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*Research Article*

# High Order Fefferman-Phong Type Inequalities in Carnot Groups and Regularity for Degenerate Elliptic Operators plus a Potential

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Let $\{X_1, X_2, \ldots, X_m\}$ be the basis of space of horizontal vector fields in a Carnot group $\mathbb{G} = (\mathbb{R}^n; \circ)$ $(m < n)$. We prove high order Fefferman-Phong type inequalities in $\mathbb{G}$. As applications, we derive a priori $L^p(\mathbb{G})$ estimates for the nondivergence degenerate elliptic operators $L = -\sum_{i,j=1}^m a_{ij}(x)X_iX_j + V(x)$ with *VMO* coefficients and a potential $V$ belonging to an appropriate Stummel type class introduced in this paper. Some of our results are also new even for the usual Euclidean space.

## 1. Introduction and the Main Results

The classical $L^p$ estimates for nondivergence elliptic operators with potentials of the form

$$\mathfrak{L}u \equiv -\sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j} + Vu, \quad x \in \mathbb{R}^n, \tag{1}$$

have been extensively investigated and many results have been proved; see [1–5] and so forth. In particular, when $(a_{ij})_{n\times n} = I$, the identity matrix, and $V$ belongs to the reverse Hölder class $B_q$ $(n/2 \le q < \infty)$, Shen [2] established $L^p$ $(1 < p \le q)$ boundedness for the Schrödinger operator $-\Delta + V$ and showed that the range of $p$ is optimal. It is noted that $V \in B_q$ $(q > 1)$ means that $V \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$, $V \ge 0$, and there exists a positive constant $c$ such that the reverse Hölder inequality

$$\left(|B|^{-1}\int_B V^q(x)\,dx\right)^{1/q} \le c\left(|B|^{-1}\int_B V(x)\,dx\right) \tag{2}$$

holds for every ball $B$ in $\mathbb{R}^n$. More recently, when $V \in B_q$ $(n/2 \le q < \infty)$, a priori $L^p(\mathbb{R}^n)$ $(1 < p \le q)$ estimate for $\mathfrak{L}$ in (1) with *VMO* coefficients has been deduced by Bramanti et al. [1] by using the representation formula for $Vu$ in terms of $\mathfrak{L}u$, which generalized the result in [2]. The aim of this paper is to establish high order Fefferman-Phong type inequalities in Carnot groups and prove $L^p$ regularity of degenerate elliptic operators plus a potential.

Let $X_1, X_2, \ldots, X_m$ be horizontal vector fields in a Carnot group $\mathbb{G} = (\mathbb{R}^n; \circ)$, $(m < n)$ (see Section 2.1). In this paper we consider the nondivergence degenerate elliptic operator of the kind

$$Lu \equiv Au + Vu \equiv -\sum_{i,j=1}^{m} a_{ij}(x)\, X_iX_ju + Vu, \tag{3}$$

where the leading coefficient $a_{ij}(x)$ satisfies $a_{ij}(x) = a_{ji}(x) \in L^\infty(\mathbb{G})$ for $i, j = 1, \ldots, m$, and there exists a constant $\mu > 0$ such that, for any $x \in \mathbb{G}$ and $\xi \in \mathbb{R}^m$,

$$\mu|\xi|^2 \le \sum_{i,j=1}^{m} a_{ij}(x)\,\xi_i\xi_j \le \mu^{-1}|\xi|^2; \tag{4}$$

furthermore, we assume

$$a_{ij}(x) \in VMO(\mathbb{G}), \tag{5}$$

which shows that, for $i, j = 1, \ldots, m$,

$$\eta_{ij} = \sup_{\rho \le r} \sup_{x \in \mathbb{G}} \left( \left| B_\rho (x) \right|^{-1} \int_{B_\rho(x)} \left| a_{ij}(y) - a_{ij}^B \right| dy \right) \longrightno 0, \quad r \longrightarrow 0^+. \tag{6}$$

Here $a_{ij}^B = |B_\rho(x)|^{-1} \int_{B_\rho(x)} a_{ij}(y) dy$. $B_\rho(x)$ denotes a metric ball of radius $r$ and center $x$ associated with the Carnot-Carathéodory distance $d$ (see Section 2) by

$$B_\rho(x) = B(x, \rho) = \{ y \in \mathbb{G} : d(x, y) < \rho \}. \tag{7}$$

As to the potential $V$, inspired by Di Fazio and Zamboni [6, Definition 2.4], we introduce the following Stummel type class $S_p$.

*Definition 1* (Stummel type class). Let $V : \mathbb{G} \to \mathbb{R}$, $1 < p < \infty$, $r > 0$. One says that $V \in S_p(\mathbb{G})$, if for every $r > 0$,

$$\begin{aligned} \varphi_V(r) := \sup_{x \in \mathbb{G}} \Bigg( \int_{d(x,y)<r} \frac{d(x,y)^2}{|B(x, d(x,y))|} & \\ \times \Bigg( \int_{d(z,x)<r} |V(z)| & \\ \times \frac{d(z,y)^2}{|B(z, d(z,y))|} dz \Bigg)^{1/(p-1)} & dy \Bigg)^{p-1} \end{aligned} \tag{8}$$

is finite and

$$\lim_{r \to 0} \varphi_V(r) = 0, \tag{9}$$

where $d(\cdot,\cdot)$ is the Carnot-Carathéodory distance; see Section 2.

Sometimes we will call $\varphi_V(r)$ the Stummel modulus of $V$.

*Remark 2.* We note that $L^\infty(\mathbb{G}) \subset S_p(\mathbb{G})$ $(1 < p < \infty)$ and $S_p(\mathbb{G})$ is the special case of the function class in [7, page 56] with $p = 2$ and $\mathbb{G} = \mathbb{R}^n$ $(n \ge 5)$. Also, note that the function $V(x) = d(0, x)^{-2}$ on $\mathbb{G}$ belongs to the classes $S_p(\mathbb{G})$ for $p > 2$, where $d$ is the Carnot-Carathéodory distance (see Section 2).

