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Research Article

Refinements of Generalized Hölder's Inequalities

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We present some new versions of generalized Hölder's inequalities. The results are used to improve Minkowski's inequality and a Beckenbach-type inequality.

1. Introduction

If $a_k \geq 0$, $b_k \geq 0$ $(k = 1, 2, \ldots, n)$, $p > 1$, $1/p + 1/q = 1$, then

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q}. \tag{1}$$

The sign of the inequality is reversed if $p < 1$, $p \neq 0$ (for $p < 0$, we assume that $a_k, b_k > 0$). Inequality (1) and its reversed version are called Hölder's inequality.

In 1979, Vasić and Pečarić [1] presented the following result.

Theorem A. *Let $A_{ij} > 0$ $(i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m)$.*

(a) *If $\beta_j > 0$ and if $\sum_{j=1}^{m}(1/\beta_j) \geq 1$, then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m} A_{ij} \leq \prod_{j=1}^{m}\left(\sum_{i=1}^{n} A_{ij}^{\beta_j}\right)^{1/\beta_j}. \tag{2}$$

(b) *If $\beta_j < 0$ $(j = 1, 2, \ldots, m)$, then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m} A_{ij} \geq \prod_{j=1}^{m}\left(\sum_{i=1}^{n} A_{ij}^{\beta_j}\right)^{1/\beta_j}. \tag{3}$$

(c) *If $\beta_1 > 0$, $\beta_j < 0$ $(j = 2, 3, \ldots, m)$, and if $\sum_{j=1}^{m}(1/\beta_j) \leq 1$, then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m} A_{ij} \geq \prod_{j=1}^{m}\left(\sum_{i=1}^{n} A_{ij}^{\beta_j}\right)^{1/\beta_j}. \tag{4}$$

Inequalities (2), (3), and (4) are called generalized Hölder's inequalities. It is well known that Hölder's inequality and generalized Hölder's inequalities are important in mathematical analysis and in the field of applied mathematics. For example, Agahi et al. [2] presented generalizations of the Hölder and the Minkowski inequality for pseudointegrals and Liu [3] established a Hölder type inequality. For a discussion on inequalities we refer the reader to [1, 4–9] and the references therein. Although generalized Hölder's inequalities play an important and basic role in many branches of mathematics, some problems can not be precisely estimated by generalized Hölder's inequalities. For example, if we set $n = 3$, $m = 2$, $A_{11} = 1$, $A_{21} = 1$, $A_{31} = 1$, $A_{12} = 1$, $A_{22} = 1$, $A_{32} = 19$, $\beta_2 = 1$, $\beta_1 = 1/2$, then from generalized Hölder's inequality (2) we obtain $21 \leq 189$. It is of interest to develop a refinement of Hölder's inequality.

In this paper we present new refinements of inequalities (2), (3), and (4) in Section 2. In Section 3, we use our results to improve the Minkowski inequality and a Beckenbach-type inequality.

2. Refinements of Generalized Hölder's Inequalities

We begin with a known result.

Lemma 1 (see [10]). *If $x > -1$, $\alpha > 1$, or $\alpha < 0$, then*

$$(1+x)^{\alpha} \geq 1 + \alpha x. \tag{5}$$

The inequality is reversed for $0 < \alpha < 1$.

Lemma 2. *Let $X_{ij} > 0$ and $1 - \sum_{i=1}^{n} X_{ij}^{\beta_j} > 0$ ($i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$).*

(a) *If $0 < \beta_1 < \beta_2 < \cdots < \beta_m$ and if $\sum_{j=1}^{m}(1/\beta_j) \geq 1$, then*

$$\prod_{j=1}^{m}\left(1-\sum_{i=1}^{n}X_{ij}^{\beta_j}\right)^{1/\beta_j} + \sum_{i=1}^{n}\prod_{j=1}^{m}X_{ij} \leq \prod_{j=1}^{[m/2]}\left[1-\left(\sum_{i=1}^{n}X_{i(2j)}^{\beta_{2j}} - \sum_{i=1}^{n}X_{i(2j-1)}^{\beta_{2j-1}}\right)^2\right]^{1/\beta_{2j}}. \tag{6}$$

(b) *If $0 > \beta_1 > \beta_2 > \cdots > \beta_m$, then*

$$\prod_{j=1}^{m}\left(1-\sum_{i=1}^{n}X_{ij}^{\beta_j}\right)^{1/\beta_j} + \sum_{i=1}^{n}\prod_{j=1}^{m}X_{ij} \geq \prod_{j=1}^{[m/2]}\left[1-\left(\sum_{i=1}^{n}X_{i(2j)}^{\beta_{2j}} - \sum_{i=1}^{n}X_{i(2j-1)}^{\beta_{2j-1}}\right)^2\right]^{1/\beta_{2j}}. \tag{7}$$

(c) *If $\beta_1 > 0$, $0 > \beta_2 > \beta_3 > \cdots > \beta_m$, and if $\sum_{j=1}^{m}(1/\beta_j) \leq 1$, then*

$$\prod_{j=1}^{m}\left(1-\sum_{i=1}^{n}X_{ij}^{\beta_j}\right)^{1/\beta_j} + \sum_{i=1}^{n}\prod_{j=1}^{m}X_{ij} \geq \prod_{j=1}^{[m/2]}\left[1-\left(\sum_{i=1}^{n}X_{i(2j)}^{\beta_{2j}} - \sum_{i=1}^{n}X_{i(2j-1)}^{\beta_{2j-1}}\right)^2\right]^{1/\beta_{2j}}. \tag{8}$$

Proof. (a) Note first that $1/\beta_1 > 1/\beta_2 > \cdots > 1/\beta_{m-1} > 1/\beta_m > 0$ and $1/\beta_j - 1/\beta_{j+1} > 0$ ($j = 1, 2, \ldots, m-1$).

