > 0$, we get

$$\frac{\partial^k}{\partial x^k} G_{n-1}(x,t) = \begin{cases} g_1(x,t) = \dfrac{(x-t)^{n-k}}{(n-k)!} - \displaystyle\sum_{j=k}^{n} B_{j-k}\left(\frac{x-a}{h}\right) \frac{h^{j-k-1}(b-t)^{n-j-1}}{(j-k)!\,(n-j-1)!} & x \geq t \\ g_2(x,t) = -\displaystyle\sum_{j=k}^{n} B_{j-k}\left(\frac{x-a}{h}\right) \frac{h^{j-k-1}(b-t)^{n-j-1}}{(j-k)!\,(n-j-1)!} & x < t. \end{cases} \tag{24}$$

From the property of Bernoulli polynomials $B_s(1) = (-1)^s B_s(0)$, we have $g_1(a,t) = g_2(b,t)$; thus

$$p_{n,i}^{(k)}(a) = \int_a^b g_1(a,t)\, l_i(t)\, dt = \int_a^b g_2(b,t)\, l_i(t)\, dt = p_{n,i}^{(k)}(b). \tag{25}$$

Hence

$$\begin{aligned} y_{n,m}^{(k)}(b) - y_{n,m}^{(k)}(a) &= y^{(k)}(b) - y^{(k)}(a) + \sum_{i=1}^{m} \left(p_{n,i}^{(k)}(b) - p_{n,i}^{(k)}(a)\right) f(x_i, \mathbf{y}_{n,m}(x_i)) \\ &= y^{(k)}(b) - y^{(k)}(a). \end{aligned} \tag{26}$$

From this, (22) follows. Next, by deriving $y_{n,m}(x)$ n times, we obtain

$$\begin{aligned} y_{n,m}^{(n)}(x) &= P_{n-1}^{(n)}[y,x] + \sum_{k=1}^{n-1} p_{n,k}^{(n)}(x)\, f(x_k, \mathbf{y}_{n,m}(x_k)) \\ &= \sum_{k=1}^{n-1} l_k(x)\, f(x_k, \mathbf{y}_{n,m}(x_k)) \end{aligned} \tag{27}$$

and this implies (23). □

4. The Error

In what follows for all $y \in C^{(q)}[a,b]$ we define the norm $\|y\| = \max_{0\leq s\leq q}\{\max_{a\leq t\leq b}|y^{(s)}(t)|\}$ [14] and the constants

$$L = \sum_{k=0}^{q} L_k, \qquad R = \max_{a\leq x\leq b} \left|\overline{R}_m(y,x)\right|. \tag{28}$$

Further, we define

$$\begin{aligned} Q_m &= \max_{0\leq s\leq q}\left\{\max_{a\leq x\leq b}\sum_{i=1}^{m}\left|p_{ni}^{(s)}(x)\right|\right\} \\ D_{n,s} &= \max_{a\leq x\leq b}\sum_{k=s}^{n-1}\frac{\left|B_{k-s}((x-a)/h)\right|}{(k-s)!\,(n-k)!}, \\ \Delta &= \max_{0\leq s\leq q}\left\{\frac{h^{n-s+1}}{(n-s+1)!} + h^{n-s-1}D_{n,s}\right\}. \end{aligned} \tag{29}$$

An a priori estimation of the global error is possible.

Theorem 4. *With the previous notations, suppose that $LQ_m < 1$. Then*

$$\|y - y_{n,m}\| \leq \frac{R\Delta}{1 - LQ_m}. \tag{30}$$

Proof. By deriving (17) and (20) we get

$$\begin{aligned} y^{(s)}(x) - y_{n,m}^{(s)}(x) &= \sum_{i=1}^{n-1} p_{ni}^{(s)}(x)\left[f(x_i, \mathbf{y}(x_i)) - f(x_i, \mathbf{y}_{n,m}(x_i))\right] \\ &\quad + \frac{\partial^s}{\partial x^s}\int_a^b G_{n-1}(x,t)\,\overline{R}_{n,m}(y,t)\,dt. \end{aligned} \tag{31}$$

Now, since

$$\begin{aligned} \frac{\partial^s}{\partial x^s}\int_a^b G_{n-1}(x,t)\,\overline{R}_{n,m}(y,t)\,dt &= \frac{1}{(n-s)!}\int_a^x (x-t)^{n-s}\overline{R}_{n,m}(y,t)\,dt \\ &\quad - \sum_{k=s}^{n-1} B_{k-s}\left(\frac{x-a}{h}\right)\frac{h^{k-s-1}}{(k-s)!\,(n-k+1)!} \\ &\quad \times \int_a^b (b-t)^{n-k+1}\overline{R}_{n,m}(y,t)\,dt, \end{aligned} \tag{32}$$

we have

$$\left|\frac{\partial^s}{\partial x^s}\int_a^b G_{n-1}(x,t)\,\overline{R}_n(y,t)\,dt\right| \leq \frac{h^{n-s+1}}{(n-s+1)!}R + h^{n-s-1}RD_{n,s}. \tag{33}$$

