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Research Article

Applications of Bregman-Opial Property to Bregman Nonspreading Mappings in Banach Spaces

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The Opial property of Hilbert spaces and some other special Banach spaces is a powerful tool in establishing fixed point theorems for nonexpansive and, more generally, nonspreading mappings. Unfortunately, not every Banach space shares the Opial property. However, every Banach space has a similar Bregman-Opial property for Bregman distances. In this paper, using Bregman distances, we introduce the classes of Bregman nonspreading mappings and investigate the Mann and Ishikawa iterations for these mappings. We establish weak and strong convergence theorems for Bregman nonspreading mappings.

1. Introduction

Let E be a (real) Banach space with norm $\|\cdot\|$ and dual space E^*. For any x in E, we denote the value of x^* in E^* at x by $\langle x, x^*\rangle$. When $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in E, we denote the strong convergence of $\{x_n\}_{n\in\mathbb{N}}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. Let C be a nonempty subset of E. Let $T : C \to E$ be a map. We denote by $F(T) = \{x \in C : Tx = x\}$ the set of *fixed points* of T. We call the map T

(i) *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all x, y in C,

(ii) *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - y\| \le \|x - y\|$ for all x in C and y in $F(T)$.

The nonexpansivity plays an important role in the study of the *Ishikawa iteration*, given by

$$\begin{aligned} y_n &= \beta_n T x_n + (1 - \beta_n) x_n, \\ x_{n+1} &= \gamma_n T y_n + (1 - \gamma_n) x_n, \end{aligned} \tag{1}$$

where the sequences $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ satisfy some appropriate conditions. When all $\beta_n = 0$, Ishikawa iteration (1) reduces to the classical Mann iteration. Construction of fixed points of nonexpansive mappings via Mann's and Ishikawa's algorithms [1] has been extensively investigated in the literature (see, e.g., [2] and the references therein).

A powerful tool in deriving weak or strong convergence of iterative sequences is due to Opial [3]. A Banach space E is said to satisfy the *Opial property* [3] if for any weakly convergent sequence $\{x_n\}_{n\in\mathbb{N}}$ in E with weak limit x we have

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \tag{2}$$

for all y in E with $y \neq x$. It is well known that all Hilbert spaces, all finite dimensional Banach spaces, and the Banach spaces l^p $(1 \le p < \infty)$ satisfy the Opial property. However, not every Banach space satisfies the Opial property; see, for example, [4, 5].

Working with the Bregman distance D_g, the following Bregman-Opial-like inequality holds for every Banach space E:

$$\limsup_{n\to\infty} D_g(x_n, x) < \limsup_{n\to\infty} D_g(x_n, y), \tag{3}$$

whenever $x_n \rightharpoonup x \neq y$. See Lemma 11 for details. The Bregman-Opial property suggests introducing the notions of Bregman nonexpansive-like mappings and developing fixed point theorems and convergence results for the Ishikawa iterations for these mappings.

We recall the definition of Bregman distances. Let $g : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function on a Banach space E. The *Bregman distance* [6] (see also [7, 8]) corresponding to g is the function $D_g : E \times E \to \mathbb{R}$ defined by

$$D_g(x, y) = g(x) - g(y) - \langle x - y, \nabla g(y) \rangle, \quad \forall x, y \in E. \tag{4}$$

It follows from the strict convexity of g that $D_g(x, y) \geq 0$ for all x, y in E. However, D_g might not be symmetric and D_g might not satisfy the triangular inequality.

When E is a smooth Banach space, setting $g(x) = \|x\|^2$ for all x in E, we have that $\nabla g(x) = 2Jx$ for all x in E. Here J is the normalized duality mapping from E into E^*. Hence, $D_g(\cdot, \cdot)$ reduces to the usual map $\phi(\cdot, \cdot)$ as

$$D_g(x, y) = \phi(x, y) := \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \tag{5}$$

If E is a Hilbert space, then $D_g(x, y) = \|x - y\|^2$.

Let $g : E \to \mathbb{R}$ be strictly convex and Gâteaux differentiable, and let $C \subseteq E$ be nonempty. A mapping $T : C \to E$ is said to be

(i) *Bregman nonexpansive* if

$$D_g(Tx, Ty) \leq D_g(x, y), \quad \forall x, y \in C; \tag{6}$$

(ii) *Bregman quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$D_g(p, Tx) \leq D_g(p, x), \quad \forall x \in C,\ \forall p \in F(T); \tag{7}$$

(iii) *Bregman skew quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$D_g(Tx, p) \leq D_g(x, p), \quad \forall x \in C,\ \forall p \in F(T), \tag{8}$$

(iv) *Bregman nonspreading* if

$$\begin{aligned} &D_g(Tx, Ty) + D_g(Ty, Tx) \\ &\quad \leq D_g(Tx, y) + D_g(Ty, x), \quad \forall x, y \in C. \end{aligned} \tag{9}$$

It is obvious that every Bregman nonspreading map T with $F(T) \neq \emptyset$ is Bregman quasi-nonexpansive. Bregman nonspreading mappings include, in particular, the class of nonspreading functions studied by Takahashi and his coauthors (see, e.g., [9, 10]), which is defined with the map ϕ in (5).

Let us give an example of a Bregman nonspreading mapping with nonempty fixed point set, which is not quasi-nonexpansive.

Example 1. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^4$. The associated Bregman distance is given by

$$\begin{aligned} D_g(x, y) &= x^4 - y^4 - 4(x - y)y^3 \\ &= x^4 + 3y^4 - 4xy^3, \quad \forall x, y \in R. \end{aligned} \tag{10}$$

Define $T : [0, 2] \to [0, 2]$ by

$$Tx = \begin{cases} 0 & \text{if } x \in [0, 2), \\ 1 & \text{if } x = 2. \end{cases} \tag{11}$$

We have $F(T) = \{0\}$. Plainly, T is neither nonexpansive nor continuous.

However, T is Bregman nonspreading. To see this, we define $f : [0, 2] \times [0, 2] \to \mathbb{R}$ by

$$\begin{aligned} f(x, y) &= D_g(Tx, Ty) + D_g(Ty, Tx) \\ &\quad - D_g(Tx, y) - D_g(Ty, x), \quad \forall x, y \in [0, 2]. \end{aligned} \tag{12}$$

Consider the following three possible cases.

Case 1. If $x = y = 2$, then we have $Tx = Ty = 1$ and hence

$$f(2, 2) = 0 + 0 - 17 - 17 = -34 < 0. \tag{13}$$

Case 2. If $x = 2$ and $y \in [0, 2)$, then we have $Tx = 1$, $Ty = 0$, and hence

$$\begin{aligned} f(2, y) &= 1 + 3 - 1 - 3y^4 + 4y^3 - 48 \\ &= -3y^4 + 4y^3 - 45 < 0. \end{aligned} \tag{14}$$

Case 3. If $x, y \in [0, 2)$, then we have $Tx = Ty = 0$ and hence

$$f(x, y) = -3\left(x^4 + y^4\right) \leq 0. \tag{15}$$

Thus we have $f(x, y) \leq 0$ for all x, y in $[0, 2]$ and hence T is a Bregman nonspreading mapping.

In Section 2, we collect and study some basic ties of Bregman distances. In Section 3, utilizing the Bregman-Opial property, we present some fixed point theorems. In Sections 4 and 5, we investigate weak and strong convergence of the Ishikawa and Bregman-Ishikawa iterations for Bregman nonspreading mappings. Our results improve and generalize some known results in the current literature; see, for example, [11].

2. Bregman Functions and Bregman Distances

Let E be a (real) Banach space, and let $g : E \to \mathbb{R}$. For any x in E, the *gradient* $\nabla g(x)$ is defined to be the linear functional in E^* such that

$$\langle y, \nabla g(x) \rangle = \lim_{t \to 0} \frac{g(x + ty) - g(x)}{t}, \quad \forall y \in E. \tag{16}$$

The function g is said to be *Gâteaux differentiable* at x if $\nabla g(x)$ is well defined, and g is *Gâteaux differentiable* if it is Gâteaux differentiable everywhere on E. We call g Fréchet differentiable at x (see, e.g., [12, page 13] or [13, page 508]) if, for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left| g(y) - g(x) - \langle y - x, \nabla g(x) \rangle \right| \leq \epsilon \|y - x\|, \quad \text{whenever } \|y - x\| \leq \delta. \tag{17}$$

The function g is said to be *Fréchet differentiable* if it is Fréchet differentiable everywhere.