Case (I). Let m be even.

Note that $(1/\beta_1 - 1/\beta_2) + 1/\beta_2 + 1/\beta_2 + (1/\beta_3 - 1/\beta_4) + 1/\beta_4 + 1/\beta_4 + \cdots + (1/\beta_{m-1} - 1/\beta_m) + 1/\beta_m + 1/\beta_m \geq 1$, and, using inequality (2), we have

$$\begin{aligned}
&\prod_{j=1}^{[m/2]}\left[1-\left(\sum_{i=1}^{n}X_{i(2j)}^{\beta_{2j}} - \sum_{i=1}^{n}X_{i(2j-1)}^{\beta_{2j-1}}\right)^2\right]^{1/\beta_{2j}} \\
&= \prod_{j=1}^{m/2}\left\{\left[\left(1-\sum_{i=1}^{n}X_{i(2j-1)}^{\beta_{2j-1}}\right) + \sum_{i=1}^{n}X_{i(2j)}^{\beta_{2j}}\right]^{1/\beta_{2j}}\right. \\
&\quad \times \left[\left(1-\sum_{i=1}^{n}X_{i(2j)}^{\beta_{2j}}\right) + \sum_{i=1}^{n}X_{i(2j-1)}^{\beta_{2j-1}}\right]^{1/\beta_{2j}} \\
&\quad \left.\times \left[\left(1-\sum_{i=1}^{n}X_{i(2j-1)}^{\beta_{2j-1}}\right) + \sum_{i=1}^{n}X_{i(2j-1)}^{\beta_{2j-1}}\right]^{1/\beta_{2j-1}-1/\beta_{2j}}\right\} \\
&= \left[\left(1-\sum_{i=1}^{n}X_{i1}^{\beta_1}\right) + \sum_{i=1}^{n}X_{i2}^{\beta_2}\right]^{1/\beta_2} \\
&\quad \times \left[\left(1-\sum_{i=1}^{n}X_{i2}^{\beta_2}\right) + \sum_{i=1}^{n}X_{i1}^{\beta_1}\right]^{1/\beta_2} \\
&\quad \times \left[\left(1-\sum_{i=1}^{n}X_{i1}^{\beta_1}\right) + \sum_{i=1}^{n}X_{i1}^{\beta_1}\right]^{1/\beta_1-1/\beta_2} \\
&\quad \times \left[\left(1-\sum_{i=1}^{n}X_{i3}^{\beta_3}\right) + \sum_{i=1}^{n}X_{i4}^{\beta_4}\right]^{1/\beta_4} \\
&\quad \times \left[\left(1-\sum_{i=1}^{n}X_{i4}^{\beta_4}\right) + \sum_{i=1}^{n}X_{i3}^{\beta_3}\right]^{1/\beta_4} \\
&\quad \times \left[\left(1-\sum_{i=1}^{n}X_{i3}^{\beta_3}\right) + \sum_{i=1}^{n}X_{i3}^{\beta_3}\right]^{1/\beta_3-1/\beta_4} \\
&\quad \vdots \\
&\quad \times \left[\left(1-\sum_{i=1}^{n}X_{i(m-1)}^{\beta_{m-1}}\right) + \sum_{i=1}^{n}X_{im}^{\beta_m}\right]^{1/\beta_m} \\
&\quad \times \left[\left(1-\sum_{i=1}^{n}X_{im}^{\beta_m}\right) + \sum_{i=1}^{n}X_{i(m-1)}^{\beta_{m-1}}\right]^{1/\beta_m} \\
&\quad \times \left[\left(1-\sum_{i=1}^{n}X_{i(m-1)}^{\beta_{m-1}}\right) + \sum_{i=1}^{n}X_{i(m-1)}^{\beta_{m-1}}\right]^{1/\beta_{m-1}-1/\beta_m} \\
&\geq \prod_{j=1}^{m/2}\left\{\left(1-\sum_{i=1}^{n}X_{i(2j-1)}^{\beta_{2j-1}}\right)^{1/\beta_{2j}}\left(1-\sum_{i=1}^{n}X_{i(2j)}^{\beta_{2j}}\right)^{1/\beta_{2j}}\right.
\end{aligned}$$

$$\times\left(1-\sum_{i=1}^{n}X_{i(2j-1)}^{\beta_{2j-1}}\right)^{1/\beta_{2j-1}-1/\beta_{2j}}\Bigg\}$$
$$+\prod_{j=1}^{m/2}\left[\left(X_{1(2j)}^{\beta_{2j}}\right)^{1/\beta_{2j}}\left(X_{1(2j-1)}^{\beta_{2j-1}}\right)^{1/\beta_{2j}}\times\left(X_{1(2j-1)}^{\beta_{2j-1}}\right)^{1/\beta_{2j-1}-1/\beta_{2j}}\right]$$
$$+\prod_{j=1}^{m/2}\left[\left(X_{2(2j)}^{\beta_{2j}}\right)^{1/\beta_{2j}}\left(X_{2(2j-1)}^{\beta_{2j-1}}\right)^{1/\beta_{2j}}\times\left(X_{2(2j-1)}^{\beta_{2j-1}}\right)^{1/\beta_{2j-1}-1/\beta_{2j}}\right]$$
$$\vdots$$
$$+\prod_{j=1}^{m/2}\left[\left(X_{n(2j)}^{\beta_{2j}}\right)^{1/\beta_{2j}}\left(X_{n(2j-1)}^{\beta_{2j-1}}\right)^{1/\beta_{2j}}\times\left(X_{n(2j-1)}^{\beta_{2j-1}}\right)^{1/\beta_{2j-1}-1/\beta_{2j}}\right]$$
$$=\prod_{j=1}^{m}\left(1-\sum_{i=1}^{n}X_{ij}^{\beta_j}\right)^{1/\beta_j}+\sum_{i=1}^{n}\prod_{j=1}^{m}X_{ij}, \tag{9}$$

so (6) holds when m is even.