Thus

$$\begin{aligned}
&\left|y^{(s)}(x)-y^{(s)}_{n,m}(x)\right| \\
&\leq \sum_{i=1}^{n-1}\left|p^{(s)}_{ni}(x)\right|\sum_{k=0}^{q}L_k\left|y^{(s)}(x_i)-y^{(s)}_{n}(x_i)\right| \\
&\quad+\frac{h^{n-s+1}}{(n-s+1)!}R+h^{n-s-1}RD_{ns} \\
&\leq L\left\|y-y_n\right\|Q_m+R\Delta
\end{aligned} \tag{34}$$

and inequality (30) follows. □

5. Particular Cases

Now we present explicitly the formulas for some values of n.

For the computation of $p_{n,i}(x)$ we need $\int_a^x t^k l_i(t)dt$ and $\int_x^b t^k l_i(t)dt$. Letting

$$\begin{aligned}
F_{i1}(x)&=\int_a^x l_i(t)\,dt, \qquad M_{i1}(x)=\int_x^b l_i(t)\,dt, \\
F_{ik}(x)&=\int_a^x F_{i,k-1}(t)\,dt, \qquad M_{ik}(x)=\int_x^b M_{i,k-1}(t)\,dt, \\
&\qquad k\geq 2,
\end{aligned} \tag{35}$$

and integrating by parts k times, we obtain

$$\begin{aligned}
\int_a^x t^k l_i(t)\,dt&=\sum_{j=0}^{k}(-1)^j\frac{k!}{(k-j)!}x^{k-j}F_{i,j+1}(x), \\
\int_x^b t^k l_i(t)\,dt&=\sum_{j=0}^{k}\frac{k!}{(k-j)!}x^{k-j}M_{i,j+1}(x).
\end{aligned} \tag{36}$$

5.1. The Fifth-Order Case. Now we consider the case of the fifth-order BVP

$$\begin{aligned}
&y^{(v)}(x)=f(x,\mathbf{y}(x)), \quad x\in[0,1], \\
&y(0)=\beta_0, \\
&y^{(k)}(1)-y^{(k)}(0)=\beta_{k+1}, \quad k=0,\ldots,3.
\end{aligned} \tag{37}$$

In this case Green's function is

$$G_4(x,t)=\begin{cases}
\frac{1}{4!}\left[t^4(1-x)+t^3\left(2x^2-2x\right)\right. \\
\quad+t^2\left(-2x^3+3x^2-x\right) \\
\quad\left.+t\left(x^4-2x^3+x^2\right)\right] \\
\qquad t\leq x \\
\frac{1}{4!}\left[-t^4x+t^3\left(2x^2+2x\right)\right. \\
\quad-t^2\left(2x^3+3x^2+x\right) \\
\quad\left.+t\left(x^4+2x^3+x^2\right)-x^4\right] \\
\qquad x\leq t,
\end{cases}$$

$$\begin{aligned}
4!p_{5,i}(x)&=\left(x^4-2x^3+x^2\right)\left[F_{i2}(x)-M_{i2}(x)\right] \\
&\quad-2x\left(2x^2-3x+1\right)\left[F_{i3}(x)+M_{i3}(x)\right] \\
&\quad+12x(x-1)\left[F_{i4}(x)-M_{i4}(x)\right] \\
&\quad+24(1-x)F_{i5}(x)-24xM_{i5}(x).
\end{aligned} \tag{38}$$

Hence

$$y_{5,m}(x)=P_4[y,x]+\sum_{i=1}^{m}p_{5,i}(x)f\left(x_i,\mathbf{y}_{5,m}(x_i)\right). \tag{39}$$

By deriving (44) we get

$$\begin{aligned}
y^{(s)}_{5,m}(x)&=P^{(s)}_4[y,x]+\sum_{i=1}^{m}p^{(s)}_{5,i}(x)f\left(x_i,\mathbf{y}_{5,m}(x_i)\right) \\
&\qquad s=1,\ldots,5,
\end{aligned} \tag{40}$$

where $p^{(s)}_{5,i}(x)$ can be easily computed using the same technique as for $p_{5,i}(x)$.