Nondivergence degenerate elliptic operators similar to (3) including the form $-\sum_{i=1}^m X_i^2 + V$ have been studied by some authors; see [8–11] and so forth. The local $L^p$ estimate for operator (3) with the vanishing potential $V = 0$ on the homogeneous group has been verified by Bramanti and Brandolini [12]. For the study of related operators, we refer to [13, 14] and references therein. We will prove regularity for the operator $L$ in (3) on $\mathbb{G}$ if $a_{ij}$ satisfy (4)-(5) and $V^p \in S_p(\mathbb{G})$; see Theorem 3 below. Our methods are different from the Euclidean case by Bramanti et al. [1], where estimates of integral operators and their commutators were used as a main tool.

Since Fefferman [15] proved the well-known imbedding inequality

$$\int_{\mathbb{R}^n} |V| |u|^2 dx \le c \int_{\mathbb{R}^n} |\nabla u|^2 dx, \quad u \in C_0^\infty(\mathbb{R}^n), \tag{10}$$

with $V$ belonging to the classical Morrey class $L^{r,n-2r}$, $1 < r \le n/2$, it has been extended to many more general settings and applied to infer regularity for partial differential operators; see [6, 16–19] and so forth. One of the main jobs of this paper is to establish a high order Fefferman-Phong type inequality in Carnot groups (see Theorem 4), which is motivated by Di Fazio and Zamboni [6, Theorem 3.1]. So far as we know, there is not any result in literature on high order Fefferman-Phong inequalities. Using this inequality and proving several estimates with the potential, a priori $L^p(\mathbb{G})$ estimate for $L$ is obtained.

We mention that the homogeneous dimension $Q$ of $\mathbb{G}$, the horizontal gradient $Xu$, the second order horizontal gradient $X^2u$, the horizontal Sobolev spaces $HW^{2,p}(\mathbb{G})$ and $HW_V^{2,p}(\mathbb{G})$, the polynomial $P_B(x)$, and the reverse Hölder class $B_q$ in our setting will be described in Section 2. Now we are in a position to state main results.

**Theorem 3.** *Under the assumptions (4)-(5), if $V^p \in S_p(\mathbb{G})$, $1 < p < \infty$, then there exists a positive constant $c = c(p, \mu, \eta, V, \mathbb{G})$ such that, for any $u \in C_0^\infty(\mathbb{G})$, it follows that*

$$\begin{aligned} &\|u\|_{HW^{2,p}(\mathbb{G})} + \|Vu\|_{L^p(\mathbb{G})} \\ &\quad \le c \left( \|Lu\|_{L^p(\mathbb{G})} + \|u\|_{L^p(\mathbb{G})} \right), \end{aligned} \tag{11}$$

*where $\eta$ in $c$ depends only on the VMO moduli $\eta_{ij}$ of the coefficients $a_{ij}$. Furthermore, (11) holds for $u \in HW_V^{2,p}(\mathbb{G})$.*

It is noted that the $L^p$ estimates of the operators similar to (3) with discontinuous leading coefficients and bounded lower terms were obtained by Bramanti and Brandolini [12, 20]. Here the potential $V$ in Theorem 3 may be unbounded on $\mathbb{G}$.

The key for the proof of Theorem 3 is the following high order Fefferman-Phong type inequality.

**Theorem 4.** *Let $B = B_{r_B}(x_0)$ be any metric ball in $\mathbb{G}$. If $V \in S_p(\mathbb{G})$ $(1 < p < \infty)$, then there exists a first order polynomial $P_B(x)$ such that, for any $u \in C^\infty(B)$, one has*

$$\int_B |u - P_B(x)|^p |V| dx \le c \varphi_V(2r_B) \int_B \left| X^2 u \right|^p dx, \tag{12}$$

*where the positive constant $c$ is independent of $u$ and $B$. Moreover, for any $u \in C_0^\infty(B)$, one has*

$$\int_B |u|^p |V| dx \le c \varphi_V(2r_B) \int_B \left| X^2 u \right|^p dx, \tag{13}$$

*where $c > 0$ is independent of $u$ and $B$.*

The above $X^2u$ is a set of $X_iX_ju$ for all $i, j = 1, \ldots, m$. We will define $X^2u$ precisely in Section 2.

*Remark 5.* The main difference between Theorem 4 and [6, Theorem 3.1] is clear; that is, the right-hand side term $\|Xu\|_{L^p(B)}$ in [6] is replaced by $\|X^2u\|_{L^p(B)}$ here. Of course, the class involving $V$ is not the same.

We observe an important relation between the Stummel class here and the reverse Hölder class: if $V \in B_q \cap L^1(\mathbb{G})$, $q > Q/2$, then $V^p \in S_p(\mathbb{G})$, $1 < p \le 2q/Q$. From it and Theorem 3, the following result follows.

**Theorem 6.** *Under the same assumptions on $a_{ij}$ as in Theorem 3, if $V \in B_q \cap L^1(\mathbb{G})$, $q > Q/2$, then for $1 < p \le 2q/Q$ and $u \in C_0^\infty(B)$, the estimate (11) holds.*

*Remark 7.* When $q = Q/2$ and $V \in B_{Q/2} \bigcap L^1(\mathbb{G})$, by the important property of the $B_q$ class (see [21]), there exists $\varepsilon > 0$ such that $V \in B_{Q/2+\varepsilon}$. Therefore, estimate (11) holds for $1 < p \le 1 + 2\varepsilon/Q$ and $u \in C_0^\infty(B)$.