Case (II). Let m be odd.

Note that $(1/\beta_1 - 1/\beta_2) + 1/\beta_2 + 1/\beta_2 + (1/\beta_3 - 1/\beta_4) + 1/\beta_4 + 1/\beta_4 + \cdots + (1/\beta_{m-2} - 1/\beta_{m-1}) + 1/\beta_{m-1} + 1/\beta_{m-1} + 1/\beta_m \geq 1$, and, using inequality (2), we have

$$\prod_{j=1}^{[m/2]}\left[1-\left(\sum_{i=1}^{n}X_{i(2j)}^{\beta_{2j}}-\sum_{i=1}^{n}X_{i(2j-1)}^{\beta_{2j-1}}\right)^2\right]^{1/\beta_{2j}}$$
$$=\prod_{j=1}^{(m-1)/2}\left[1-\left(\sum_{i=1}^{n}X_{i(2j)}^{\beta_{2j}}-\sum_{i=1}^{n}X_{i(2j-1)}^{\beta_{2j-1}}\right)^2\right]^{1/\beta_{2j}}$$
$$=\left\{\prod_{j=1}^{(m-1)/2}\left[1-\left(\sum_{i=1}^{n}X_{i(2j)}^{\beta_{2j}}-\sum_{i=1}^{n}X_{i(2j-1)}^{\beta_{2j-1}}\right)^2\right]^{1/\beta_{2j}}\right\}$$
$$\times\left[\left(1-\sum_{i=1}^{n}X_{im}^{\beta_m}\right)+\sum_{i=1}^{n}X_{im}^{\beta_m}\right]^{1/\beta_m}$$
$$=\left\{\prod_{j=1}^{(m-1)/2}\left\{\left[\left(1-\sum_{i=1}^{n}X_{i(2j-1)}^{\beta_{2j-1}}\right)+\sum_{i=1}^{n}X_{i(2j)}^{\beta_{2j}}\right]^{1/\beta_{2j}}\times\left[\left(1-\sum_{i=1}^{n}X_{i(2j)}^{\beta_{2j}}\right)+\sum_{i=1}^{n}X_{i(2j-1)}^{\beta_{2j-1}}\right]^{1/\beta_{2j}}\times\left[\left(1-\sum_{i=1}^{n}X_{i(2j-1)}^{\beta_{2j-1}}\right)+\sum_{i=1}^{n}X_{i(2j-1)}^{\beta_{2j-1}}\right]^{1/\beta_{2j-1}-1/\beta_{2j}}\right\}\right\}$$
$$\times\left[\left(1-\sum_{i=1}^{n}X_{im}^{\beta_m}\right)+\sum_{i=1}^{n}X_{im}^{\beta_m}\right]^{1/\beta_m}$$
$$\geq\left\{\prod_{j=1}^{(m-1)/2}\left[\left(1-\sum_{i=1}^{n}X_{i(2j-1)}^{\beta_{2j-1}}\right)^{1/\beta_{2j}}\left(1-\sum_{i=1}^{n}X_{i(2j)}^{\beta_{2j}}\right)^{1/\beta_{2j}}\times\left(1-\sum_{i=1}^{n}X_{i(2j-1)}^{\beta_{2j-1}}\right)^{1/\beta_{2j-1}-1/\beta_{2j}}\right]\right\}$$
$$\times\left(1-\sum_{i=1}^{n}X_{im}^{\beta_m}\right)^{1/\beta_m}$$
$$+\left\{\prod_{j=1}^{(m-1)/2}\left[\left(X_{1(2j)}^{\beta_{2j}}\right)^{1/\beta_{2j}}\left(X_{1(2j-1)}^{\beta_{2j-1}}\right)^{1/\beta_{2j}}\times\left(X_{1(2j-1)}^{\beta_{2j-1}}\right)^{1/\beta_{2j-1}-1/\beta_{2j}}\right]\right\}$$
$$\times\left(X_{1m}^{\beta_m}\right)^{1/\beta_m}$$
$$+\left\{\prod_{j=1}^{(m-1)/2}\left[\left(X_{2(2j)}^{\beta_{2j}}\right)^{1/\beta_{2j}}\left(X_{2(2j-1)}^{\beta_{2j-1}}\right)^{1/\beta_{2j}}\times\left(X_{2(2j-1)}^{\beta_{2j-1}}\right)^{1/\beta_{2j-1}-1/\beta_{2j}}\right]\right\}$$
$$\times\left(X_{2m}^{\beta_m}\right)^{1/\beta_m}$$
$$\vdots$$
$$+\left\{\prod_{j=1}^{(m-1)/2}\left[\left(X_{n(2j)}^{\beta_{2j}}\right)^{1/\beta_{2j}}\left(X_{n(2j-1)}^{\beta_{2j-1}}\right)^{1/\beta_{2j}}\times\left(X_{n(2j-1)}^{\beta_{2j-1}}\right)^{1/\beta_{2j-1}-1/\beta_{2j}}\right]\right\}$$
$$\times\left(X_{nm}^{\beta_m}\right)^{1/\beta_m}$$
$$=\prod_{j=1}^{m}\left(1-\sum_{i=1}^{n}X_{ij}^{\beta_j}\right)^{1/\beta_j}+\sum_{i=1}^{n}\prod_{j=1}^{m}X_{ij}, \tag{10}$$

so (6) holds for m is odd.