5.2. The Seventh-Order Case. Consider

$$\begin{aligned}
&y^{(vii)}(x)=f(x,\mathbf{y}(x)), \quad x\in[0,1], \\
&y(0)=\beta_0, \\
&y^{(k)}(1)-y^{(k)}(0)=\beta_{k+1}, \quad k=0,\ldots,5.
\end{aligned} \tag{41}$$

In this case Green's function is

$$G_6(x,t)=\frac{1}{6!}\begin{cases}
t^6(1-x)+3t^5\left(x^2-x\right)+5t^4\left(-x^3+\frac{3}{2}x^2-\frac{x}{2}\right) \\
\quad+5t^3\left(5x^4-2x^3+x^2\right) \\
\quad+t^2\left(-3x^5+\frac{15}{2}x^4-5x^3+\frac{x}{2}\right) \\
\quad+t\left(x^6-3x^5+\frac{5}{2}x^4-\frac{x^2}{2}\right) \\
\qquad t\leq x \\
-t^6x+3t^5\left(x^2+x\right)+5t^4\left(-x^3-\frac{3}{2}x^2-\frac{x}{2}\right) \\
\quad+5t^3\left(5x^4+2x^3+x^2\right) \\
\quad+t^2\left(-3x^5+\frac{15}{2}x^4-5x^3+\frac{x}{2}\right) \\
\quad+t\left(x^6-3x^5+\frac{5}{2}x^4-\frac{x^2}{2}\right)-x^6 \\
\qquad x\leq t.
\end{cases} \tag{42}$$

Hence

$$6!p_{7,i}(x) = \left(x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{x^2}{2}\right)\left[F_{i2}(x) - M_{i2}(x)\right] - x\left(6x^4 - 15x^3 + 10x^2 - 1\right)\left[F_{i3}(x) + M_{i3}(x)\right] + 30(x-1)^2\left[F_{i4}(x) - M_{i4}(x)\right] - 60x\left(2x^2 - 3x + 1\right)\left[F_{i5}(x) + M_{i5}(x)\right] + 360x(x-1)\left[F_{i6}(x) - M_{i6}(x)\right] + 6!(1-x)F_{i7}(x) - 6!xM_{i7}(x), \quad (43)$$

$$y_{7,m}(x) = P_6[y,x] + \sum_{i=1}^{m} p_{7,i}(x) f\left(x_i, \mathbf{y}_{7,m}(x_i)\right). \quad (44)$$

5.3. Order $n = 8, 9, 10$. For $n = 8$, we have

$$7!p_{8,i}(x) = \left(\frac{x^8}{4} - x^7 + \frac{7}{6}x^6 - \frac{7}{12}x^4 + \frac{x^2}{6}\right)\left[F_{i1}(x) + M_{i1}(x)\right] + \left(x^7 - \frac{7}{2}x^6 + \frac{7}{2}x^5 - \frac{7}{6}x^3 + \frac{x}{6}\right)\left[F_{i2}(x) - M_{i2}(x)\right] - 7x\left(x^5 - 3x^4 + \frac{5}{2}x^3 - \frac{x}{2}\right)\left[F_{i3}(x) + M_{i3}(x)\right] - 7x\left(6x^4 - 15x^3 + 10x^2 - 1\right)\left[F_{i4}(x) - M_{i4}(x)\right] + 210x^2(x-1)^2\left[F_{i5}(x) + M_{i5}(x)\right] - 420x\left(2x^2 - 3x + 1\right)\left[F_{i6}(x) - M_{i6}(x)\right] + 2520x(x-1)\left[F_{i7}(x) + M_{i7}(x)\right] + 7!(1-x)F_{i8}(x) + 7!xM_{i8}(x). \quad (45)$$

For $n = 9$, we get

$$8!p_{9,i}(x) = \left(x^8 - 4x^7 + \frac{14}{3}x^6 - \frac{7}{3}x^4 + \frac{2}{3}x^2\right)\left[F_{i2}(x) - M_{i2}(x)\right] + 4x\left(-2x^6 + 7x^5 - 7x^4 + \frac{7}{3}x^2 - \frac{x}{3}\right)\left[F_{i3}(x) + M_{i3}(x)\right] + 28x^2\left(2x^4 - 6x^3 + 5x^4 - 1\right)\left[F_{i4}(x) - M_{i4}(x)\right] + 56x\left(-5x^4 + 15x^3 - 10x^2 + 1\right)\left[F_{i5}(x) + M_{i5}(x)\right] + 1680x^2(x-1)^2\left[F_{i6}(x) - M_{i6}(x)\right] + 3360x\left(-2x^2 + 3x - 1\right)\left[F_{i7}(x) + M_{i7}(x)\right] + 20160x(x-1)\left[F_{i8}(x) - M_{i8}(x)\right] + 8!(1-x)F_{i9}(x) - 8!xM_{i9}(x). \quad (46)$$