Let B be the closed unit ball of a Banach space E. A function $g : E \to \mathbb{R}$ is said to be

(i) *strongly coercive* if

$$\lim_{\|x_n\| \to +\infty} \frac{g(x_n)}{\|x_n\|} = +\infty; \tag{18}$$

(ii) *locally bounded* if $g(rB)$ is bounded for all $r > 0$;

(iii) *locally uniformly smooth* on E ([14, pp. 207, 221]) if the function $\sigma_r : [0, +\infty) \to [0, +\infty]$, defined by

$$\begin{aligned} \sigma_r(t) &= \sup_{x \in rB, y \in S_E, \alpha \in (0,1)} (\alpha g(x + (1-\alpha) ty) \\ &\qquad + (1-\alpha) g(x - \alpha t y) - g(x)) \\ &\quad \times (\alpha(1-\alpha))^{-1/2}, \end{aligned} \tag{19}$$

satisfies

$$\lim_{t \downarrow 0} \frac{\sigma_r(t)}{t} = 0, \quad \forall r > 0; \tag{20}$$

(iv) *locally uniformly convex* on E (or *uniformly convex on bounded subsets* of E ([14, pp. 203, 221])) if *the gauge* $\rho_r : [0, +\infty) \to [0, +\infty]$ of *uniform convexity* of g, defined by

$$\begin{aligned} \rho_r(t) &= \inf_{x, y \in rB, \|x - y\| = t, \alpha \in (0,1)} (\alpha g(x) + (1-\alpha) g(y) \\ &\qquad - g(\alpha x + (1-\alpha) y)) \\ &\quad \times (\alpha(1-\alpha))^{-1/2}, \end{aligned} \tag{21}$$

satisfies

$$\rho_r(t) > 0, \quad \forall r, t > 0. \tag{22}$$

For a locally uniformly convex map $g : E \to \mathbb{R}$, we have

$$\begin{aligned} g(\alpha x + (1-\alpha) y) &\le \alpha(x) g + (1-\alpha) g(y) \\ &\quad - \alpha(1-\alpha) \rho_r(\|x - y\|), \end{aligned} \tag{23}$$

for all x, y in rB and for all α in $(0, 1)$.

Let E be a Banach space and $g : E \to \mathbb{R}$ a strictly convex and Gâteaux differentiable function. By (4), the Bregman distance satisfies that [6]

$$\begin{aligned} D_g(x, z) &= D_g(x, y) + D_g(y, z) \\ &\quad + \langle x - y, \nabla g(y) - \nabla g(z) \rangle, \quad \forall x, y, z \in E. \end{aligned} \tag{24}$$

In particular,

$$\begin{aligned} D_g(x, y) &= -D_g(y, x) \\ &\quad + \langle y - x, \nabla g(y) - \nabla g(x) \rangle, \quad \forall x, y \in E. \end{aligned} \tag{25}$$

Lemma 2 (see [15]). *Let E be a Banach space and $g : E \to \mathbb{R}$ a Gâteaux differentiable function which is locally uniformly convex on E. Let $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ be bounded sequences in E. Then the following assertions are equivalent:*

(1) $\lim_{n \to \infty} D_g(x_n, y_n) = 0$,

(2) $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

The following Bregman-Opial-like inequality has been proved in [16].

Lemma 3 (see [16]). *Let E be a Banach space and let $g : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function. Suppose $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in E such that $x_n \rightharpoonup x$ for some x in E. Then*

$$\limsup_{n \to \infty} D_g(x_n, x) < \limsup_{n \to \infty} D_g(x_n, y), \tag{26}$$

for all y in the interior of dom g with $y \neq x$.

We call a function $g : E \to (-\infty, +\infty]$ *lower semicontinuous* if $\{x \in E : g(x) \le r\}$ is closed for all r in $\mathbb{R}$. For a lower semicontinuous convex function $g : E \to \mathbb{R}$, the *subdifferential* ∂g of g is defined by

$$\begin{aligned} \partial g(x) = \{x^* \in E^* : g(x) \\ + \langle y - x, x^* \rangle \le g(y), \forall y \in E\}, \end{aligned} \tag{27}$$

for all x in E. It is well known that $\partial g \subset E \times E^*$ is maximal monotone [17, 18]. For any lower semicontinuous convex function $g : E \to (-\infty, +\infty]$, the *conjugate function* g^* of g is defined by

$$g^*(x^*) = \sup_{x \in E} \{\langle x, x^* \rangle - g(x)\}, \quad \forall x^* \in E^*. \tag{28}$$

It is well known that

$$g(x) + g^*(x^*) \ge \langle x, x^* \rangle, \quad \forall (x, x^*) \in E \times E^*, \tag{29}$$

$$(x, x^*) \in \partial g \text{ is equivalent to } g(x) + g^*(x^*) = \langle x, x^* \rangle. \tag{30}$$

We also know that if $g : E \to (-\infty, +\infty]$ is a proper lower semicontinuous convex function, then $g^* : E^* \to (-\infty, +\infty]$ is a proper weak* lower semicontinuous convex function. Here, saying g is *proper* we mean that dom $g := \{x \in E : g(x) < +\infty\} \neq \emptyset$.

The following definition is slightly different from that in Butnariu and Iusem [12].

Definition 4 (see [13]). Let E be a Banach space. A function $g : E \to \mathbb{R}$ is said to be a Bregman function if the following conditions are satisfied:

(1) g is continuous, strictly convex, and Gâteaux differentiable;

(2) the set $\{y \in E : D_g(x, y) \le r\}$ is bounded for all x in E and $r > 0$.

The following lemma follows from Butnariu and Iusem [12] and Zălinescu [14].

Lemma 5. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a strongly coercive Bregman function. Then*

(1) *$\nabla g : E \to E^*$ is one-to-one, onto, and norm-to-weak* continuous;*

(2) *$\langle x - y, \nabla g(x) - \nabla g(y)\rangle = 0$ if and only if $x = y$;*

(3) *$\{x \in E : D_g(x, y) \le r\}$ is bounded for all y in E and $r > 0$;*

(4) *$\operatorname{dom} g^* = E^*$, g^* is Gâteaux differentiable, and $\nabla g^* = (\nabla g)^{-1}$.*

The following two results follow from [14, Proposition 3.6.4].

Proposition 6. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex function which is locally bounded. The following assertions are equivalent:*

(1) *g is strongly coercive and locally uniformly convex on E;*

(2) *$\operatorname{dom} g^* = E^*$ and g^* is locally bounded and locally uniformly smooth on E;*

(3) *$\operatorname{dom} g^* = E^*$, g^* is Fréchet differentiable, and ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^*.*

Proposition 7. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a continuous convex function which is strongly coercive. The following assertions are equivalent:*

(1) *g is locally bounded and locally uniformly smooth on E;*

(2) *g^* is Fréchet differentiable and ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E;*

(3) *$\operatorname{dom} g^* = E^*$ and g^* is strongly coercive and locally uniformly convex on E.*

Lemma 8 (see [13, 19]). *Let E be a reflexive Banach space, $g : E \to \mathbb{R}$ a strongly coercive Bregman function, and V the function defined by*

$$V(x, x^*) = g(x) - \langle x, x^*\rangle + g^*(x^*), \quad \forall x \in E,\ \forall x^* \in E^*. \tag{31}$$

The following assertions hold:

(1) *$D_g(x, \nabla g^*(x^*)) = V(x, x^*)$ for all x in E and x^* in E^*,*

(2) *$V(x, x^*) + \langle \nabla g^*(x^*) - x, y^*\rangle \le V(x, x^* + y^*)$ for all x in E and x^*, y^* in E^*.*

It also follows from the definition that V is convex in the second variable x^*, and

$$V(x, \nabla g(y)) = D_g(x, y). \tag{32}$$

Let E be a Banach space and let C be a nonempty convex subset of E. Let $g : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function. Then, we know from [20] that, for x in E and x_0 in C, we have

$$\begin{gathered} D_g(x_0, x) = \min_{y\in C} D_g(y, x) \text{ if and only if} \\ \langle y - x_0, \nabla g(x) - \nabla g(x_0)\rangle \le 0, \quad \forall y \in C. \end{gathered} \tag{33}$$

Further, if C is a nonempty, closed, and convex subset of a reflexive Banach space E and $g : E \to \mathbb{R}$ is a strongly coercive Bregman function, then, for each x in E, there exists a unique x_0 in C such that

$$D_g(x_0, x) = \min_{y\in C} D_g(y, x). \tag{34}$$

The *Bregman projection* proj_C^g from E onto C defined by $\operatorname{proj}_C^g(x) = x_0$ has the following property:

$$\begin{gathered} D_g(y, \operatorname{proj}_C^g x) + D_g(\operatorname{proj}_C^g x, x) \\ \le D_g(y, x), \quad \forall y \in C,\ \forall x \in E. \end{gathered} \tag{35}$$

See [12] for details.

Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a lower semicontinuous, strictly convex, and Gâteaux differentiable function. Let C be a nonempty, closed, and convex subset of E and let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence in E. For any x in E, we set

$$\operatorname{Br}(x, \{x_n\}) = \limsup_{n\to\infty} D_g(x_n, x). \tag{36}$$

The *Bregman asymptotic radius* of $\{x_n\}_{n\in\mathbb{N}}$ relative to C is defined by

$$\operatorname{Br}(C, \{x_n\}) = \inf\{\operatorname{Br}(x, \{x_n\}) : x \in C\}. \tag{37}$$

The *Bregman asymptotic center* of $\{x_n\}_{n\in\mathbb{N}}$ relative to C is the set

$$\operatorname{BA}(C, \{x_n\}) = \{x \in C : \operatorname{Br}(x, \{x_n\}) = \operatorname{Br}(C, \{x_n\})\}. \tag{38}$$

Proposition 9. *Let C be a nonempty, closed, and convex subset of a reflexive Banach space E, and let $g : E \to \mathbb{R}$ be strictly convex, Gâteaux differentiable, and locally bounded on E. If $\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence of C, then $BA(C, \{x_n\}_{n\in\mathbb{N}})$ is a singleton.*

Proof. In view of the definition of Bregman asymptotic radius, we may assume that $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to z in C. By Lemma 3, we conclude that $\operatorname{BA}(C, \{x_n\}_{n\in\mathbb{N}}) = \{z\}$. □

3. Fixed Point Theorems

Lemma 10 (see [21]). *Let C be a nonempty, closed, and convex subset of a reflexive Banach space E. Let $g : E \to \mathbb{R}$ be strictly convex, continuous, strongly coercive, Gâteaux differentiable, and locally bounded on E. Let $T : C \to E$ be a Bregman quasi-nonexpansive mapping. Then $F(T)$ is closed and convex.*

Lemma 11. *Let C be a nonempty, closed, and convex subset of a reflexive Banach space E. Let $g : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function. Let $T : C \to E$ be a Bregman nonspreading mapping. Then*

$$\begin{aligned} D_g(x, Ty) \le{}& D_g(x, y) + D_g(Tx, x) \\ &+ \langle x - Tx, \nabla g(y) - \nabla g(Ty)\rangle \\ &+ \langle Tx - Ty, \nabla g(x) - \nabla g(Tx)\rangle, \\ &\qquad \forall x, y \in C. \end{aligned} \tag{39}$$

Proof. Let $x, y \in C$. In view of (24), we have

$$\begin{aligned} D_g(Tx, Ty) &\le D_g(Tx, y) + D_g(Ty, x) - D_g(Ty, Tx) \\ &= D_g(Tx, x) + D_g(x, y) \\ &\quad + \langle Tx - x, \nabla g(x) - \nabla g(y)\rangle \\ &\quad + D_g(Ty, Tx) + D_g(Tx, x) \\ &\quad + \langle Ty - Tx, \nabla g(Tx) - \nabla g(x)\rangle - D_g(Ty, Tx) \\ &= D_g(x, y) + 2D_g(Tx, x) \\ &\quad + \langle Tx - x, \nabla g(x) - \nabla g(y)\rangle \\ &\quad + \langle Tx - Ty, \nabla g(x) - \nabla g(Tx)\rangle. \end{aligned} \tag{40}$$

This, together with (24), implies that

$$\begin{aligned} D_g(x, Ty) &= D_g(x, Tx) + D_g(Tx, Ty) \\ &\quad + \langle x - Tx, \nabla g(Tx) - \nabla g(Ty)\rangle \\ &\le D_g(x, Tx) + D_g(x, y) + 2D_g(Tx, x) \\ &\quad + \langle Tx - x, \nabla g(x) - \nabla g(y)\rangle \\ &\quad + \langle Tx - Ty, \nabla g(x) - \nabla g(Tx)\rangle \\ &\quad + \langle x - Tx, \nabla g(Tx) - \nabla g(Ty)\rangle \\ &= D_g(x, y) + D_g(Tx, x) \\ &\quad + \langle x - Tx, \nabla g(x) - \nabla g(Tx)\rangle \\ &\quad + \langle Tx - Ty, \nabla g(x) - \nabla g(Tx)\rangle \\ &\quad + \langle Tx - x, \nabla g(x) - \nabla g(y)\rangle \\ &\quad + \langle x - Tx, \nabla g(Tx) - \nabla g(Ty)\rangle \\ &= D_g(x, y) + D_g(Tx, x) \\ &\quad + \langle x - Tx, \nabla g(y) - \nabla g(Ty)\rangle \\ &\quad + \langle Tx - Ty, \nabla g(x) - \nabla g(Tx)\rangle. \end{aligned} \tag{41}$$

□

Proposition 12 (demiclosedness principle). *Let C be a nonempty subset of a reflexive Banach space E. Let $g : E \to \mathbb{R}$ be a strictly convex, Gâteaux differentiable, and locally bounded function. Let $T : C \to E$ be a Bregman nonspreading mapping. If $x_n \rightharpoonup z$ in C and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$, then $Tz = z$. That is, $I - T$ is demiclosed at zero, where I is the identity mapping on E.*

Proof. Since $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to z and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$, both the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{Tx_n\}_{n\in\mathbb{N}}$ are bounded. Since ∇g is uniformly norm-to-norm continuous on bounded subsets of E (see, e.g., [14]), we arrive at

$$\lim_{n\to\infty} \|\nabla g(x_n) - \nabla g(Tx_n)\| = 0. \tag{42}$$

In view of Lemma 2, we deduce that $\lim_{n\to\infty} D_g(x_n, Tx_n) = 0$. Set

$$\begin{aligned} M_1 = {}& \sup\{\|Tx_n\|, \|Tz\|, \|\nabla g(z)\|, \|\nabla g(Tz)\| : n \in \mathbb{N}\} \\ &< +\infty. \end{aligned} \tag{43}$$

By Lemma 11, for all n in $\mathbb{N}$,

$$\begin{aligned} & D_g(x_n, Tz) \\ &\quad \le D_g(x_n, z) + D_g(Tx_n, x_n) \\ &\qquad + \langle x_n - Tx_n, \nabla g(z) - \nabla g(Tz)\rangle \\ &\qquad + \langle Tx_n - Tz, \nabla g(x_n) - \nabla g(Tx_n)\rangle \\ &\quad \le D_g(x_n, z) + D_g(Tx_n, x_n) \\ &\qquad + \|x_n - Tx_n\| \|\nabla g(z) - \nabla g(Tz)\| \\ &\qquad + \|Tx_n - Tz\| \|\nabla g(x_n) - \nabla g(Tx_n)\| \\ &\quad \le D_g(x_n, z) + D_g(Tx_n, x_n) \\ &\qquad + 2M_1\|x_n - Tx_n\| + 2M_1\|\nabla g(x_n) - \nabla g(Tx_n)\|. \end{aligned} \tag{44}$$

This implies

$$\limsup_{n\to\infty} D_g(x_n, Tz) \le \limsup_{n\to\infty} D_g(x_n, z). \tag{45}$$

From the Bregman-Opial-like property, we obtain $Tz = z$. □

Let ℓ^∞ be the Banach lattice of bounded real sequences with the supremum norm. It is well known that there exists a bounded linear functional μ on ℓ^∞ such that the following three conditions hold:

(1) if $\{t_n\}_{n\in\mathbb{N}} \in \ell^\infty$ and $t_n \ge 0$ for every n in $\mathbb{N}$, then $\mu(\{t_n\}) \ge 0$;

(2) if $t_n = 1$ for every n in $\mathbb{N}$, then $\mu(\{t_n\}) = 1$;

(3) $\mu(\{t_{n+1}\}) = \mu(\{t_n\})$ for all $\{t_n\}_{n\in\mathbb{N}}$ in ℓ^∞.