(b) Using similar reasoning as in Case (a) and using inequality (3), we obtain inequality (7).

(c) The proof of inequality (8) is similar to the reasoning used to prove inequality (6) so we omit it.

The proof of Lemma 2 is complete. □

Next, we present new refinements of inequalities (2), (3), and (4).

Theorem 3. *Let $A_{ij} > 0$ $(i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m)$, and let s be any given natural number $(1 \le s \le n)$.*

(a) *If $0 < \beta_1 < \beta_2 < \cdots < \beta_m$ and if $\sum_{j=1}^{m}(1/\beta_j) \ge 1$, then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m}A_{ij} \le \left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\beta_j}\right)^{1/\beta_j}\right] \times \prod_{j=1}^{[m/2]}\left[1-\left(\frac{A_{s(2j)}^{\beta_{2j}}}{\sum_{k=1}^{n}A_{k(2j)}^{\beta_{2j}}}-\frac{A_{s(2j-1)}^{\beta_{2j-1}}}{\sum_{k=1}^{n}A_{k(2j-1)}^{\beta_{2j-1}}}\right)^2\right]^{1/\beta_{2j}} \le \left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\beta_j}\right)^{1/\beta_j}\right]. \tag{11}$$

(b) *If $\beta_1 > 0$, $0 > \beta_2 > \beta_3 > \cdots > \beta_m$, and if $\sum_{j=1}^{m}(1/\beta_j) \le 1$, then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m}A_{ij} \ge \left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\beta_j}\right)^{1/\beta_j}\right] \times \prod_{j=1}^{[m/2]}\left[1-\left(\frac{A_{s(2j)}^{\beta_{2j}}}{\sum_{k=1}^{n}A_{k(2j)}^{\beta_{2j}}}-\frac{A_{s(2j-1)}^{\beta_{2j-1}}}{\sum_{k=1}^{n}A_{k(2j-1)}^{\beta_{2j-1}}}\right)^2\right]^{1/\beta_{2j}} \ge \left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\beta_j}\right)^{1/\beta_j}\right]. \tag{12}$$

(c) *If $0 > \beta_1 > \beta_2 > \cdots > \beta_m$, then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m}A_{ij} \ge \left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\beta_j}\right)^{1/\beta_j}\right] \times \prod_{j=1}^{[m/2]}\left[1-\left(\frac{A_{s(2j)}^{\beta_{2j}}}{\sum_{k=1}^{n}A_{k(2j)}^{\beta_{2j}}}-\frac{A_{s(2j-1)}^{\beta_{2j-1}}}{\sum_{k=1}^{n}A_{k(2j-1)}^{\beta_{2j-1}}}\right)^2\right]^{1/\beta_{2j}} \ge \left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\beta_j}\right)^{1/\beta_j}\right]. \tag{13}$$

Proof. (a) Consider the substitution

$$X_{ij} = \frac{A_{ij}}{\left(\sum_{k=1}^{n}A_{kj}^{\beta_j}\right)^{1/\beta_j}} \quad (i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m). \tag{14}$$

It is easy to see that, for any given natural number s $(1 \le s \le n)$, the following inequalities hold:

$$X_{ij} > 0, \qquad 1 - \sum_{1\le i\le n,\, i\ne s} X_{ij}^{\beta_j} > 0. \tag{15}$$

Consequently, by using substitution (14) and inequality (6), we have

$$\prod_{j=1}^{m}\left[1-\sum_{1\le i\le n,\, i\ne s}\left(\frac{A_{ij}^{\beta_j}}{\sum_{k=1}^{n}A_{kj}^{\beta_j}}\right)\right]^{1/\beta_j} + \sum_{1\le i\le n,\, i\ne s}\left(\prod_{j=1}^{m}\frac{A_{ij}}{\left(\sum_{k=1}^{n}A_{kj}^{\beta_j}\right)^{1/\beta_j}}\right) \le \prod_{j=1}^{[m/2]}\left\{1-\left[\sum_{1\le i\le n,\, i\ne s}\left(\frac{A_{i(2j)}^{\beta_{2j}}}{\sum_{k=1}^{n}A_{k(2j)}^{\beta_{2j}}}\right) - \sum_{1\le i\le n,\, i\ne s}\left(\frac{A_{i(2j-1)}^{\beta_{2j-1}}}{\sum_{k=1}^{n}A_{k(2j-1)}^{\beta_{2j-1}}}\right)\right]^2\right\}^{1/\beta_{2j}}, \tag{16}$$

and thus we have

$$\frac{\prod_{j=1}^{m}A_{sj}}{\prod_{j=1}^{m}\left(\sum_{k=1}^{n}A_{kj}^{\beta_j}\right)^{1/\beta_j}} + \frac{\sum_{1\le i\le n,\, i\ne s}\prod_{j=1}^{m}A_{ij}}{\prod_{j=1}^{m}\left(\sum_{k=1}^{n}A_{kj}^{\beta_j}\right)^{1/\beta_j}} \le \prod_{j=1}^{[m/2]}\left[1-\left(\frac{A_{s(2j)}^{\beta_{2j}}}{\sum_{k=1}^{n}A_{k(2j)}^{\beta_{2j}}}-\frac{A_{s(2j-1)}^{\beta_{2j-1}}}{\sum_{k=1}^{n}A_{k(2j-1)}^{\beta_{2j-1}}}\right)^2\right]^{1/\beta_{2j}}; \tag{17}$$

that is,

$$\frac{\sum_{i=1}^{n}\left(\prod_{j=1}^{m}A_{ij}\right)}{\prod_{j=1}^{m}\left(\sum_{k=1}^{n}A_{kj}^{\beta_j}\right)^{1/\beta_j}} \leq \prod_{j=1}^{[m/2]}\left[1-\left(\frac{A_{s(2j)}^{\beta_{2j}}}{\sum_{k=1}^{n}A_{k(2j)}^{\beta_{2j}}}-\frac{A_{s(2j-1)}^{\beta_{2j-1}}}{\sum_{k=1}^{n}A_{k(2j-1)}^{\beta_{2j-1}}}\right)^2\right]^{1/\beta_{2j}}. \quad (18)$$

We have the desired inequality (11). The proof of inequalities (12) and (13) is similar to the reasoning used to prove inequality (11) so we omit the proof. □

From Theorem 3, we obtain the following new refinements of generalized Hölder inequalities (2), (3), and (4).