For $n = 10$, we obtain

$$9!p_{10,i}(x) = \left(\frac{x^{10}}{5} - x^9 + \frac{3}{2}x^8 - \frac{7}{5}x^6 + x^4 - \frac{3}{10}x^3\right) \times \left[F_{i1}(x) + M_{i1}(x)\right] - x\left(x^8 - \frac{9}{2}x^7 + 6x^6 - \frac{21}{5}x^4 + 2x^2 - \frac{3}{10}\right) \times \left[F_{i2}(x) - M_{i2}(x)\right] + 3x^2\left(3x^6 - 12x^5 + 14x^4 - 7x^2 + 2\right)\left[F_{i3}(x) + M_{i3}(x)\right] + 12x\left(-6x^6 + 21x^5 - 21x^4 + 7x^2 - 1\right)\left[F_{i4}(x) - M_{i4}(x)\right] + 252x^2\left(2x^4 - 6x^3 + 5x^2 - 1\right)\left[F_{i5}(x) + M_{i5}(x)\right] + 504x\left(-6x^4 + 15x^3 - 10x^2 + 1\right)\left[F_{i6}(x) - M_{i6}(x)\right] + 15120x^2(x-1)^2\left[F_{i7}(x) + M_{i7}(x)\right] + 30240x\left(-2x^2 + 3x - 1\right)\left[F_{i8}(x) - M_{i8}(x)\right] + 181440x(x-1)\left[F_{i9}(x) + M_{i9}(x)\right] + 9!(1-x)F_{i10}(x) + 9!xM_{i10}(x). \quad (47)$$

6. Algorithms

To calculate the approximate solution of problem (1)–(5) by (20) at $x \in [a,b]$, we need the values $y_j^{(k)} = y_{n,m}^{(k)}(x_j)$, $j = 1,\ldots,m$, and $k = 0,\ldots,q$. These values can be calculated by solving the following system:

$$y_i^{(k)} = P_{n-1}^{(k)}[y,x_i] + \sum_{j=1}^{m} p_{nj}^{(k)}(x_i) f\left(x_j, \mathbf{y}_j\right), \quad i = 1,\ldots,m, \quad k = 0,\ldots,q \quad (48)$$

with $\mathbf{y}_j = (y_j, y_j', \ldots, y_j^{(q)})$, $0 \le q \le n-1$.

To solve it, if we put

$$Y = \left(Y_0,\ldots,Y_q\right)^T, \qquad Y_k = \left(y_1^{(k)},\ldots,y_m^{(k)}\right),$$
$$F(Y) = \left(F_m,\ldots,F_m\right)^T \in \mathbb{R}^{q+1},$$
$$F_m = \left(f_1,\ldots,f_m\right), \qquad f_i = f\left(x_i,\mathbf{y}_i\right),$$
$$C = \left(B_0,\ldots,B_q\right)^T,$$
$$B_k = \left(P_m^{(k)}[y,x_1],\ldots,P_{n-1}^{(k)}[y,x_m]\right),$$
$$A = \begin{pmatrix} A_0 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & A_q \end{pmatrix}, \qquad A_k = \begin{pmatrix} p_{1,1}^{(k)} & \cdots & p_{1,m}^{(k)} \\ \vdots & & \vdots \\ p_{m,1}^{(k)} & \cdots & p_{m,m}^{(k)} \end{pmatrix} \quad (49)$$

with $p_{i,j}^{(k)} = p_{n,j}^{(k)}(x_i)$, $k = 0, \dots, q$, we write (48) as

$$Y - AF(Y) = C \tag{50}$$

or, equivalently, $Y = G(Y)$, where

$$G(Z) = AF(Z) + C. \tag{51}$$

For the existence and the uniqueness of the solution of (50) the following result holds.

Proposition 5. *Let L be defined as in* (28). *If $T = L\|A\|_\infty < 1$, the system* (50) *has a unique solution which can be calculated by an iterative method*

$$Y^{(\nu+1)} = G\left(Y^{(\nu)}\right), \quad \nu = 1, 2, \dots \tag{52}$$

with a fixed $Y^{(0)} \in \mathbb{R}^s$, $s = m(q+1)$, and G defined as in (51). *Moreover, if Y is the exact solution of the system,*

$$\left\|Y^{(\nu)} - Y\right\|_\infty \le \frac{T^\nu}{1-T}\left\|Y^{(1)} - Y^{(0)}\right\|_\infty. \tag{53}$$

Proof. If $V = (V_0, \dots, V_q)^T$, $V_k = (v_1^{(k)}, \dots, v_m^{(k)})$ and $W = (W_0, \dots, W_q)^T$, $W_k = (w_1^{(k)}, \dots, w_m^{(k)})$, then $\|G(V) - G(W)\|_\infty \le \|A\|_\infty L\|V - W\|_\infty$; hence G is contractive. Thus the result follows from the well-known contraction mapping theorem. □

To calculate the elements $A_0, \dots, A_q$ of the matrix A we need the values $F_{is}(x_j)$ and $M_{is}(x_j)$, $s = 1, \dots, n$, $i, j = 1, \dots m$, where $F_{is}(x)$ and $M_{is}(x)$ are defined in (35).