Here, $\{t_{n+1}\}$ denotes the sequence $(t_2, t_3, t_4, \ldots, t_{n+1}, \ldots)$ in ℓ^∞. Such a functional μ is called a *Banach limit* and the value of μ at $\{t_n\}_{n\in\mathbb{N}}$ in ℓ^∞ is denoted by $\mu_n t_n$. Therefore, condition (3) means $\mu_n t_n = \mu_n t_{n+1}$. If μ satisfies conditions (1) and (2), we call μ a *mean* on ℓ^∞. See, for example, [22].

To see some examples of those mappings T satisfying all the stated hypotheses in the following result, we refer the reader to [23].

Theorem 13 (see [23]). *Let C be a nonempty, closed, and convex subset of a reflexive Banach space E. Let $g : E \to \mathbb{R}$ be strictly convex, continuous, strongly coercive, Gâteaux differentiable, locally bounded and locally uniformly convex on E. Let $T : C \to C$ be a mapping. Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence of C and let μ be a mean on ℓ^∞. Suppose that*

$$\mu_n D_g(x_n, Ty) \le \mu_n D_g(x_n, y), \quad \forall y \in C. \tag{46}$$

Then T has a fixed point in C.

Corollary 14. *Let C be a nonempty, bounded, closed, and convex subset of a reflexive Banach space E. Let $g : E \to \mathbb{R}$ be strictly convex, continuous, strongly coercive, Gâteaux differentiable function, locally bounded, and locally uniformly convex on E. Let $T : C \to C$ be a Bregman nonspreading mapping. Then T has a fixed point.*

Proof. Let μ a Banach limit on ℓ^∞ and $x \in C$ be such that $\{T^n x\}_{n\in\mathbb{N}}$ is bounded. For any n in $\mathbb{N}$ we have

$$\begin{aligned} &D_g(T^n x, Ty) + D_g(Ty, T^n x) \\ &\quad \le D_g(T^n x, y) + D_g(Ty, T^{n-1} x), \quad \forall y \in C. \end{aligned} \tag{47}$$

This implies that

$$\begin{aligned} &\mu_n D_g(T^n x, Ty) + \mu_n D_g(Ty, T^n x) \\ &\quad \le \mu_n D_g(T^n x, y) + \mu_n D_g(Ty, T^{n-1} x), \quad \forall y \in C. \end{aligned} \tag{48}$$

Thus we have

$$\mu_n D_g(T^n x, Ty) \le \mu_n D_g(T^n x, y), \quad \forall y \in C. \tag{49}$$

It follows from Theorem 13 that $F(T) \ne \emptyset$. □

4. Weak and Strong Convergence Theorems for Bregman Nonspreading Mappings

In this section, we prove weak and strong convergence theorems concerning Bregman nonspreading mappings in a reflexive Banach space.

Lemma 15. *Let C be a nonempty, closed, and convex subset of a reflexive Banach space E. Let $g : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function. Let $T : C \to C$ be a Bregman skew quasi-nonexpansive mapping with a nonempty fixed point set $F(T)$. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be two sequences defined by* (1) *such that $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ are arbitrary sequences in $[0,1]$. Then the following assertions hold:*

(1) *$\max\{D_g(x_{n+1}, z), D_g(y_n, z)\} \le D_g(x_n, z)$ for all z in $F(T)$ and $n = 1, 2, \ldots$,*

(2) *$\lim_{n\to\infty} D_g(x_n, z)$ exists for any z in $F(T)$.*

Proof. Let $z \in F(T)$. In view of (23), we have

$$\begin{aligned} D_g(y_n, z) &= D_g(\beta_n T x_n + (1-\beta_n) x_n, z) \\ &\le \beta_n D_g(T x_n, z) + (1-\beta_n) D_g(x_n, z) \\ &\le \beta_n D_g(x_n, z) + (1-\beta_n) D_g(x_n, z) \\ &= D_g(x_n, z). \end{aligned} \tag{50}$$

Consequently,

$$\begin{aligned} D_g(x_{n+1}, z) &= D_g(\gamma_n T y_n + (1-\gamma_n) x_n, z) \\ &\le \gamma_n D_g(T y_n, z) + (1-\gamma_n) D_g(x_n, z) \\ &\le \gamma_n D_g(y_n, z) + (1-\gamma_n) D_g(x_n, z) \\ &\le \gamma_n D_g(x_n, z) + (1-\gamma_n) D_g(x_n, z) \\ &= D_g(x_n, z). \end{aligned} \tag{51}$$

This implies that $\{D_g(x_n, z)\}_{n\in\mathbb{N}}$ is a bounded and nonincreasing sequence for all z in $F(T)$. Thus we have that $\lim_{n\to\infty} D_g(x_n, z)$ exists for any z in $F(T)$. □

Theorem 16. *Let C be a nonempty, closed, and convex subset of a reflexive Banach space E. Let $g : E \to \mathbb{R}$ be strictly convex, Gâteaux differentiable, locally bounded, and locally uniformly convex on E. Let $T : C \to C$ be a Bregman nonspreading and Bregman skew quasi-nonexpansive mapping. Let $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ be sequences in $[0,1]$, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence with x_1 in C defined by* (1).

(a) *If $\{x_n\}_{n\in\mathbb{N}}$ is bounded and $\liminf_{n\to\infty}\|Tx_n - x_n\| = 0$, then the fixed point set $F(T) \ne \emptyset$.*

(b) *Assume $F(T) \ne \emptyset$. Then $\{x_n\}_{n\in\mathbb{N}}$ is bounded.*

  (i) *$\lim_{n\to\infty}\|Tx_n - x_n\| = 0$ when $\liminf_{n\to\infty}\gamma_n(1-\gamma_n) > 0$ and $\lim_{n\to\infty}\beta_n = 1$.*

  (ii) *$\liminf_{n\to\infty}\|Tx_n - x_n\| = 0$ when either*

    (1) *$\limsup_{n\to\infty}\gamma_n(1-\gamma_n) > 0$ and $\lim_{n\to\infty}\beta_n = 1$ or*

    (2) *$\liminf_{n\to\infty}\gamma_n(1-\gamma_n) > 0$ and $\limsup_{n\to\infty}\beta_n = 1$.*

Proof. Assume that $\{x_n\}_{n\in\mathbb{N}}$ is bounded and $\liminf_{n\to\infty}\|Tx_n - x_n\| = 0$. Consequently, there is a bounded subsequence $\{Tx_{n_k}\}_{k\in\mathbb{N}}$ of $\{Tx_n\}_{n\in\mathbb{N}}$ such that $\lim_{k\to\infty}\|Tx_{n_k} - x_{n_k}\| = 0$. Since ∇g is uniformly norm-to-norm continuous on bounded subsets of E (see, e.g., [14]),

$$\lim_{k\to\infty}\left\|\nabla g(Tx_{n_k}) - \nabla g(x_{n_k})\right\| = 0. \tag{52}$$

In view of Proposition 9, we conclude that $\mathrm{BA}(C,\{x_{n_k}\})=\{z\}$ for some z in C. Let

$$M_2=\sup\left\{\|T(z)\|,\left\|Tx_{n_k}\right\|,\|\nabla g(z)\|,\right.\\ \left.\|\nabla g(Tz)\|:k\in\mathbb{N}\right\}<+\infty. \tag{53}$$

It follows from Lemma 11 that

$$\begin{aligned}D_g&\left(x_{n_k},Tz\right)\\ &\le D_g\left(x_{n_k},z\right)+D_g\left(Tx_{n_k},x_{n_k}\right)\\ &\quad+\left\langle x_{n_k}-Tx_{n_k},\nabla g(z)-\nabla g(Tz)\right\rangle\\ &\quad+\left\langle Tx_{n_k}-Tz,\nabla g\left(x_{n_k}\right)-\nabla g\left(Tx_{n_k}\right)\right\rangle\\ &\le D_g\left(x_{n_k},z\right)+D_g\left(Tx_{n_k},x_{n_k}\right)\\ &\quad+\left\|x_{n_k}-Tx_{n_k}\right\|\|\nabla g(z)-\nabla g(Tz)\|\\ &\quad+\left\|Tx_{n_k}-Tz\right\|\left\|\nabla g\left(x_{n_k}\right)-\nabla g\left(Tx_{n_k}\right)\right\|\\ &\le D_g\left(x_{n_k},z\right)+D_g\left(Tx_{n_k},x_{n_k}\right)\\ &\quad+2M_2\left\|x_{n_k}-Tx_{n_k}\right\|\\ &\quad+2M_2\left\|\nabla g\left(x_{n_k}\right)-\nabla g\left(Tx_{n_k}\right)\right\|,\quad k=1,2,\ldots.\end{aligned} \tag{54}$$

This implies

$$\limsup_{k\to\infty}D_g\left(x_{n_k},Tz\right)\le\limsup_{k\to\infty}D_g\left(x_{n_k},z\right). \tag{55}$$

From the Bregman-Opial-like property, we obtain $Tz=z$.