The paper is organized as follows. In Section 2 we recall some basic facts about Carnot groups and function spaces. In Section 3 we first give the proof of Theorem 4. Then combining with the known result in [12, Theorem 2] and proving an estimate with the potential $V$, we finish the proof of Theorem 3. The proof of Theorem 6 is given in Section 4. In Section 5, we restate Theorems 3 and 4 for the Euclidean case and elliptic operators without proofs.

*Dependence of Constants.* Throughout this paper, the letter $c$ denotes a positive constant which may vary from line to line.

## 2. Preliminaries

*2.1. Background on Carnot Groups.* We collect some facts about Carnot groups that will be needed in the sequel and refer the readers to [22–25] for further details.

*Definition 8* (Carnot group). A Carnot group $\mathbb{G} = (\mathbb{R}^n; \circ)$ is a simply connected nilpotent Lie group such that its Lie algebra $\tilde{g}$ admits a stratification

$$\tilde{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_r = \oplus_{j=1}^r V_j, \tag{14}$$

where $[V_1, V_j] = V_{j+1}$, $j = 1, \ldots, r-1$, and $[V_1, V_r] = \{0\}$. Here $r$ is called the step of $\mathbb{G}$.

For $k = 1, \ldots, r$, let $X_{1,k}, \ldots, X_{m_k,k}$ be a basis of $V_k$ consisting of commutators of length $k$, where $m_k$ is the dimension of $V_k$. The horizontal vector fields are ones in the first layer $V_1$ and for convenience, we set $m_1 = m$ and denote $X_{i,1} = X_i$, $i = 1, \ldots, m$. Clearly, vector fields $X_1, \ldots, X_m$ satisfy Hormander's condition [26].

Let $\{\delta_\lambda\}_{\lambda>0}$ be a family of nonisotropic dilations on $\mathbb{G}$ defined by

$$\delta_\lambda : \mathbb{G} \longrightarrow \mathbb{G}, \quad \delta_\lambda(\xi) = \left(\lambda\xi_1, \lambda^2\xi_2, \ldots, \lambda^r\xi_r\right), \tag{15}$$

for any $\lambda > 0$ and $\xi = (\xi_1, \ldots, \xi_r) = (x_{1,1}, \ldots, x_{m_1,1}, \ldots, x_{1,r}, \ldots, x_{m_r,r}) \in \mathbb{G}$. The integer $Q = \sum_{k=1}^r k m_k$ is said to be the homogeneous dimension of $\mathbb{G}$. In general, we assume $Q \ge 4$. We call that a vector field $X_{i,k} \in \tilde{g}$ is left invariant if for any smooth function $f$ one has

$$X_{i,k}^x(f(y \circ x)) = (X_{i,k}f)(y \circ x), \quad \text{for any } x \in \mathbb{G}, \tag{16}$$

and $X_{i,k}$ is $s$th homogeneous if for any smooth function $f$, it follows that

$$X_{i,k}(f(\delta_\lambda(x))) = \lambda^s (X_{i,k}f)(\delta_\lambda(x)), \quad \text{for any } x \in \mathbb{G}. \tag{17}$$

As in [23], the homogeneous norm of $\xi \in \mathbb{G}$ is defined by

$$d_{\mathbb{G}}(\xi) = \left[\sum_{k=1}^r \left(\sum_{i=1}^{m_k} |\xi_{i,k}|^2\right)^{r!/k}\right]^{1/2r!}. \tag{18}$$

It is natural to define the pseudo distance by the homogeneous norm

$$d_{\mathbb{G}}(\xi, \eta) = d_{\mathbb{G}}\left(\eta^{-1} \circ \xi\right), \quad \text{for any } \xi, \eta \in \mathbb{G}, \tag{19}$$

where $\eta^{-1}$ is the inverse of $\eta$. A polynomial on $\mathbb{G}$ [24] is a function which can be expressed in exponential coordinates (see, e.g., [22, Section 2.1.9] and [27]) as

$$P(x) = \sum_I a_I x^I, \quad a_I \in \mathbb{R}, \tag{20}$$

where $I = (i_{j,k})_{j=1,\ldots,m_k}^{k=1,\ldots,r}$ are multi-indices and

$$x^I = \prod_{k=1,\ldots,r;\, j=1,\ldots,m_k} x_{j,k}^{i_{j,k}}. \tag{21}$$

The homogeneous degree of monomial $x^I$ is the sum $d(I) = \sum_{k=1}^r \sum_{j=1}^{m_k} k i_{j,k}$ and the homogeneous degree of $P(x)$ is $\max\{d(I) \mid a_I \ne 0\}$.

From [28], the left invariant vector fields $X_1, \ldots, X_m$ can induce the corresponding Carnot-Carathéodory distance $d$: for any $\delta > 0$, let $A(\delta)$ be the set of absolutely continuous curves $\gamma : [0,1] \to \mathbb{G}$ such that for a.e. $t \in [0,1]$,

$$\begin{gathered} \gamma'(t) = \sum_{i=1}^m a_i(t) X_i(\gamma(t)) \\ \text{with } \sum_{i=1}^m |a_i(t)| \le \delta. \end{gathered} \tag{22}$$

By [29], it is known that for $\delta$ large enough the set $A(\delta)$ is nonempty. We define the Carnot-Carathéodory distance by

$$d(\xi, \eta) = \inf\{\delta > 0 \mid \exists \delta \in A(\delta) \text{ with } \gamma(0) = \xi, \gamma(1) = \eta\}. \tag{23}$$

It is well known that the distance $d$ is equivalent to the pseudo distance $d_{\mathbb{G}}$ (see [28]). In this paper, we will mainly use the Carnot-Carathéodory distance $d$ to study regularity of (3).