Since $l_i(t) = \prod_{k=1, k\ne i}^{m}((t - x_k)/(x_i - x_k))$, it suffices to compute

$$\int_c^{x_j = t_k}\int_c^{t_{k-1}} \cdots \int_c^{t_1} r_{m,i}(t)\, dt\, dt_1 \cdots dt_{k-1}, \tag{54}$$

where $c = a$ or $c = b$,$r_{0,0}(t) = 1$, and

$$r_{m,i}(t) = (t - x_1)\cdots(t - x_{i-1})(t - x_{i+1})\cdots(t - x_m) \qquad i = 1, 2, \dots, m. \tag{55}$$

Let us define

$$g_{0,1,c}^{(i)}(x) = x - c \tag{56}$$

and, for $s = 1, \dots, m-1$,

$$g_{s,j,c}^{(i)}(x) = \int_c^{x = t_j}\int_c^{t_{j-1}} \cdots \int_c^{t_1} \left(t - z_1^{(i)}\right)\left(t - z_2^{(i)}\right) \cdots \left(t - z_s^{(i)}\right) dt\, dt_1 \cdots dt_{j-1}, \tag{57}$$

where

$$z_j^{(i)} = \begin{cases} x_j & \text{if } j < i \\ x_{j+1} & \text{if } j \ge i \end{cases} \quad j = 1, \dots, m-1. \tag{58}$$

We can easily compute

$$g_{0,j,c}^{(i)}(x) = \frac{(x-c)^j}{j!}. \tag{59}$$

For the computation of (57) we use the recursive algorithm [15]

$$g_{s,j,c}^{(i)}(x) = \left(x - z_s^{(i)}\right) g_{s-1,j,c}^{(i)}(x) - j g_{s-1,j+1,c}^{(i)}(x). \tag{60}$$

Thus, if $W_i = \prod_{k=1, k\ne i}^{m}(x_i - x_k)$, we get

$$F_{ik}\left(x_j\right) = \frac{g_{m-1,k,a}^{(i)}\left(x_j\right)}{W_i}, \qquad M_{ik}\left(x_j\right) = (-1)^k \frac{g_{m-1,k,b}^{(i)}\left(x_j\right)}{W_i}. \tag{61}$$

7. Numerical Examples

Now we present some numerical results obtained by applying method (20) to find numerical approximations of the solutions of some test problems. As the true solutions are known, we considered the error function $E(x) = |y(x) - y_{n,m}(x)|$. To solve the nonlinear system (48) we used the so-called modified Newton method [16] (the same Jacobian matrix is used for more than one iteration) and algorithm (60) for the computation of the entries of the matrix. Equidistant points are used as nodal points. Analogous results are obtained in the considered examples by using as nodes the zeros of Chebyshev polynomials of first and second kind.

Example 1. Consider the following

$$\begin{aligned} & y^{(\nu)}(x) = y(x) - (15 + 10x)e^x, \quad x \in [0,1], \\ & y(0) = 0, \qquad y(1) = 0, \\ & y'(1) - y'(0) = -(e+1), \qquad y''(1) - y''(0) = -4e, \\ & y'''(1) - y'''(0) = 3 - 9e, \end{aligned} \tag{62}$$

with solution $y(x) = x(1-x)e^x$. Figure 1 shows the graph of the error function $E(x)$ for two different values of m.

Example 2. Consider the following

$$\begin{aligned} & y^{(\nu)}(x) = -24e^{-5y} + \frac{48}{1 + x^5}, \quad x \in [0,1], \\ & y(0) = 0, \qquad y(1) = \log 2, \\ & y'(1) - y'(0) = -\frac{1}{2}, \qquad y''(1) - y''(0) = \frac{3}{4}, \\ & y'''(1) - y'''(0) = -\frac{7}{4}, \end{aligned} \tag{63}$$

with solution $y(x) = \log(x+1)$. The graph of $E(x)$, for two different numbers of nodes, is plotted in Figure 2.