Theorem 4. *Let* $A_{ij}>0$ $(i=1,2,\dots,n,\ j=1,2,\dots,m)$, *and let s be any given natural number* $(1\le s\le n)$.

(a) *If* $0<\beta_1<\beta_2<\cdots<\beta_m$ *and if* $\sum_{j=1}^{m}(1/\beta_j)\ge 1$, *then*

$$\begin{aligned}\sum_{i=1}^{n}\prod_{j=1}^{m}A_{ij} &\le \left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\beta_j}\right)^{1/\beta_j}\right] \\ &\quad\times\min_{1\le s\le n}\prod_{j=1}^{[m/2]}\left[1-\left(\frac{A_{s(2j)}^{\beta_{2j}}}{\sum_{k=1}^{n}A_{k(2j)}^{\beta_{2j}}}-\frac{A_{s(2j-1)}^{\beta_{2j-1}}}{\sum_{k=1}^{n}A_{k(2j-1)}^{\beta_{2j-1}}}\right)^2\right]^{1/\beta_{2j}} \\ &\le \left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\beta_j}\right)^{1/\beta_j}\right].\end{aligned} \quad (19)$$

(b) *If* $\beta_1>0$, $0>\beta_2>\beta_3>\cdots>\beta_m$, *and if* $\sum_{j=1}^{m}(1/\beta_j)\le 1$, *then*

$$\begin{aligned}\sum_{i=1}^{n}\prod_{j=1}^{m}A_{ij} &\ge \left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\beta_j}\right)^{1/\beta_j}\right] \\ &\quad\times\max_{1\le s\le n}\prod_{j=1}^{[m/2]}\left[1-\left(\frac{A_{s(2j)}^{\beta_{2j}}}{\sum_{k=1}^{n}A_{k(2j)}^{\beta_{2j}}}-\frac{A_{s(2j-1)}^{\beta_{2j-1}}}{\sum_{k=1}^{n}A_{k(2j-1)}^{\beta_{2j-1}}}\right)^2\right]^{1/\beta_{2j}} \\ &\ge \left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\beta_j}\right)^{1/\beta_j}\right].\end{aligned} \quad (20)$$

(c) *If* $0>\beta_1>\beta_2>\cdots>\beta_m$, *then*

$$\begin{aligned}\sum_{i=1}^{n}\prod_{j=1}^{m}A_{ij} &\ge \left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\beta_j}\right)^{1/\beta_j}\right] \\ &\quad\times\max_{1\le s\le n}\prod_{j=1}^{[m/2]}\left[1-\left(\frac{A_{s(2j)}^{\beta_{2j}}}{\sum_{k=1}^{n}A_{k(2j)}^{\beta_{2j}}}-\frac{A_{s(2j-1)}^{\beta_{2j-1}}}{\sum_{k=1}^{n}A_{k(2j-1)}^{\beta_{2j-1}}}\right)^2\right]^{1/\beta_{2j}} \\ &\ge \left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\beta_j}\right)^{1/\beta_j}\right].\end{aligned} \quad (21)$$

From Lemma 1 and Theorem 4, we obtain the following refinements of Hölder's inequality.

Theorem 5. *Let* $A_{ij}>0$ $(i=1,2,\dots,n,\ j=1,2,\dots,m)$, *and let s be any given natural number* $(1\le s\le n)$.

(a) *If* $0<\beta_1<\beta_2<\cdots<\beta_m$ *and if* $\sum_{j=1}^{m}(1/\beta_j)\ge 1$, *then*

$$\begin{aligned}\sum_{i=1}^{n}\prod_{j=1}^{m}A_{ij} &\le \left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\beta_j}\right)^{1/\beta_j}\right] \\ &\quad\times\min_{1\le s\le n}\prod_{j=1}^{[m/2]}\left[1-\frac{1}{\beta_{2j}}\left(\frac{A_{s(2j)}^{\beta_{2j}}}{\sum_{k=1}^{n}A_{k(2j)}^{\beta_{2j}}}\right.\right.\end{aligned}$$

$$- \frac{A_{s(2j-1)}^{\beta_{2j-1}}}{\sum_{k=1}^{n} A_{k(2j-1)}^{\beta_{2j-1}}} \Bigg)^2 \Bigg]$$

$$\leq \left[\prod_{j=1}^{m} \left(\sum_{i=1}^{n} A_{ij}^{\beta_j} \right)^{1/\beta_j} \right]. \tag{22}$$