Let $F(T)\neq\emptyset$ and let $z\in F(T)$. It follows from Lemma 15 that $\lim_{n\to\infty}\|x_n-z\|$ exists and hence $\{x_n\}_{n\in\mathbb{N}}$ is bounded. This implies that the sequence $\{Ty_n\}_{n\in\mathbb{N}}$ is bounded too. Let $s_1=\sup\{\|x_n\|,\|Ty_n\|:n\in\mathbb{N}\}<\infty$. In view of (23), we obtain a continuous, strictly increasing, and convex function $\rho_{s_1}:[0,+\infty)\to[0,+\infty)$ with $\rho_{s_1}(0)=0$ such that

$$\begin{aligned}D_g(x_{n+1},z)&=D_g(\gamma_nTy_n+(1-\gamma_n)x_n,z)\\ &\le\gamma_nD_g(Ty_n,z)+(1-\gamma_n)D_g(x_n,z)\\ &\quad-\gamma_n(1-\gamma_n)\rho_{s_1}(\|Ty_n-x_n\|)\\ &\le\gamma_nD_g(y_n,z)+(1-\gamma_n)D_g(x_n,z)\\ &\quad-\gamma_n(1-\gamma_n)\rho_{s_1}(\|Ty_n-x_n\|)\\ &\le\gamma_nD_g(x_n,z)+(1-\gamma_n)D_g(x_n,z)\\ &\quad-\gamma_n(1-\gamma_n)\rho_{s_1}(\|Ty_n-x_n\|)\\ &=D_g(x_n,z)-\gamma_n(1-\gamma_n)\rho_{s_1}(\|Ty_n-x_n\|).\end{aligned} \tag{56}$$

Consequently, we conclude that

$$\begin{aligned}\gamma_n&(1-\gamma_n)\rho_{s_1}(\|Ty_n-x_n\|)\\ &\le D_g(x_n,z)-D_g(x_{n+1},z)\\ &\longrightarrow 0,\quad\text{as } n\longrightarrow\infty.\end{aligned} \tag{57}$$

It follows that

$$\liminf_{n\to\infty}\rho_{s_1}(\|Ty_n-x_n\|)=0\ \text{whenever}\ \limsup_{n\to\infty}\gamma_n(1-\gamma_n)>0. \tag{58}$$

From the property of ρ_{s_1} we deduce that

$$\liminf_{n\to\infty}\|Ty_n-x_n\|=0\ \text{whenever}\ \limsup_{n\to\infty}\gamma_n(1-\gamma_n)>0. \tag{59}$$

In the same manner, we also obtain that

$$\lim_{n\to\infty}\|Ty_n-x_n\|=0\ \text{whenever}\ \liminf_{n\to\infty}\gamma_n(1-\gamma_n)>0. \tag{60}$$

Since ∇g is uniformly norm-to-norm continuous on bounded subsets of E (see, e.g., [14]), we arrive at

$$\lim_{n\to\infty}\|\nabla g(Ty_n)-\nabla g(x_n)\|=0. \tag{61}$$

On the other hand, from (1) we get

$$\begin{aligned}Tx_n-y_n&=(1-\beta_n)(Tx_n-x_n),\\ x_n-y_n&=\beta_n(x_n-Tx_n).\end{aligned} \tag{62}$$

Assuming first $\liminf_{n\to\infty}\gamma_n(1-\gamma_n)>0$. By (60) we see that

$$M_3:=\sup\{\|\nabla g(x_n)\|,\|\nabla g(Tx_n)\|,\\ \|\nabla g(Ty_n)\|:n\in\mathbb{N}\}<+\infty. \tag{63}$$

Since T is Bregman nonspreading, in view of (24), (25), and (62), we obtain

$$\begin{aligned}D_g&(x_n,Tx_n)\\ &=D_g(x_n,Ty_n)+D_g(Ty_n,Tx_n)\\ &\quad+\langle x_n-Ty_n,\nabla g(Ty_n)-\nabla g(Tx_n)\rangle\\ &\le D_g(x_n,Ty_n)\\ &\quad+\left[D_g(Ty_n,x_n)+D_g(Tx_n,y_n)-D_g(Tx_n,Ty_n)\right]\\ &\quad+\|x_n-Ty_n\|\|\nabla g(Ty_n)-\nabla g(Tx_n)\|\end{aligned}$$

$$
\begin{aligned}
&\leq D_g(x_n, Ty_n) \\
&\quad + \left[-D_g(x_n, Ty_n) + \langle x_n - Ty_n, \nabla g(x_n) - \nabla g(Ty_n)\rangle\right] \\
&\quad + \left[-D_g(y_n, Tx_n) + \langle y_n - Tx_n, \nabla g(y_n) - \nabla g(Tx_n)\rangle\right] \\
&\quad + \|x_n - Ty_n\| \|\nabla g(Ty_n) - \nabla g(Tx_n)\| \\
&\leq \|x_n - Ty_n\| \|\nabla g(x_n) - \nabla g(Ty_n)\| \\
&\quad + \|y_n - Tx_n\| \|\nabla g(y_n) - \nabla g(Tx_n)\| \\
&\quad + \|x_n - Ty_n\| \|\nabla g(Ty_n) - \nabla g(Tx_n)\| \\
&= (1-\beta_n) \|x_n - Tx_n\| \|\nabla g(y_n) - \nabla g(Tx_n)\| \\
&\quad + \|x_n - Ty_n\| [\|\nabla g(x_n) - \nabla g(Ty_n) \\
&\qquad\qquad + \|\nabla g(Ty_n) - \nabla g(Tx_n)\|] \\
&\leq 2(1-\beta_n) M_3 \|x_n - Tx_n\| + 4M_3 \|x_n - Ty_n\|.
\end{aligned}
\tag{64}
$$

When $\lim_{n\to\infty} \beta_n = 1$, we conclude that

$$\lim_{n\to\infty} D_g(x_n, Tx_n) = 0. \tag{65}$$

In view of Lemma 2, we have that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{66}$$

Finally, we assume $\limsup_{n\to\infty} \gamma_n(1-\gamma_n) > 0$ and $\lim_{n\to\infty}\beta_n = 1$ instead. By (59) we have subsequences $\{x_{n_k}\}_{k\in\mathbb{N}}$ and $\{y_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$, respectively, such that

$$\lim_{k\to\infty} \left\|Ty_{n_k} - x_{n_k}\right\| = 0. \tag{67}$$

Replacing M_3 with the finite number $\sup\{\|\nabla g(x_{n_k})\|, \|\nabla g(Tx_{n_k})\|, \|\nabla g(Ty_{n_k})\| : k \in \mathbb{N}\} < +\infty$, and dealing with the subsequences $\{x_{n_k}\}_{k\in\mathbb{N}}$ and $\{y_{n_k}\}_{k\in\mathbb{N}}$ in (60) and (62). Passing to a further subsequence if necessary, we will arrive at the desired conclusion with (66) that $\lim_{k\to\infty}\|Tx_{n_k} - x_{n_k}\| = 0$. Hence, $\liminf_{n\to\infty}\|Tx_n - x_n\| = 0$. The other case can be argued similarly. □

Theorem 17. *Let C be a nonempty, closed, and convex subset of a reflexive Banach space E. Let $g : E \to \mathbb{R}$ be strictly convex, Gâteaux differentiable, locally bounded, and locally uniformly convex on E. Let $T : C \to C$ be a Bregman nonspreading and Bregman skew quasi-nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ be sequences in $[0,1]$, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence with x_1 in C defined by (1). Assume that $\liminf_{n\to\infty}\gamma_n(1-\gamma_n) > 0$ and $\lim_{n\to\infty}\beta_n = 1$. Then $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to a fixed point of T.*