Associated with the distance, we define the metric ball of center $\xi$ and radius $r$ in $\mathbb{G}$ by

$$B_r(\xi) = \{\eta \in \mathbb{G} \mid d(\xi,\eta) < r\}. \tag{24}$$

The Lebesgue measure in $\mathbb{R}^n$ is the Haar measure on $\mathbb{G}$ ([25, page 619]). Due to (15), one has

$$\left|B_r(\xi)\right| = C_Q r^Q, \tag{25}$$

where $|B_r(\xi)|$ is the measure of $B_r(\xi)$ and $C_Q$ is a positive constant.

*2.2. Function Spaces.* Denote $X = (X_1,\ldots,X_m)$, $Xu = (X_1u,\ldots,X_mu)$, $X^2u = \{X_iX_ju\}_{i,j=1}^m$, $|Xu| = \sum_{i=1}^m |X_iu|$, and $|X^2u| = \sum_{i,j=1}^m |X_iX_ju|$.

*Definition 9* (Horizontal Sobolev space). For any $p \geq 1$ and a domain $\Omega \subseteq \mathbb{G}$, one defines the Horizontal Sobolev spaces by

$$HW^{2,p}(\Omega) = \{u \in L^p(\Omega) \mid \|u\|_{HW^{2,p}(\Omega)} < +\infty\} \tag{26}$$

with the norm

$$\|u\|_{HW^{2,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|Xu\|_{L^p(\Omega)} + \left\|X^2u\right\|_{L^p(\Omega)}, \tag{27}$$

where $\|Xu\|_{L^p(\Omega)} = \sum_{i=1}^m \|X_iu\|_{L^p(\Omega)}$, $\|X^2u\|_{L^p(\Omega)} = \sum_{i,j=1}^m \|X_iX_ju\|_{L^p(\Omega)}$.

Analogously to [1], the space $HW_V^{2,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_{HW_V^{2,p}(\Omega)} = \|u\|_{HW^{2,p}(\Omega)} + \|Vu\|_{L^p(\Omega)}. \tag{28}$$

*Definition 10* (Reverse Hölder class). (1) A nonnegative locally $L^q$ integrable function $V(x)$ on $\mathbb{G}$ is said to belong to the reverse Hölder class $B_q$ $(1 < q < \infty)$, if there exists a positive constant $c$ such that

$$\left(|B|^{-1}\int_B V^q(x)\,dx\right)^{1/q} \leq c\left(|B|^{-1}\int_B V(x)\,dx\right), \tag{29}$$

for any metric ball $B$ in $\mathbb{G}$.

(2) Let $V(x) > 0$ a.e. and $V(x) \in L^\infty_{\mathrm{loc}}(\mathbb{G})$; one says $V(x) \in B_\infty$ if there exists a positive constant $c$ such that

$$\sup_B V(x) \leq c\left(|B|^{-1}\int_B V(x)\,dx\right), \quad \text{for any metric ball } B \text{ in } \mathbb{G}. \tag{30}$$

It is easy to see that $B_\infty \subset B_q \subset B_p, 1 < p < q < \infty$.

## 3. Proofs of Theorems 3 and 4

We first prove Theorem 4 and then prove Theorem 3.

*3.1. Proof of Theorem 4.* The following lemma is due to Lu and Wheeden [30, 31]. It will play a key role in our proof.

**Lemma 11.** *Let $B = B_{r_B}(x_0)$ be a metric ball in $\mathbb{G}$. If $u \in C^\infty(B)$, then there exists a first order polynomial $P_B(x)$ such that, for a.e. $x \in B$,*

$$\left|u(x) - P_B(x)\right| \leq c\int_B \left|X^2u(y)\right| \frac{d(x,y)^2}{\left|B(x,d(x,y))\right|}dy, \tag{31}$$

*where the positive constant $c$ is independent of $u$, $x$, and $B$. Moreover, if $u \in C_0^\infty(B)$, then for a.e. $x \in B$,*

$$|u(x)| \leq c\int_B \left|X^2u(y)\right| \frac{d(x,y)^2}{\left|B(x,d(x,y))\right|}dy, \tag{32}$$

*where $c > 0$ is independent of $u$, $x$, and $B$.*

*Proof of Theorem 4.* By (31), Fubini's Theorem, and Hölder's inequality, we have

$$\begin{aligned}
&\int_B |u(x) - P_B(x)|^p\,|V(x)|\,dx \\
&\quad\leq c\int_B |V(x)|\,|u(x) - P_B(x)|^{p-1} \\
&\qquad\times\left(\int_B \left|X^2u(y)\right| \frac{d(x,y)^2}{\left|B(x,d(x,y))\right|}dy\right)dx \\
&\quad= c\int_B \left|X^2u(y)\right| \\
&\qquad\times\left(\int_B |V(x)|\,|u(x) - P_B(x)|^{p-1}\right. \\
&\qquad\qquad\left.\times\frac{d(x,y)^2}{\left|B(x,d(x,y))\right|}dx\right)dy \\
&\quad\leq c\left(\int_B \left|X^2u(y)\right|^p dy\right)^{1/p} \\
&\qquad\cdot\left(\int_B\left(\int_B |V(x)|\,|u(x) - P_B(x)|^{p-1}\right.\right. \\
&\qquad\qquad\left.\left.\times\frac{d(x,y)^2}{\left|B(x,d(x,y))\right|}dx\right)^{p/(p-1)} dy\right)^{(p-1)/p} \\
&\quad\equiv c\left(\int_B \left|X^2u(y)\right|^p dy\right)^{1/p}\cdot I.
\end{aligned} \tag{33}$$