(b) *If* $\beta_1 > 0$, $0 > \beta_2 > \beta_3 > \cdots > \beta_m$, *and if* $\sum_{j=1}^{m}(1/\beta_j) \leq 1$, *then*

$$\begin{aligned}\sum_{i=1}^{n} \prod_{j=1}^{m} A_{ij} &\geq \left[\prod_{j=1}^{m} \left(\sum_{i=1}^{n} A_{ij}^{\beta_j} \right)^{1/\beta_j} \right] \\ &\quad \times \max_{1\leq s\leq n} \prod_{j=1}^{[m/2]} \left[1 - \frac{1}{\beta_{2j}} \left(\frac{A_{s(2j)}^{\beta_{2j}}}{\sum_{k=1}^{n} A_{k(2j)}^{\beta_{2j}}} - \frac{A_{s(2j-1)}^{\beta_{2j-1}}}{\sum_{k=1}^{n} A_{k(2j-1)}^{\beta_{2j-1}}} \right)^2 \right] \\ &\geq \left[\prod_{j=1}^{m} \left(\sum_{i=1}^{n} A_{ij}^{\beta_j} \right)^{1/\beta_j} \right].\end{aligned} \tag{23}$$

(c) *If* $0 > \beta_1 > \beta_2 > \cdots > \beta_m$, *then*

$$\begin{aligned}\sum_{i=1}^{n} \prod_{j=1}^{m} A_{ij} &\geq \left[\prod_{j=1}^{m} \left(\sum_{i=1}^{n} A_{ij}^{\beta_j} \right)^{1/\beta_j} \right] \\ &\quad \times \max_{1\leq s\leq n} \prod_{j=1}^{[m/2]} \left[1 - \frac{1}{\beta_{2j}} \left(\frac{A_{s(2j)}^{\beta_{2j}}}{\sum_{k=1}^{n} A_{k(2j)}^{\beta_{2j}}} - \frac{A_{s(2j-1)}^{\beta_{2j-1}}}{\sum_{k=1}^{n} A_{k(2j-1)}^{\beta_{2j-1}}} \right)^2 \right] \\ &\geq \left[\prod_{j=1}^{m} \left(\sum_{i=1}^{n} A_{ij}^{\beta_j} \right)^{1/\beta_j} \right].\end{aligned} \tag{24}$$

In particular, putting $m = 2$, $\beta_2 = p$, $\beta_1 = q$, $A_{r1} = b_r$, $A_{r2} = a_r$ $(r = 1, 2, \ldots, n)$ in inequality (19) and putting $m = 2$, $\beta_1 = p$, $\beta_2 = q$, $A_{r1} = a_r$, $A_{r2} = b_r$ $(r = 1, 2, \ldots, n)$ in inequalities (20) and (21), respectively, we obtain the following corollary.

Corollary 6. *Let* $a_r, b_r > 0$ $(r = 1, 2, \ldots, n)$, *and let* s *be any given natural number* $(1 \leq s \leq n)$.

(a) *If* $p > q > 0$, $1/p + 1/q \geq 1$, *then*

$$\begin{aligned}\sum_{k=1}^{n} a_k b_k &\leq \left(\sum_{k=1}^{n} a_k^p \right)^{1/p} \left(\sum_{k=1}^{n} b_k^q \right)^{1/q} \\ &\quad \times \min_{1\leq s\leq n} \left[1 - \left(\frac{a_s^p}{\sum_{k=1}^{n} a_k^p} - \frac{b_s^q}{\sum_{k=1}^{n} b_k^q} \right)^2 \right]^{1/p}.\end{aligned} \tag{25}$$

(b) *If* $p > 0$, $q < 0$, $1/p + 1/q \leq 1$, *then*

$$\begin{aligned}\sum_{k=1}^{n} a_k b_k &\geq \left(\sum_{k=1}^{n} a_k^p \right)^{1/p} \left(\sum_{k=1}^{n} b_k^q \right)^{1/q} \\ &\quad \times \max_{1\leq s\leq n} \left[1 - \left(\frac{a_s^p}{\sum_{k=1}^{n} a_k^p} - \frac{b_s^q}{\sum_{k=1}^{n} b_k^q} \right)^2 \right]^{1/q}.\end{aligned} \tag{26}$$

(c) *If* $0 > p > q$, *then*

$$\begin{aligned}\sum_{k=1}^{n} a_k b_k &\geq \left(\sum_{k=1}^{n} a_k^p \right)^{1/p} \left(\sum_{k=1}^{n} b_k^q \right)^{1/q} \\ &\quad \times \max_{1\leq s\leq n} \left[1 - \left(\frac{a_s^p}{\sum_{k=1}^{n} a_k^p} - \frac{b_s^q}{\sum_{k=1}^{n} b_k^q} \right)^2 \right]^{1/q}.\end{aligned} \tag{27}$$

Remark 7. Let $n = 3$, $b_1 = 1$, $b_2 = 1$, $b_3 = 1$, $a_1 = 19$, $a_2 = 1$, and $a_3 = 1$, and let $p = 1$, $q = 1/2$. Then from inequality (25) we obtain $21 \leq 891/7 \approx 127.28571$.

Similarly, putting $m = 2$, $\beta_2 = p$, $\beta_1 = q$, $A_{r1} = b_r$, $A_{r2} = a_r$ $(r = 1, 2, \ldots, n)$ in inequality (22) and putting $m = 2$, $\beta_1 = p$, $\beta_2 = q$, $A_{r1} = a_r$, $A_{r2} = b_r$ $(r = 1, 2, \ldots, n)$ in inequalities (23) and (24), respectively, we obtain the following corollary.

Corollary 8. *Let* $a_r, b_r > 0$ $(r = 1, 2, \ldots, n)$, *and let* s *be any given natural number* $(1 \leq s \leq n)$.