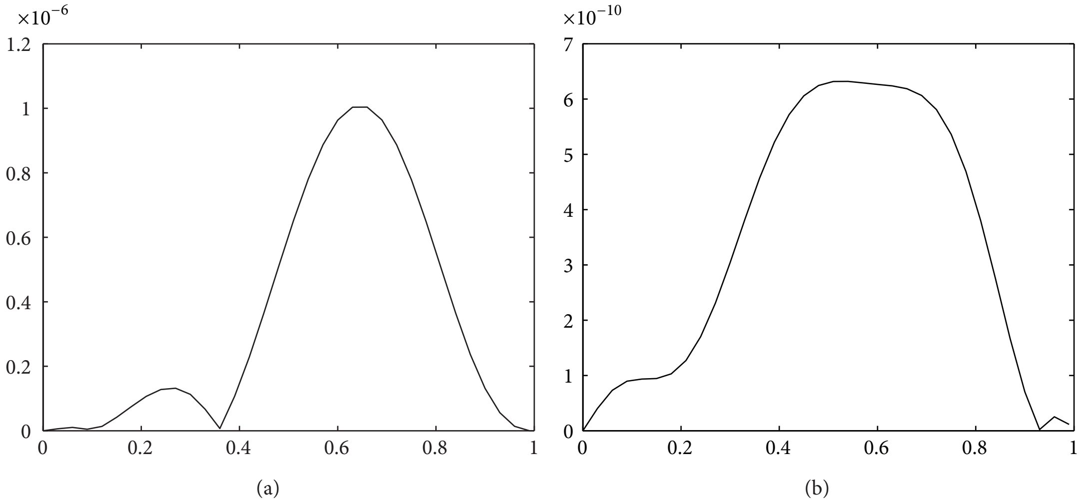


Figure 1: Error function of problem (62) for $m = 4$ (a) and for $m = 6$ (b).

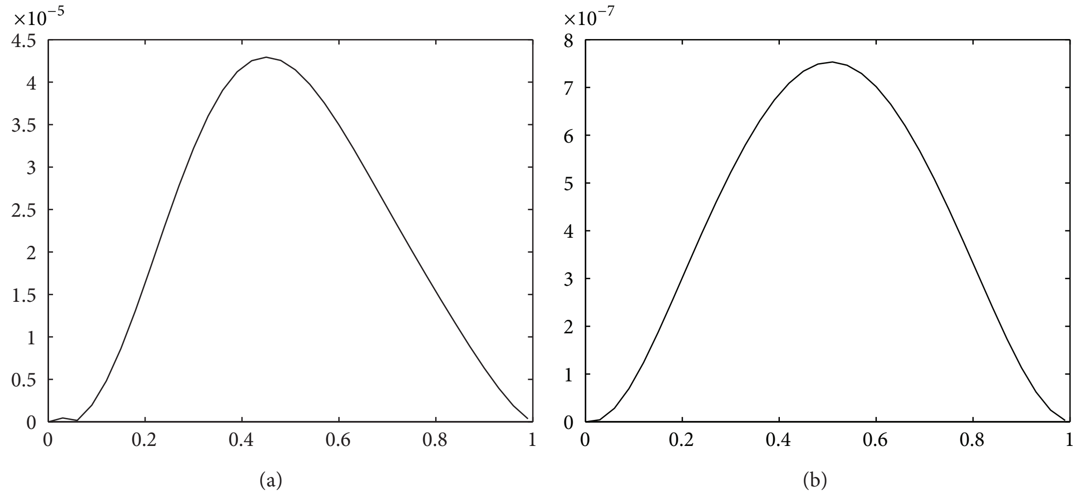


Figure 2: Error function of problem (63) for $m = 4$ (a) and for $m = 7$ (b).

Example 3. Consider

$$\begin{aligned}
&y^{(vii)}(x) = -y - e^{x}\left(2x^{2} + 12x + 35\right), \quad x \in [0,1], \\
&y(0) = y(1) = 0, \\
&y'(1) - y'(0) = -(e+1), \qquad y''(1) - y''(0) = -4e, \\
&y'''(1) - y'''(0) = 3(1-3e), \\
&y^{(iv)}(1) - y^{(iv)}(0) = 8(1-2e), \\
&y^{(v)}(1) - y^{(v)}(0) = 5(3-5e),
\end{aligned} \tag{64}$$