Proof. It follows from Theorem 16 that $\{x_n\}_{n\in N}$ is bounded and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. Since E is reflexive, then there exists a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{n_i} \rightharpoonup p \in C$ as $i \to \infty$. By Proposition 12, $p \in F(T)$. We claim that $x_n \rightharpoonup p$ as $n \to \infty$. If not, then there exists a subsequence $\{x_{n_j}\}_{j\in\mathbb{N}}$ of $\{x_n\}_{n\in N}$ such that $\{x_{n_j}\}_{j\in\mathbb{N}}$ converges weakly to some q in C with $p \neq q$. In view of Proposition 12 again, we conclude that $q \in F(T)$. By Lemma 15, $\lim_{n\to\infty}D_g(x_n, z)$ exists for all z in $F(T)$. Thus we obtain by the Bregman-Opial-like property that

$$
\begin{aligned}
&\lim_{n\to\infty} D_g(x_n, p) \\
&\quad = \lim_{i\to\infty} D_g\left(x_{n_i}, p\right) < \lim_{i\to\infty} D_g\left(x_{n_i}, q\right) \\
&\quad = \lim_{n\to\infty} D_g(x_n, q) = \lim_{j\to\infty} D_g\left(x_{n_j}, q\right) \\
&\quad < \lim_{j\to\infty} D_g\left(x_{n_j}, p\right) = \lim_{n\to\infty} D_g(x_n, p).
\end{aligned}
\tag{68}
$$

This is a contradiction. Thus we have $p = q$, and the desired assertion follows. □

Theorem 18. *Let C be a nonempty, compact, and convex subset of a reflexive Banach space E. Let $g : E \to \mathbb{R}$ be strictly convex, Gâteaux differentiable, locally bounded, and uniformly convex on bounded sets. Let $T : C \to C$ be a Bregman nonspreading and Bregman skew quasi-nonexpansive mapping. Let $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ be sequences in $[0,1]$. Assume that either $\limsup_{n\to\infty}\gamma_n(1-\gamma_n) > 0$ and $\lim_{n\to\infty}\beta_n = 1$ or $\liminf_{n\to\infty}\gamma_n(1-\gamma_n) > 0$ and $\limsup_{n\to\infty}\beta_n = 1$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence with x_1 in C defined by (1). Then $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to a fixed point z of T.*

Proof. By Corollary 14, we see that the fixed point set $F(T)$ of T is nonempty. In view of Theorem 16, we obtain that $\{x_n\}_{n\in\mathbb{N}}$ is bounded and $\liminf_{n\to\infty}\|Tx_n - x_n\| = 0$. By the compactness of C, there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $\{x_{n_k}\}_{k\in\mathbb{N}}$ converges strongly to some z in C. In view of Lemma 2 we deduce that $\lim_{k\to\infty}D_g(x_{n_k}, z) = 0$. We can even assume that $\lim_{k\to\infty}\|Tx_{n_k} - x_{n_k}\| = 0$, and in particular, $\{Tx_{n_k}\}_{k\in\mathbb{N}}$ is bounded. Since ∇g is uniformly norm-to-norm continuous on bounded subsets of E (see, e.g., [14]),

$$\lim_{k\to\infty} \left\|\nabla g\left(Tx_{n_k}\right) - \nabla g\left(x_{n_k}\right)\right\| = 0. \tag{69}$$

Let $M_4 = \sup\{\|Tz\|, \|Tx_{n_k}\|, \|\nabla g(z)\|, \|\nabla g(Tz)\| : k \in \mathbb{N}\} < +\infty$. In view of Lemma 11, we obtain

$$
\begin{aligned}
&D_g\left(x_{n_k}, Tz\right) \\
&\quad \leq D_g\left(x_{n_k}, z\right) + D_g\left(Tx_{n_k}, x_{n_k}\right) \\
&\qquad + \left\langle x_{n_k} - Tx_{n_k}, \nabla g(z) - \nabla g(Tz)\right\rangle \\
&\qquad + \left\langle Tx_{n_k} - Tz, \nabla g\left(x_{n_k}\right) - \nabla g\left(Tx_{n_k}\right)\right\rangle \\
&\quad \leq D_g\left(x_{n_k}, z\right) + D_g\left(Tx_{n_k}, x_{n_k}\right) \\
&\qquad + 2M_4\left[\left\|x_{n_k} - Tx_{n_k}\right\| + \left\|\nabla g\left(x_{n_k}\right) - \nabla g\left(Tx_{n_k}\right)\right\|\right]
\end{aligned}
\tag{70}
$$

for all k in $\mathbb{N}$.

It follows that $\lim_{k\to\infty}\|x_{n_k} - Tz\| = 0$. Thus we have $Tz = z$. In view of Lemmas 15 and 2, we conclude that $\lim_{n\to\infty}\|x_n - z\| = 0$. Therefore, z is the strong limit of the sequence $\{x_n\}_{n\in\mathbb{N}}$. □

5. Bregman-Ishikawa's Type Iteration for Bregman Nonspreading Mappings

We propose the following Bregman-Ishikawa's type iteration. Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function. Let C be a nonempty, closed, and convex subset of E. Let $T : C \to C$ be a Bregman nonspreading mapping such that the fixed point set $F(T)$ is nonempty. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be two sequences defined by

$$\begin{aligned} y_n &= \nabla g^* \left[\beta_n \nabla g(Tx_n) + (1-\beta_n)\nabla g(x_n)\right], \\ x_{n+1} &= \mathrm{proj}_C^g \left(\nabla g^* \left[\gamma_n \nabla g(Ty_n) + (1-\gamma_n)\nabla g(x_n)\right]\right), \end{aligned} \tag{71}$$

where $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ are arbitrary sequences in $[0,1]$.

Lemma 19. *Let C be a nonempty, closed, and convex subset of a reflexive Banach space E. Let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function. Let $T : C \to C$ be a Bregman quasi-nonexpansive mapping. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be two sequences defined by (71) such that $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ are arbitrary sequences in $[0,1]$. Then the following assertions hold:*

(1) $\max\{D_g(z,x_{n+1}), D_g(z,y_n)\} \le D_g(z,x_n)$ *for all z in $F(T)$ and $n = 1,2,\ldots$,*

(2) $\lim_{n\to\infty} D_g(z,x_n)$ *exists for any z in $F(T)$.*

Proof. Let $z \in F(T)$. In view of Lemma 8 and (71), we conclude that

$$\begin{aligned} D_g(z,y_n) &= D_g\left(z, \nabla g^*\left[\beta_n \nabla g(Tx_n) + (1-\beta_n)\nabla g(x_n)\right]\right) \\ &= V\left(z, \beta_n \nabla g(Tx_n) + (1-\beta_n)\nabla g(x_n)\right) \\ &\le \beta_n V\left(z, \nabla g(Tx_n)\right) + (1-\beta_n) V\left(z, \nabla g(x_n)\right) \\ &= \beta_n D_g(z, Tx_n) + (1-\beta_n) D_g(z, x_n) \\ &\le \beta_n D_g(z, x_n) + (1-\beta_n) D_g(z, x_n) \\ &= D_g(z, x_n). \end{aligned} \tag{72}$$

Consequently, using (35) we have

$$\begin{aligned} & D_g(z, x_{n+1}) \\ &= D_g\left(z, \mathrm{proj}_C^g\left(\nabla g^*\left[\gamma_n \nabla g(Ty_n) + (1-\gamma_n)\nabla g(x_n)\right]\right)\right) \\ &\le D_g\left(z, \nabla g^*\left[\gamma_n \nabla g(Ty_n) + (1-\gamma_n)\nabla g(x_n)\right]\right) \\ &= V\left(z, \gamma_n \nabla g(Ty_n) + (1-\gamma_n)\nabla g(x_n)\right) \\ &\le \gamma_n V\left(z, \nabla g(Ty_n)\right) + (1-\gamma_n) V\left(z, \nabla g(x_n)\right) \\ &= \gamma_n D_g(z, Ty_n) + (1-\gamma_n) D_g(z, x_n) \\ &\le \gamma_n D_g(z, y_n) + (1-\gamma_n) D_g(z, x_n) \\ &\le \gamma_n D_g(z, x_n) + (1-\gamma_n) D_g(z, x_n) \\ &= D_g(z, x_n). \end{aligned} \tag{73}$$

This implies that $\{D_g(z,x_n)\}_{n\in\mathbb{N}}$ is a bounded and nonincreasing sequence for all z in $F(T)$. Thus we have that $\lim_{n\to\infty} D_g(z,x_n)$ exists for any z in $F(T)$. □

Theorem 20. *Let C be a nonempty, closed, and convex subset of a reflexive Banach space E. Let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function which is locally bounded, locally uniformly convex, and locally uniformly smooth on E. Let $T : C \to C$ be a Bregman nonspreading mapping. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ be two sequences in $[0,1]$ satisfying the control condition:*

$$\sum_{n=1}^{\infty} \gamma_n \beta_n (1-\beta_n) = +\infty. \tag{74}$$

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by algorithm (71). Then the following are equivalent.