(a) *If* $p > q > 0$, $1/p + 1/q \geq 1$, *then*

$$\begin{aligned}\sum_{k=1}^{n} a_k b_k &\leq \left(\sum_{k=1}^{n} a_k^p \right)^{1/p} \left(\sum_{k=1}^{n} b_k^q \right)^{1/q} \\ &\quad \times \min_{1\leq s\leq n} \left[1 - \frac{1}{p} \left(\frac{a_s^p}{\sum_{k=1}^{n} a_k^p} - \frac{b_s^q}{\sum_{k=1}^{n} b_k^q} \right)^2 \right].\end{aligned} \tag{28}$$

(b) *If* $p > 0$, $q < 0$, $1/p + 1/q \leq 1$, *then*

$$\begin{aligned}\sum_{k=1}^{n} a_k b_k &\geq \left(\sum_{k=1}^{n} a_k^p \right)^{1/p} \left(\sum_{k=1}^{n} b_k^q \right)^{1/q} \\ &\quad \times \max_{1\leq s\leq n} \left[1 - \frac{1}{q} \left(\frac{a_s^p}{\sum_{k=1}^{n} a_k^p} - \frac{b_s^q}{\sum_{k=1}^{n} b_k^q} \right)^2 \right].\end{aligned} \tag{29}$$

(c) *If* $0 > q > p$, *then*

$$\sum_{k=1}^{n} a_k b_k \geq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q} \times \max_{1\leq s\leq n}\left[1 - \frac{1}{q}\left(\frac{a_s^p}{\sum_{k=1}^{n} a_k^p} - \frac{b_s^q}{\sum_{k=1}^{n} b_k^q}\right)^2\right]. \tag{30}$$

3. Applications

In this section, we give two applications of our new inequalities. Firstly, we present a refinement of Minkowski's inequality.

Theorem 9. *Let* $a_k > 0$, $b_k > 0$ $(k = 1, 2, \ldots, n)$, *and let* s *be any given natural number* $(1 \leq s \leq n)$. *If* $p > 1$, *then*

$$\begin{aligned}\left[\sum_{k=1}^{n}(a_k+b_k)^p\right]^{1/p} &\leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p} \\ &\quad - \max_{1\leq s\leq n}\frac{1}{p}\left(\sum_{k=1}^{n} a_k^p\right)^{1/p}\left[\frac{a_s^p}{\sum_{k=1}^{n} a_k^p} - \frac{(a_s+b_s)^p}{\sum_{k=1}^{n}(a_k+b_k)^p}\right]^2 \\ &\quad - \max_{1\leq s\leq n}\frac{1}{p}\left(\sum_{k=1}^{n} b_k^p\right)^{1/p}\left[\frac{b_s^p}{\sum_{k=1}^{n} b_k^p} - \frac{(a_s+b_s)^p}{\sum_{k=1}^{n}(a_k+b_k)^p}\right]^2 \\ &\leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p}.\end{aligned} \tag{31}$$

If $0 < p < 1$, *then*

$$\begin{aligned}\left[\sum_{k=1}^{n}(a_k+b_k)^p\right]^{1/p} &\geq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p} \\ &\quad + \max_{1\leq s\leq n}\frac{1-p}{p}\left(\sum_{k=1}^{n} a_k^p\right)^{1/p}\left[\frac{a_s^p}{\sum_{k=1}^{n} a_k^p} - \frac{(a_s+b_s)^p}{\sum_{k=1}^{n}(a_k+b_k)^p}\right]^2 \\ &\quad + \max_{1\leq s\leq n}\frac{1-p}{p}\left(\sum_{k=1}^{n} b_k^p\right)^{1/p}\left[\frac{b_s^p}{\sum_{k=1}^{n} b_k^p} - \frac{(a_s+b_s)^p}{\sum_{k=1}^{n}(a_k+b_k)^p}\right]^2 \\ &\geq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p}.\end{aligned} \tag{32}$$

Proof. Consider the following.

Case (i). Let $p > 1$.

Now

$$\sum_{k=1}^{n}(a_k+b_k)^p = \sum_{k=1}^{n} a_k (a_k+b_k)^{p-1} + \sum_{k=1}^{n} b_k (a_k+b_k)^{p-1}, \tag{33}$$

and apply Corollary 6 with indices p and $p/(p-1)$ to each sum on the right so

$$\begin{aligned}\sum_{k=1}^{n}(a_k+b_k)^p &\leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p}\left[\sum_{k=1}^{n}(a_k+b_k)^p\right]^{(p-1)/p} \\ &\quad \times \min_{1\leq s\leq n}\left[1-\left(\frac{a_s^p}{\sum_{k=1}^{n} a_k^p} - \frac{(a_s+b_s)^p}{\sum_{k=1}^{n}(a_k+b_k)^p}\right)^2\right]^{1/p} \\ &\quad + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p}\left[\sum_{k=1}^{n}(a_k+b_k)^p\right]^{(p-1)/p} \\ &\quad \times \min_{1\leq s\leq n}\left[1-\left(\frac{b_s^p}{\sum_{k=1}^{n} b_k^p} - \frac{(a_s+b_s)^p}{\sum_{k=1}^{n}(a_k+b_k)^p}\right)^2\right]^{1/p}.\end{aligned} \tag{34}$$

From Lemma 1 we have

$$\begin{aligned}\sum_{k=1}^{n}(a_k+b_k)^p &\leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p}\left[\sum_{k=1}^{n}(a_k+b_k)^p\right]^{(p-1)/p} \\ &\quad \times \min_{1\leq s\leq n}\left[1-\frac{1}{p}\left(\frac{a_s^p}{\sum_{k=1}^{n} a_k^p} - \frac{(a_s+b_s)^p}{\sum_{k=1}^{n}(a_k+b_k)^p}\right)^2\right] \\ &\quad + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p}\left[\sum_{k=1}^{n}(a_k+b_k)^p\right]^{(p-1)/p} \\ &\quad \times \min_{1\leq s\leq n}\left[1-\frac{1}{p}\left(\frac{b_s^p}{\sum_{k=1}^{n} b_k^p} - \frac{(a_s+b_s)^p}{\sum_{k=1}^{n}(a_k+b_k)^p}\right)^2\right].\end{aligned} \tag{35}$$

Dividing both sides by $(\sum_{k=1}^{n}(a_k+b_k)^p)^{(p-1)/p}$, we obtain the desired inequality.