Now a computation yields

$$
\begin{aligned}
& I^{p/(p-1)} \\
& \leq \int_B \left( \int_B |V(z)| \frac{d(z,y)^2}{|B(z,d(z,y))|} dz \right)^{1/(p-1)} \\
& \quad \cdot \left( \int_B |u(x) - P_B(x)|^p |V(x)| \right. \\
& \quad \left. \times \frac{d(x,y)^2}{|B(x,d(x,y))|} dx \right) dy \\
& = \int_B |u(x) - P_B(x)|^p |V(x)| \int_B \frac{d(x,y)^2}{|B(x,d(x,y))|} \\
& \quad \cdot \left( \int_B |V(z)| \frac{d(z,y)^2}{|B(z,d(z,y))|} dz \right)^{1/(p-1)} dy\, dx \\
& \leq \int_{B(x_0,r_B)} |u(x) - P_B(x)|^p |V(x)| \\
& \quad \times \int_{B(x,2r_B)} \frac{d(x,y)^2}{|B(x,d(x,y))|} \\
& \quad \cdot \left( \int_{B(x,2r_B)} |V(z)| \right. \\
& \quad \left. \times \frac{d(z,y)^2}{|B(z,d(z,y))|} dz \right)^{1/(p-1)} dy\, dx \\
& \leq (\varphi_V(2r_B))^{1/(p-1)} \int_B |u(x) - P_B(x)|^p |V(x)|\, dx.
\end{aligned} \tag{34}
$$

Therefore,

$$
\begin{aligned}
& \int_B |u(x) - P_B(x)|^p |V(x)|\, dx \\
& \leq c (\varphi_V(2r_B))^{1/p} \left( \int_B |u(x) - P_B(x)|^p |V(x)|\, dx \right)^{(p-1)/p} \\
& \quad \times \left( \int_B \left| X^2 u(y) \right|^p dy \right)^{1/p}.
\end{aligned} \tag{35}
$$

It implies (12).

By using (32) and repeating the argument above for (12), we immediately obtain (13). □

*3.2. Proof of Theorem 3.* Let us recall $L^p$ estimates for the operator $A = \sum_{i,j=1}^{m} a_{ij} X_i X_j$ by Bramanti and Brandolini [12, Theorem 2].

**Lemma 12.** *Under the assumptions (4) and (5), for every $p \in (1,\infty)$, there exist positive constants $c = c(p,\mu,\mathbb{G})$ and $r = r(p,\mu,\eta,\mathbb{G})$, where $\eta$ denotes the VMO moduli of coefficients $a_{ij}$, such that, for any $u \in C_0^\infty(\mathbb{G})$ and $\operatorname{sprt} u \subset B_r$ ($B_r$ any metric ball of radius $r$),*

$$
\left\| X^2 u \right\|_{L^p(B_r)} \leq c \|Au\|_{L^p(B_r)}. \tag{36}
$$

Based on it and Theorem 4, we have the following $L^p$ estimates for $L$ in (3).

**Lemma 13.** *Under the assumptions (4) and (5), for every $p \in (1,\infty)$ and $V^p \in S_p(\mathbb{G})$, there exist positive constants $r_0 = r_0(p,\mu,\eta,\mathbb{G})$ and $c = c(p,r_0,\mu,V,\mathbb{G})$ such that, for any $u \in C_0^\infty(\mathbb{G})$ and $\operatorname{sprt} u \subset B_{r_0}$,*

$$
\left\| X^2 u \right\|_{L^p(B_{r_0})} + \|Vu\|_{L^p(B_{r_0})} \leq c \|Lu\|_{L^p(B_{r_0})}. \tag{37}
$$

*Proof.* By Theorem 4,

$$
\|Vu\|_{L^p(B_{r_0})}^p \leq c\varphi_{V^p}(2r_0) \int_{B_{r_0}} \left| X^2 u(x) \right|^p dx. \tag{38}
$$

Applying Lemma 12, it follows that

$$
\begin{aligned}
& \left\| X^2 u \right\|_{L^p(B_{r_0})} + \|Vu\|_{L^p(B_{r_0})} \\
& \quad \leq c \left( \|Au\|_{L^p(B_{r_0})} + \|Vu\|_{L^p(B_{r_0})} \right) \\
& \quad \leq c \left( \|Lu\|_{L^p(B_{r_0})} + \|Vu\|_{L^p(B_{r_0})} \right) \\
& \quad \leq c \left( \|Lu\|_{L^p(B_{r_0})} + \varphi_{V^p}(2r_0)^{1/p} \left\| X^2 u \right\|_{L^p(B_{r_0})} \right).
\end{aligned} \tag{39}
$$

Choosing $r_0 > 0$ such that $c\varphi_{V^p}(2r_0)^{1/p} \leq 1/2$, we derive (37). □

*Proof of Theorem 3.* We consult the way in [1, pages 342-343] and apply our previous results. By the basic theorem on the partition of unity (e.g., see [32, page 66]), there exists a partition of unity of nonnegative functions $\{\varphi_i\}_{i=1}^\infty$ in $\mathbb{G}$ such that $\varphi_i \in C_0^\infty(B_{r_0}(z_i))$ with $r_0$ in Lemma 13 and a family of metric balls $\{B_{r_0}(z_i)\}_{i=1}^\infty$ satisfying the finite overlapping property. We have from Lemma 13 that

$$
\begin{aligned}
& \left\| X^2 u \right\|_{L^p(\mathbb{G})} + \|Vu\|_{L^p(\mathbb{G})} \\
& \quad = \left\| \sum_i X^2(\varphi_i u) \right\|_{L^p(\mathbb{G})} + \left\| \sum_i V\varphi_i u \right\|_{L^p(\mathbb{G})} \\
& \quad \leq c \left( \sum_i \left\| X^2(\varphi_i u) \right\|_{L^p(B_{r_0}(z_i))} + \sum_i \|V\varphi_i u\|_{L^p(B_{r_0}(z_i))} \right) \\
& \quad \leq c \sum_i \|L(\varphi_i u)\|_{L^p(B_{r_0}(z_i))} \\
& \quad \leq \sum_i \left( \|Lu\|_{L^p(B_{r_0}(z_i))} + \|Du\|_{L^p(B_{r_0}(z_i))} + \|u\|_{L^p(B_{r_0}(z_i))} \right) \\
& \quad \leq c \left( \|Lu\|_{L^p(\mathbb{G})} + \|Xu\|_{L^p(\mathbb{G})} + \|u\|_{L^p(\mathbb{G})} \right).
\end{aligned} \tag{40}
$$