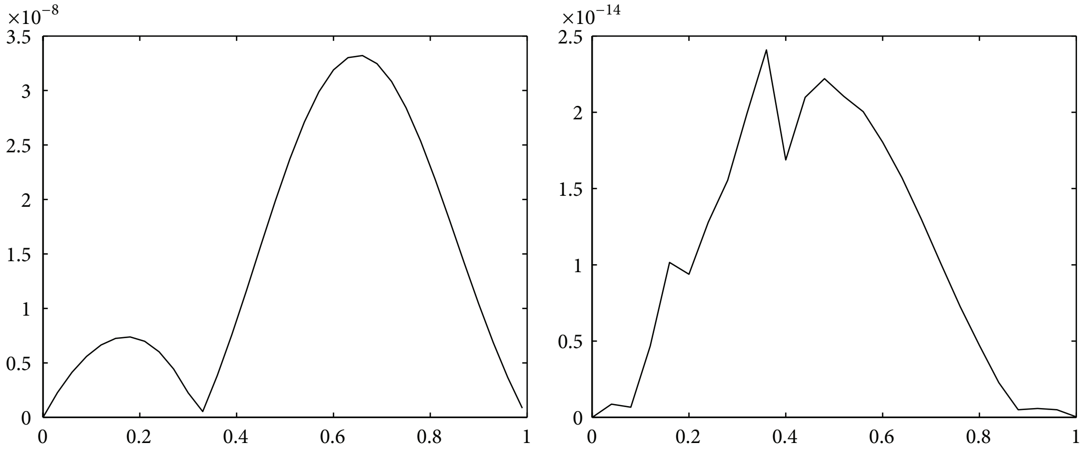
with solution $y(x) = x(1-x)e^{x}$. Figure 3 shows the graph of $E(x)$.

Note that the equation in (64) is the same as that in Example 2 of [1], but the boundary conditions are different.

Example 4. Consider

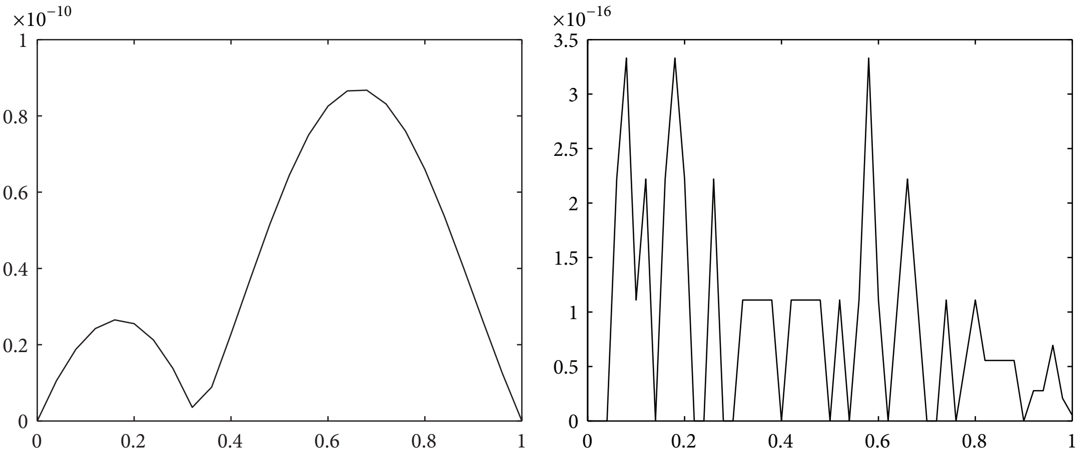
$$\begin{aligned}
&y^{(ix)}(x) = y(x) - 9e^{x}, \quad x \in [0,1], \\
&y(0) = 1, \qquad y(1) = 0, \\
&y'(1) - y'(0) = -e, \qquad y''(1) - y''(0) = 1 - 2e, \\
&y'''(1) - y'''(0) = 2 - 3e, \\
&y^{(iv)}(1) - y^{(iv)}(0) = 3 - 4e, \\
&y^{(v)}(1) - y^{(v)}(0) = 4 - 5e, \\
&y^{(vi)}(1) - y^{(vi)}(0) = 5 - 6e, \\
&y^{(vii)}(1) - y^{(vii)}(0) = 6 - 7e,
\end{aligned} \tag{65}$$

with solution $y(x) = (1-x)e^{x}$. Figure 4 shows the graph of $E(x)$.



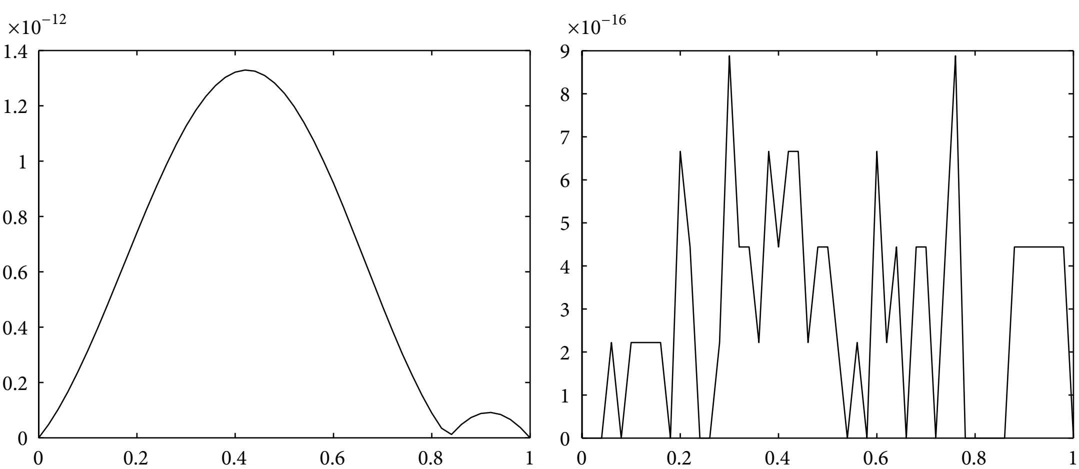
(a) (b)

Figure 3: Error function of problem (64) for $m = 4$ (a) and for $m = 8$ (b).