Case (ii). Let $0 < p < 1$.

Similar reasoning as in Case (i) yields inequality (32). □

Now, we give a sharpened version of a Beckenbach-type inequality. The Beckenbach inequality [11] was generalized and extended in several directions (see, e.g., [12–15]). In 1983, Wang [16] presented the following Beckenbach-type inequality.

Theorem B. *Let $p > q > 0$, $1/p+1/q = 1$, let a, b, c be positive numbers, and let $f(x)$, $g(x)$ be positive integrable functions defined on $[0, T]$. Then*

$$\frac{\left(a + c\int_0^T \varphi^p(x)\,dx\right)^{1/p}}{b + c\int_0^T \varphi(x)\,g(x)\,dx} \le \frac{\left(a + c\int_0^T f^p(x)\,dx\right)^{1/p}}{b + c\int_0^T f(x)\,g(x)\,dx}, \quad (36)$$

where $\varphi(x) = (ag(x)/b)^{q/p}$. The sign of the inequality in (36) is reversed if $0 < p < 1$.

From Corollary 8, we obtain a new refinement of Beckenbach-type inequality (36).

Theorem 10. *Let a, b, c be positive numbers, and let $f(x)$, $g(x)$ be positive integrable functions defined on $[0, T]$. If $p > q > 0$, $1/p + 1/q = 1$, then*

$$\begin{aligned}\frac{\left(a + c\int_0^T \varphi^p(x)\,dx\right)^{1/p}}{b + c\int_0^T \varphi(x)\,g(x)\,dx} &\le \frac{\left(a + c\int_0^T f^p(x)\,dx\right)^{1/p}}{b + c\int_0^T f(x)\,g(x)\,dx} \\ &\quad\times\left[1 - \frac{1}{p}\left(\frac{a}{a + c\int_0^T f^p(x)\,dx} - \frac{a^{-q/p}b^q}{a^{-q/p}b^q + c\int_0^T g^q(x)\,dx}\right)^2\right],\end{aligned} \quad (37)$$

where $\varphi(x) = (ag(x)/b)^{q/p}$.

If $0 < p < 1$, $q < 0$, $1/p + 1/q = 1$, then

$$\begin{aligned}\frac{\left(a + c\int_0^T \varphi^p(x)\,dx\right)^{1/p}}{b + c\int_0^T \varphi(x)\,g(x)\,dx} &\ge \frac{\left(a + c\int_0^T f^p(x)\,dx\right)^{1/p}}{b + c\int_0^T f(x)\,g(x)\,dx} \\ &\quad\times\left[1 - \frac{1}{q}\left(\frac{a}{a + c\int_0^T f^p(x)\,dx} - \frac{a^{-q/p}b^q}{a^{-q/p}b^q + c\int_0^T g^q(x)\,dx}\right)^2\right],\end{aligned} \quad (38)$$

where $\varphi(x) = (ag(x)/b)^{q/p}$.

Proof. We first consider the case $p > q > 0$, $1/p + 1/q = 1$. Using simple computations, we have

$$\frac{\left(a + c\int_0^T \varphi^p(x)\,dx\right)^{1/p}}{b + c\int_0^T \varphi(x)\,g(x)\,dx} = \left(a^{-q/p}b^q + c\int_0^T g^q(x)\,dx\right)^{-1/q}. \quad (39)$$

Moreover, from Corollary 8 we obtain

$$\begin{aligned}b + c\int_0^T f(x)\,g(x)\,dx &\le b + c\left(\int_0^T f^p(x)\,dx\right)^{1/p}\left(\int_0^T g^q(x)\,dx\right)^{1/q} \\ &= a^{1/p}\left(ba^{-1/p}\right) + c\left(\int_0^T f^p(x)\,dx\right)^{1/p}\left(\int_0^T g^q(x)\,dx\right)^{1/q} \\ &\le \left(a + c\int_0^T f^p(x)\,dx\right)^{1/p}\left(a^{-q/p}b^q + c\int_0^T g^q(x)\,dx\right)^{1/q} \\ &\quad\times\left[1 - \frac{1}{p}\left(\frac{a}{a + c\int_0^T f^p(x)\,dx} - \frac{a^{-q/p}b^q}{a^{-q/p}b^q + c\int_0^T g^q(x)\,dx}\right)^2\right];\end{aligned} \quad (40)$$

that is,

$$\begin{aligned}\left(a^{-q/p}b^q + c\int_0^T g^q(x)\,dx\right)^{-1/q} &\le \frac{\left(a + c\int_0^T f^p(x)\,dx\right)^{1/p}}{b + c\int_0^T f(x)\,g(x)\,dx} \\ &\quad\times\left[1 - \frac{1}{p}\left(\frac{a}{a + c\int_0^T f^p(x)\,dx} - \frac{a^{-q/p}b^q}{a^{-q/p}b^q + c\int_0^T g^q(x)\,dx}\right)^2\right].\end{aligned} \quad (41)$$

Then combining inequalities (39) and (41) yields inequality (37).

Using the above reasoning and applying inequality (26), it is easy to obtain inequality (38). □

4. Conclusions

In this paper we presented some new refinements of Hölder's inequality and we obtained a refinement of Cauchy's inequality. We improved Minkowski inequality and a Beckenbach-type inequality. In future research we hope to obtain new results using inequalities (11), (12), (13), and (19)–(30).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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