Combining with the interpolation inequality (see [12, Proposition 2])

$$\|Xu\|_{L^p(\mathbb{G})} \le \varepsilon \left\|X^2u\right\|_{L^p(\mathbb{G})} + \frac{2}{\varepsilon}\|u\|_{L^p(\mathbb{G})}, \quad \text{for any } \varepsilon > 0, \tag{41}$$

we obtain (11). □

## 4. Proof of Theorem 6

Several preliminary conclusions are necessary.

**Lemma 14.** *If $V \in B_q$, $q > 1$, then there exists a constant $c > 0$ such that, for any $1 \le p < q < \infty$ and $0 < r < R < \infty$,*

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} V(y)^p\,dy \le c\left(\frac{R}{r}\right)^{pQ/q}\left(\frac{1}{|B_R(x)|}\int_{B_R(x)} V(y)dy\right)^p. \tag{42}$$

*Proof.* By Hölder's inequality and (29), it yields

$$\begin{aligned}\frac{1}{|B_r(x)|}\int_{B_r(x)} V(y)^p\,dy &\le \left(\frac{1}{|B_r(x)|}\int_{B_r(x)} V(y)^q\,dy\right)^{p/q}\\ &\le c\left(\frac{R}{r}\right)^{Q}\left(\frac{1}{|B_R(x)|}\int_{B_R(x)} V(y)^q\,dy\right)^{p/q}\\ &\le c\left(\frac{R}{r}\right)^{pQ/q}\left(\frac{1}{|B_R(x)|}\int_{B_R(x)} V(y)dy\right)^{p}.\end{aligned} \tag{43}$$

□

*Remark 15.* If we take $\mathbb{G} = \mathbb{R}^n$ and $p = 1$, then Lemma 14 gives the version in [2, Lemma 1.2].

**Lemma 16.** *If $V \in B_q \cap L^1(\mathbb{G})$, $q > Q/2$, then $V^p \in S_p(\mathbb{G})$, $1 < p \le 2q/Q$.*

*Proof.* For any $x \in \mathbb{G}$, it follows that

$$\begin{aligned}I =: &\int_{d(x,y)<r} \frac{d^2(x,y)}{|B(x,d(x,y))|}\\ &\times\left(\int_{d(z,x)<r} V^p(z)\frac{d^2(z,y)}{|B(z,d(z,y))|}dz\right)^{1/(p-1)} dy\\ \le &\int_{d(x,y)<r} \frac{d^2(x,y)}{|B(x,d(x,y))|}\\ &\times\left(\int_{d(z,y)<2r} V^p(z)\frac{d^2(z,y)}{|B(z,d(z,y))|}dz\right)^{1/(p-1)} dy\\ \equiv &\int_{d(x,y)<r} \frac{d^2(x,y)}{|B(x,d(x,y))|}(I_1)^{1/(p-1)}\,dy.\end{aligned} \tag{44}$$

By (42), it yields

$$\begin{aligned}I_1 &\le \sum_{k=0}^{\infty}\int_{r/2^k\le d(z,y)<r/2^{k-1}} V^p(z)\frac{d^2(z,y)}{|B(z,d(z,y))|}dz\\ &\le c\sum_{k=0}^{\infty}\left(\frac{r}{2^k}\right)^{2-Q}\int_{d(z,y)<r/2^{k-1}} V^p(z)\,dz\\ &\le cr^{2-pQ/q}R^{pQ/q-pQ}\sum_{k=0}^{\infty}\left(\frac{1}{2^k}\right)^{2-pQ/q}\left(\int_{B(y,R)} V(z)\,dz\right)^p\\ &\le cr^{2-pQ/q}R^{pQ/q-pQ}\left(\int_{\mathbb{G}} Vdz\right)^p.\end{aligned} \tag{45}$$

Also, we have

$$\begin{aligned}\int_{d(x,y)<r}\frac{d^2(x,y)}{|B(x,d(x,y))|}dy &\le \sum_{k=1}^{\infty}\int_{r/2^k\le d(x,y)<r/2^{k-1}}\frac{d^2(x,y)}{|B(x,d(x,y))|}dy\\ &\le c\sum_{k=1}^{\infty}\left(\frac{r}{2^k}\right)^{2-Q}\int_{d(x,y)<r/2^{k-1}} dy\\ &\le cr^2.\end{aligned} \tag{46}$$

Therefore combining (45) and (46) gets

$$\begin{aligned}I &\le cr^{(1/(p-1))(2-pQ/q)}R^{(pQ/(p-1))(1/q-1)}\\ &\quad\times\left(\int_{\mathbb{G}} V(z)dz\right)^{p/(p-1)}\cdot cr^2\\ &\le cr^{(1/(p-1))(2p-pQ/q)}R^{(pQ/(p-1))(1/q-1)}\\ &\quad\times\left(\int_{\mathbb{G}} V(z)\,dz\right)^{p/(p-1)}\longrightarrow 0, \quad \text{as } r\longrightarrow 0.\end{aligned} \tag{47}$$

The result is proved. □

*Proof of Theorem 6.* By Lemma 16 and Theorem 3, we immediately obtain Theorem 6. □

*Remark 17.* In order to assure the convergence of the series $\sum_{k=0}^{\infty}(2^{-k})^{2-pQ/q}$ in the proof of Lemma 16, we require the assumption $p \le 2q/Q$, which leads to the range of $p$ in Theorem 6 smaller than [1, Theorem 1].