(a) (b)

Figure 4: Error function of problem (65) for $m = 4$ (a) and for $m = 8$ (b).



(a) (b)

Figure 5: Error function of problem (66) for $m = 3$ (a) and for $m = 6$ (b).

The equation in (65) is the same as that in [1, 2], but the boundary conditions are different.

Example 5. Consider

$$y^{(x)}(x) = e^{-x} y^2(x), \quad x \in [0,1],$$
$$y(0) = 1, \quad (66)$$
$$y^{(k)}(1) - y^{(k)}(0) = e - 1, \quad k = 0, \ldots, 8$$

with solution $y(x) = e^x$. Figure 5 shows the graph of $E(x)$.

Note that the equation in (66) is the same as that in [2], but the boundary conditions are different. The conditions in [2] are the so-called Lidstone-type conditions. Problems of this type have been analyzed in [13] using a similar technique.

8. Conclusions

This paper presents a class of collocation methods for nth order differential equations with Bernoulli boundary conditions. For two positive integers n, m a polynomial of degree $n + m$ approximating the exact solution is given explicitly. Numerical experiments support theoretical results. Further developments can be done, concerning particularly numerical estimates of the error.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] B. Hossain and S. Islam, "A novel numerical approach for odd higher order boundary value problems," *Mathematical Theory and Modeling*, vol. 4, no. 5, pp. 1–11, 2014.

[2] A. Wazwaz, "Approximate solutions to boundary value problems of higher order by the modified decomposition method," *Computers & Mathematics with Applications*, vol. 40, no. 6-7, pp. 679–691, 2000.

[3] A. Karageorghis, T. N. Phillips, and A. R. Davies, "Spectral collocation methods for the primary two-point boundary value problem in modelling viscoelastic flows," *International Journal for Numerical Methods in Engineering*, vol. 26, no. 4, pp. 805–813, 1988.

[4] G. Richards and P. R. R. Sarma, "Reduced order models for induction motors with two rotor circuits," *IEEE Transactions on Energy Conversion*, vol. 9, no. 4, pp. 673–678, 1994.

[5] S. S. Siddiqi, A. Ghazala, and I. Muzammal, "Solution of seventh order boundary value problem by differential transformation method," *World Applied Sciences Journal*, vol. 16, no. 11, pp. 1521–1526, 2012.

[6] S. S. Siddiqi and E. H. Twizell, "Spline solutions of linear twelfth-order boundary-value problems," *Journal of Computational and Applied Mathematics*, vol. 78, no. 2, pp. 371–390, 1997.

[7] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, Dover, New York, NY, USA, 1981.

[8] F. Costabile and A. Napoli, "A class of collocation methods for high order boundary value problems," submitted.

[9] F. Costabile and A. Serpe, "On Bernoulli boundary value problems," *Le Matematiche*, vol. 62, no. 2, pp. 163–173, 2007.

[10] F. Costabile, A. Serpe, and A. Bruzio, "No classic boundary conditions," in *Proceedings of the World Congress on Engineering*, L. Gelman, D. W. L. Hukins, A. Hunter, and A. Korsunsky, Eds., pp. 918–921, London, UK, 2007.

[11] C. Jordan, *Calculus of Finite Differences*, Chelsea, New York, NY, USA, 3rd edition, 1965.

[12] F. Costabile, "Expansion of real functions in Bernoulli polynomials and applications," in *Conferences Seminars Mathematics*, vol. 273, pp. 1–13, Univesty of Bari, 1999.

[13] F. Costabile and A. Napoli, "Collocation for high-order differential equations with Lidstone boundary conditions," *Journal of Applied Mathematics*, vol. 2012, Article ID 120792, 20 pages, 2012.

[14] R. P. Agarwal, *Boundary Value Problems for Higher Order Differential Equations*, World Scientific, Singapore, 1986.

[15] F. Costabile and A. Napoli, "A class of collocation methods for numerical integration of initial value problems," *Computers & Mathematics with Applications*, vol. 62, no. 8, pp. 3221–3235, 2011.

[16] A. Quarteroni, R. Sacco, and F. Saleri, *Numerical Mathematics*, Springer, New York, NY, USA, 2000.