(1) *There exists a bounded sequence $\{x_n\}_{n\in\mathbb{N}} \subset C$ such that $\liminf_{n\to\infty}\|Tx_n - x_n\| = 0$.*

(2) *The fixed point set $F(T) \ne \emptyset$.*

Proof. The implication (1) ⇒ (2) follows similarly as in the first part of the proof of Theorem 16.

For the implication (2) ⇒ (1), we assume $F(T) \ne \emptyset$. The boundedness of the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ follows from Lemma 19 and Definition 4. Since T is a Bregman quasi-nonexpansive mapping, for any q in $F(T)$, we have

$$D_g(q, Tx_n) \le D_g(q, x_n), \quad \forall n \in \mathbb{N}. \tag{75}$$

This, together with Definition 4 and the boundedness of $\{x_n\}_{n\in\mathbb{N}}$, implies that $\{Tx_n\}_{n\in\mathbb{N}}$ is bounded.

The function g is bounded on bounded subsets of E and therefore ∇g is also bounded on bounded subsets of E^* (see, e.g., [12, Proposition 1.1.11] for more details). This implies that the sequences $\{\nabla g(x_n)\}_{n\in\mathbb{N}}$, $\{\nabla g(y_n)\}_{n\in\mathbb{N}}$, $\{\nabla g(Ty_n)\}_{n\in\mathbb{N}}$, and $\{\nabla g(Tx_n)\}_{n\in\mathbb{N}}$ are bounded in E^*.

In view of Proposition 7, we have that $\mathrm{dom}\, g^* = E^*$ and g^* is strongly coercive and uniformly convex on bounded subsets of E^*. Let $s_2 = \sup\{\|\nabla g(x_n)\|, \|\nabla g(Tx_n)\| : n \in \mathbb{N}\} < \infty$ and let $\rho_{s_2}^* : E^* \to \mathbb{R}$ be the gauge of uniform convexity of the conjugate function g^*.

Claim. For any p in $F(T)$ and n in $\mathbb{N}$,

$$\begin{aligned} D_g(p, y_n) \le{}& D_g(p, x_n) \\ & - \beta_n (1-\beta_n) \rho_{s_2}^* \left(\|\nabla g(x_n) - \nabla g(Tx_n)\|\right). \end{aligned} \tag{76}$$

Let $p \in F(T)$. For each n in $\mathbb{N}$, it follows from the definition of Bregman distance (4), Lemma 8, (23), and (71) that

$$\begin{aligned} & D_g(p, y_n) \\ &= g(p) - g(y_n) - \langle p - y_n, \nabla g(y_n)\rangle \\ &= g(p) + g^*(\nabla g(y_n)) - \langle y_n, \nabla g(y_n)\rangle \\ &\quad - \langle p, \nabla g(y_n)\rangle + \langle y_n, \nabla g(y_n)\rangle \end{aligned}$$

$$
\begin{aligned}
&= g(p) + g^{*}((1-\beta_n)\nabla g(x_n) + \beta_n \nabla g(Tx_n)) \\
&\quad - \langle p, (1-\beta_n)\nabla g(x_n) + \beta_n \nabla g(Tx_n)\rangle \\
&\le (1-\beta_n) g(p) + \beta_n g(p) \\
&\quad + (1-\beta_n) g^{*}(\nabla g(x_n)) + \beta_n g^{*}(\nabla g(Tx_n)) \\
&\quad - \beta_n (1-\beta_n) \rho_{s_2}^{*}(\|\nabla g(x_n) - \nabla g(Tx_n)\|) \\
&\quad - (1-\beta_n)\langle p, \nabla g(x_n)\rangle - \beta_n \langle p, \nabla g(Tx_n)\rangle \\
&= (1-\beta_n)[g(p) + g^{*}(\nabla g(x_n)) - \langle p, \nabla g(x_n)\rangle] \\
&\quad + \beta_n [g(p) + g^{*}(\nabla g(Tx_n)) - \langle p, \nabla g(Tx_n)\rangle] \\
&\quad - \beta_n (1-\beta_n) \rho_{s_2}^{*}(\|\nabla g(x_n) - \nabla g(Tx_n)\|) \\
&= (1-\beta_n)[g(p) - g(x_n) \\
&\qquad + \langle x_n, \nabla g(x_n)\rangle - \langle p, \nabla g(x_n)\rangle] \\
&\quad + \beta_n [g(p) - g(Tx_n) \\
&\qquad + \langle Tx_n, \nabla g(Tx_n)\rangle - \langle p, \nabla g(Tx_n)\rangle] \\
&\quad - \beta_n (1-\beta_n) \rho_{s_2}^{*}(\|\nabla g(x_n) - \nabla g(Tx_n)\|) \\
&= (1-\beta_n) D(p, x_n) + \beta_n D(p, Tx_n) \\
&\quad - \beta_n (1-\beta_n) \rho_{s_2}^{*}(\|\nabla g(x_n) - \nabla g(Tx_n)\|) \\
&\le (1-\beta_n) D(p, x_n) + \beta_n D(p, x_n) \\
&\quad - \beta_n (1-\beta_n) \rho_{s_2}^{*}(\|\nabla g(x_n) - \nabla g(Tx_n)\|) \\
&= D(p, x_n) - \beta_n (1-\beta_n) \rho_{s_2}^{*}(\|\nabla g(x_n) - \nabla g(Tx_n)\|).
\end{aligned} \tag{77}
$$

In view of Lemma 8 and (76), we obtain

$$
\begin{aligned}
D_g(p, x_{n+1}) &= D_g(p, \nabla g^{*}[\gamma_n \nabla g(Ty_n) + (1-\gamma_n)\nabla g(x_n)]) \\
&= V(p, \gamma_n \nabla g(Ty_n) + (1-\gamma_n)\nabla g(x_n)) \\
&\le \gamma_n V(p, \nabla g(Ty_n)) + (1-\gamma_n) V(p, \nabla g(x_n)) \\
&= \gamma_n D_g(p, Ty_n) + (1-\gamma_n) D_g(p, x_n) \\
&\le \gamma_n D_g(p, y_n) + (1-\gamma_n) D_g(p, x_n) \\
&\le D_g(p, x_n) - \gamma_n \beta_n (1-\beta_n) \rho_{s_2}^{*} \\
&\qquad \times (\|\nabla g(x_n) - \nabla g(Tx_n)\|).
\end{aligned} \tag{78}
$$

Thus we have

$$
\begin{aligned}
&\gamma_n \beta_n (1-\beta_n) \rho_{s_2}^{*}(\|\nabla g(x_n) - \nabla g(T_n x_n)\|) \\
&\quad \le D_g(p, x_n) - D_g(p, x_{n+1}).
\end{aligned} \tag{79}
$$

Since $\{D_g(p, x_n)\}_{n\in\mathbb{N}}$ converges, together with the control condition (74), we have

$$
\liminf_{n\to\infty} \rho_{s_2}^{*}(\|\nabla g(x_n) - \nabla g(Tx_n)\|) = 0. \tag{80}
$$

Therefore, from the property of $\rho_{s_2}^{*}$ we deduce that

$$
\liminf_{n\to\infty} \|\nabla g(x_n) - \nabla g(Tx_n)\| = 0. \tag{81}
$$

Since ∇g^{*} is uniformly norm-to-norm continuous on bounded subsets of E^{*} (see, e.g., [14]), we arrive at

$$
\liminf_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{82}
$$

□

Theorem 21. *Let C be a nonempty, closed, and convex subset of a reflexive Banach space E. Let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function which is locally bounded, locally uniformly convex, and locally uniformly smooth on E. Let $T : C \to C$ be a Bregman nonspreading mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ be two sequences in $[0,1]$ satisfying the control conditions $\sum_{n=1}^{\infty} \gamma_n \beta_n (1-\beta_n) = +\infty$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by the algorithm (71). Then, there exists a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ which converges weakly to a fixed point of T as $i \to \infty$.*

Proof. It follows from Theorem 20 that $\{x_n\}_{n\in\mathbb{N}}$ is bounded and $\liminf_{n\to\infty}\|Tx_n - x_n\| = 0$. Since E is reflexive, then there exists a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{ni} \rightharpoonup p \in C$ as $i \to \infty$. In view of Proposition 12, we conclude that $p \in F(T)$ and the desired conclusion follows. □

The construction of fixed points of nonexpansive mappings via Halpern's algorithm [24] has been extensively investigated recently in the current literature (see, e.g., [2] and the references therein). Numerous results have been proved on Halpern's iterations for nonexpansive mappings in Hilbert and Banach spaces (see, e.g., [11, 25, 26]).