## 5. Results to the Euclidean Case and Elliptic Operators

Here for convenience of readers, we restate Theorems 3 and 4 corresponding to the Euclidean case but omit their proofs because the proofs are analogous to Theorems 3 and 4. It will be assumed for the leading coefficients $a_{ij}$ in (1) that

$(H_1)$ $a_{ij}(x) = a_{ji}(x) \in L^\infty(\mathbb{R}^n)$ for all $i, j = 1, \ldots, n$ and there exists a positive constant $\mu$ such that, for any $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$,

$$\mu^{-1} |\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \le \mu |\xi|^2; \tag{48}$$

$(H_2)$ $a_{ij}(x) \in VMO(\mathbb{R}^n)$; that is, for $i, j = 1, \ldots, n$,

$$\eta_{ij} = \sup_{\rho \le r} \sup_{x \in \mathbb{R}^n} \left( \left| B_\rho(x) \right|^{-1} \int_{B_\rho(x)} \left| a_{ij}(y) - a_{ij}^B \right| dy \right) \longrightarrow 0, \quad \text{as } r \longrightarrow 0^+, \tag{49}$$

where $a_{ij}^B = |B_\rho(x)|^{-1} \int_{B_\rho(x)} a_{ij}(y) dy$.

A function $V \in S_p(\mathbb{R}^n)$ for $1 < p < \infty$ means that, for each $r > 0$,

$$\varphi_V(r) := \sup_{x \in \mathbb{R}^n} \left( \int_{|x-y|<r} \frac{1}{|x-y|^{n-2}} \times \left( \int_{|z-y|<r} \frac{|V(z)|}{|z-y|^{n-2}} dz \right)^{1/(p-1)} dy \right)^{p-1} \tag{50}$$

is finite and

$$\lim_{r \to 0} \varphi_V(r) = 0. \tag{51}$$

**Theorem 18.** *Under assumptions $(H_1)$ and $(H_2)$, if $V^p \in S_p(\mathbb{R}^n)$, $1 < p < \infty$, then there exists a positive constant $c = c(n, p, \mu, \eta, V)$ such that, for any $u \in C_0^\infty(\mathbb{R}^n)$, one has*

$$\|u\|_{W^{2,p}(\mathbb{R}^n)} + \|Vu\|_{L^p(\mathbb{R}^n)} \le c \left( \|\mathfrak{L}u\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)} \right), \tag{52}$$

*where $\eta$ depends only on the VMO moduli of the coefficients $a_{ij}$.*

*Remark 19.* If $|V| \equiv 0$, $x \in \mathbb{R}^n$ or $|V| \le \text{const.}$, $x \in \mathbb{R}^n$, the $L^p$ theory of (1) with discontinuous leading coefficients was intensively studied and the result was proved in [33–36] and so forth. Bramanti et al. [1] obtained a prior $L^p(\mathbb{R}^n)$ estimate for (1) with $a_{ij} \in VMO$ and $V \in B_q$. Here $B_q \not\subset S_p(\mathbb{R}^n)$ and $S_p(\mathbb{R}^n) \not\subset B_q$.

**Theorem 20.** *Let $B = B_{r_B}(x_0)$ be any ball in $\mathbb{R}^n$. If $V \in S_p(\mathbb{R}^n)$, then there exists a first order polynomial $P_B(x)$ in $\mathbb{R}^n$ such that, for any $u \in C^\infty(B)$, one has*

$$\int_B |u - P_B(x)|^p |V| dx \le c\varphi_V(2r_B) \int_B \left| X^2 u \right|^p dx, \tag{53}$$

*where the positive constant $c$ is independent of $u$ and $B$. Moreover, for any $u \in C_0^\infty(B)$, one has*

$$\int_B |u|^p |V| dx \le c\varphi_V(2r_B) \int_B \left| X^2 u \right|^p dx, \tag{54}$$

*where $c > 0$ is independent of $u$ and $B$.*

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## References

[1] M. Bramanti, L. Brandolini, E. Harboure, and B. Viviani, "Global $W^{2,p}$ estimates for nondivergence elliptic operators with potentials satisfying a reverse Hölder condition," *Annali di Matematica Pura ed Applicata*, vol. 191, no. 2, pp. 339–362, 2012.

[2] Z. W. Shen, "$L^p$ estimates for Schrödinger operators with certain potentials," *Annales de L'Institut Fourier*, vol. 45, no. 2, pp. 513–546, 1995.

[3] S. Thangavelu, "Riesz transforms and the wave equation for the Hermite operator," *Communications in Partial Differential Equations*, vol. 15, no. 8, pp. 1199–1215, 1990.

[4] C. Vitanza, "A new contribution to the $W^{2,p}$-regularity for a class of elliptic second order equations with discontinuous coefficients," *Le Matematiche*, vol. 47, no. 1, pp. 177–186, 1992.

[5] J. Zhong, *Harmonic analysis for some Schrödinger type operators [Ph.D. thesis]*, Princeton University, Princeton, NJ, USA, 1993.

[6] G. Di Fazio and P. Zamboni, "A Fefferman-Poincaré type inequality for Carnot-Carathéodory vector fields," *Proceedings of the American Mathematical Society*, vol. 130, no. 9, pp. 2655–2660, 2002.

[7] M. A. Ragusa and P. Zamboni, "A potential theoretic inequality," *Czechoslovak Mathematical Journal*, vol. 51(126), no. 1, pp. 55–65, 2001.

[8] G. Citti, N. Garofalo, and E. Lanconelli, "Harnack's inequality for sum of squares of vector fields plus a potential," *The American Journal of Mathematics*, vol. 115, no. 3, pp. 699–734, 1993.

[9] H.-Q. Li, "Estimations $L^p$ des opérateurs de Schrödinger sur les groupes nilpotents," *Journal of Functional Analysis*, vol. 161, no. 1, pp. 152–218, 1999.

[10] G. Lu, "A Fefferman-Phong type inequality for degenerate vector fields and applications," *Panamerican Mathematical Journal*, vol. 6, no. 4, pp. 37–57, 1996.

[11] G. Lu, "On Harnack's inequality for a class of strongly degenerate Schrödinger operators formed by vector fields," *Differential and Integral Equations*, vol. 7, no. 1, pp. 73–100, 1994.

[12] M. Bramanti and L. Brandolini, “$L^p$ estimates for uniformly hypoelliptic operators with discontinuous coefficients on homogeneous groups,” *Università e Politecnico di Torino. Seminario Matematico. Rendiconti*, vol. 58, no. 4, pp. 389–433 (2003), 2000.

[13] X. Feng and P. Niu, “Global Orlicz regularity for sub-Laplace equations on homogeneous groups,” *Bulletin des Sciences Mathematiques*, vol. 136, no. 6, pp. 648–665, 2012.

[14] X. Feng and P. Niu, “Interior regularity for degenerate elliptic equations with drift on homogeneous groups,” *Journal of Lie Theory*, vol. 23, no. 3, pp. 803–825, 2013.

[15] C. L. Fefferman, “The uncertainty principle,” *Bulletin of the American Mathematical Society*, vol. 9, no. 2, pp. 129–206, 1983.

[16] F. Chiarenza and M. Frasca, “A remark on a paper by C. Fefferman,” *Proceedings of the American Mathematical Society*, vol. 108, no. 2, pp. 407–409, 1990.

[17] D. Danielli, “A Fefferman-Phong type inequality and applications to quasilinear subelliptic equations,” *Potential Analysis*, vol. 11, no. 4, pp. 387–413, 1999.

[18] D. Danielli, N. Garofalo, and D.-M. Nhieu, “Trace inequalities for Carnot-Carathéodory spaces and applications,” *Annali della Scuola Normale Superiore di Pisa*, vol. 27, no. 2, pp. 195–252, 1998.

[19] G. Lu, “Embedding theorems into Lipschitz and BMO spaces and applications to quasilinear subelliptic differential equations,” *Publicacions Matemàtiques*, vol. 40, no. 2, pp. 301–329, 1996.

[20] M. Bramanti and L. Brandolini, “$L^P$ estimates for nonvariational hypoelliptic operators with *VMO* coefficients,” *Transactions of the American Mathematical Society*, vol. 352, no. 2, pp. 781–822, 2000.

[21] A. Calderón, “Inequalities for the maximal function relative to a metric,” *Studia Mathematica*, vol. 57, no. 3, pp. 297–306, 1976.

[22] A. Bonfiglioli, E. Lanconelli, and F. Uguzzoni, *Stratified Lie Groups and Potential Theory for Their Sub-Laplacians*, Springer Monographs in Mathematics, Springer, Berlin , Germany, 2007.

[23] G. B. Folland, “Subelliptic estimates and function spaces on nilpotent Lie groups,” *Arkiv för Matematik*, vol. 13, no. 2, pp. 161–207, 1975.

[24] G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups*, Princeton University Press, Princeton, NJ, USA, 1980.

[25] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton University Press, Princeton, NJ, USA, 1993.

[26] L. Hörmander, “Hypoelliptic second order differential equations,” *Acta Mathematica*, vol. 119, pp. 147–171, 1967.

[27] W. S. Cohn, G. Lu, and P. Wang, “Sub-elliptic global high order Poincaré inequalities in stratified Lie groups and applications,” *Journal of Functional Analysis*, vol. 249, no. 2, pp. 393–424, 2007.

[28] A. Nagel, E. M. Stein, and S. Wainger, “Balls and metrics defined by vector fields. I: basic properties,” *Acta Mathematica*, vol. 155, no. 1, pp. 130–147, 1985.

[29] W.-L. Chow, “Über Systeme von liearren partiellen Differentialgleichungen erster Ordnung,” *Mathematische Annalen*, vol. 117, pp. 98–105, 1939.

[30] G. Lu and R. L. Wheeden, “High order representation formulas and embedding theorems on stratified groups and generalizations,” *Studia Mathematica*, vol. 142, no. 2, pp. 101–133, 2000.

[31] G. Lu and R. L. Wheeden, “Simultaneous representation and approximation formulas and high-order Sobolev embedding theorems on stratified groups,” *Constructive Approximation*, vol. 20, no. 4, pp. 647–668, 2004.

[32] W. Chen, *An Introduction to Differentiable Manifold*, Higher Education Press, Beijing, China, 2nd edition, 2001.

[33] F. Chiarenza, M. Frasca, and P. Longo, “Interior $W^{2,p}$ estimates for nondivergence elliptic equations with discontinuous coefficients,” *Ricerche di Matematica*, vol. 40, no. 1, pp. 149–168, 1991.

[34] F. Chiarenza, M. Frasca, and P. Longo, “$W^{2,p}$-solvability of the Dirichlet problem for nondivergence elliptic equations with *VMO* coefficients,” *Transactions of the American Mathematical Society*, vol. 336, no. 2, pp. 841–853, 1993.

[35] D. Kim and N. V. Krylov, “Elliptic differential equations with coefficients measurable with respect to one variable and VMO with respect to the others,” *SIAM Journal on Mathematical Analysis*, vol. 39, no. 2, pp. 489–506, 2007.

[36] N. V. Krylov, “Second-order elliptic equations with variably partially *VMO*coefficients,” *Journal of Functional Analysis*, vol. 257, no. 6, pp. 1695–1712, 2009.