Before dealing with the strong convergence of a Halpern-type iterative algorithm, we need the following lemmas.

Lemma 22 (see [27]). *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ with a subsequence $\{a_{n_i}\}_{i\in\mathbb{N}}$ such that $a_{n_i} < a_{n_i+1}$ for all i in $\mathbb{N}$. Then there exists another subsequence $\{a_{m_k}\}_{k\in\mathbb{N}}$ such that for all (sufficiently large) number k one have*

$$
a_{m_k} \le a_{m_k+1}, \qquad a_k \le a_{m_k+1}. \tag{83}
$$

In fact, one can set $m_k = \max\{j \le k : a_j < a_{j+1}\}$.

Lemma 23 (see [28]). *Let $\{s_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers satisfying*

$$
s_{n+1} \le (1-\gamma_n) s_n + \gamma_n \delta_n, \quad \forall n \ge 1, \tag{84}
$$

where $\{\gamma_n\}_{n\in\mathbb{N}}$ and $\{\delta_n\}_{n\in\mathbb{N}}$ satisfy the following conditions:

(i) *$\{\gamma_n\}_{n\in\mathbb{N}} \subset [0,1]$ and $\sum_{n=1}^{\infty} \gamma_n = +\infty$, or, equivalently, $\prod_{n=1}^{\infty}(1-\gamma_n) = 0$,*

(ii) *$\limsup_{n\to\infty} \delta_n \le 0$, or*

(iii) *$\sum_{n=1}^{\infty} \gamma_n \delta_n < \infty$.*

Then, $\lim_{n\to\infty} s_n = 0$.

Theorem 24. *Let C be a nonempty, closed, and convex subset of a reflexive Banach space E. Let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function which is locally bounded, locally uniformly convex, and locally uniformly smooth on E. Let $T : C \to C$ be a Bregman nonspreading mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ be two sequences in $[0,1]$ satisfying the following control conditions:*

(a) $\lim_{n\to\infty}\alpha_n = 0$;

(b) $\sum_{n=1}^{\infty}\alpha_n = +\infty$;

(c) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by

$$\begin{aligned} &u \in C, x_1 \in C \text{ chosen arbitrarily}, \\ &y_n = \nabla g^* \left[\beta_n \nabla g(x_n) + (1-\beta_n)\nabla g(Tx_n)\right], \\ &x_{n+1} = \operatorname{proj}_C^g \left(\nabla g^* \left[\alpha_n \nabla g(u) \right.\right. \\ &\qquad \left.\left. + (1-\alpha_n)\nabla g(y_n)\right]\right) \quad \text{for } n \text{ in } \mathbb{N}. \end{aligned} \tag{85}$$

Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined in (85) converges strongly to $\operatorname{proj}_{F(T)}^g u$ as $n \to \infty$.

Proof. We divide the proof into several steps. In view of Lemma 10, we conclude that $F(T)$ is closed and convex. Set

$$z = \operatorname{proj}_{F(T)}^g u. \tag{86}$$

Step 1. We prove that $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are bounded sequences in C.

We first show that $\{x_n\}_{n\in\mathbb{N}}$ is bounded. Let $p \in F(T)$ be fixed. In view of Lemma 8 and (85), we have

$$\begin{aligned} &D_g(p, y_n) \\ &\quad = D_g\left(p, \nabla g^* \left[(1-\beta_n)\nabla g(x_n) + \beta_n \nabla g(Tx_n)\right]\right) \\ &\quad = V\left(p, \left[(1-\beta_n)\nabla g(x_n) + \beta_n \nabla g(Tx_n)\right]\right) \\ &\quad \leq (1-\beta_n) V(p, \nabla g(x_n)) + \beta_n V(p, \nabla g(Tx_n)) \\ &\quad = (1-\beta_n) D_g(p, x_n) + \beta_n D_g(p, Tx_n) \\ &\quad \leq (1-\beta_n) D_g(p, x_n) + \beta_n D_g(p, x_n) \\ &\quad = D_g(p, x_n). \end{aligned} \tag{87}$$

This, together with (71), implies that

$$\begin{aligned} &D_g(p, x_{n+1}) \\ &\quad = D_g\left(p, \operatorname{proj}_C^g\left(\nabla g^* \left[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)\right]\right)\right) \\ &\quad \leq D_g\left(p, \nabla g^* \left[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)\right]\right) \\ &\quad = V\left(p, \alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)\right) \\ &\quad \leq \alpha_n V(p, \nabla g(u)) + (1-\alpha_n) V(p, \nabla g(y_n)) \\ &\quad = \alpha_n D_g(p, u) + (1-\alpha_n) D_g(p, y_n) \\ &\quad \leq \alpha_n D_g(p, u) + (1-\alpha_n) D_g(p, y_n) \\ &\quad \leq \alpha_n D_g(p, u) + (1-\alpha_n) D_g(p, x_n) \\ &\quad \leq \max\left\{D_g(p, u), D_g(p, x_n)\right\}. \end{aligned} \tag{88}$$

By induction, we obtain

$$D_g(p, x_{n+1}) \leq \max\left\{D_g(p, u), D_g(p, x_1)\right\} \tag{89}$$

for all n in $\mathbb{N}$. It follows from (89) that the sequence $\{D_g(p, x_n)\}_{n\in\mathbb{N}}$ is bounded and hence there exists $M_7 > 0$ such that

$$D_g(p, x_n) \leq M_7, \quad \forall n \in \mathbb{N}. \tag{90}$$

In view of Definition 4, we deduce that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded. Since T is a Bregman quasi-nonexpansive mapping from C into itself, we conclude that

$$D_g(p, Tx_n) \leq D_g(p, x_n), \quad \forall n \in \mathbb{N}. \tag{91}$$

This, together with Definition 4 and the boundedness of $\{x_n\}_{n\in\mathbb{N}}$, implies that $\{Tx_n\}_{n\in\mathbb{N}}$ is bounded. The function g is bounded on bounded subsets of E and therefore ∇g is also bounded on bounded subsets of E^* (see, e.g., [12, Proposition 1.1.11] for more details). This, together with Step 1, implies that the sequences $\{\nabla g(x_n)\}_{n\in\mathbb{N}}$, $\{\nabla g(y_n)\}_{n\in\mathbb{N}}$, and $\{\nabla g(Tx_n)\}_{n\in\mathbb{N}}$ are bounded in E^*. In view of Proposition 7, we obtain that $\operatorname{dom} g^* = E^*$ and g^* is strongly coercive and uniformly convex on bounded subsets of E. Let $s_3 = \sup\{\|\nabla g(x_n)\|, \|\nabla g(Tx_n)\| : n \in \mathbb{N}\}$ and let $\rho_{s_3}^* : E^* \to \mathbb{R}$ be the gauge of uniform convexity of the conjugate function g^*.

Step 2. We prove that

$$\begin{aligned} D_g(z, y_n) \leq{}& D_g(z, x_n) - \beta_n(1-\beta_n)\rho_{s_3}^* \\ &\times \left(\|\nabla g(x_n) - \nabla g(Tx_n)\|\right), \quad \forall n \in \mathbb{N}. \end{aligned} \tag{92}$$

For each n in $\mathbb{N}$, in view of the definition of Bregman distance (4), Lemma 8, and (30), we obtain

$$\begin{aligned} &D_g(z, y_n) \\ &\quad = g(z) - g(y_n) - \langle z - y_n, \nabla g(y_n)\rangle \\ &\quad = g(z) + g^*(\nabla g(y_n)) - \langle y_n, \nabla g(y_n)\rangle \\ &\qquad - \langle z, \nabla g(y_n)\rangle + \langle y_n, \nabla g(y_n)\rangle \\ &\quad = g(z) + g^*\left((1-\beta_n)\nabla g(x_n) + \beta_n \nabla g(Tx_n)\right) \\ &\qquad - \langle z, (1-\beta_n)\nabla g(x_n) + \beta_n \nabla g(Tx_n)\rangle \end{aligned}$$

$$\begin{aligned}
&\le (1-\beta_n) g(z) + \beta_n g(z) + (1-\beta_n) g^* \\
&\quad \times (\nabla g(x_n)) + \beta_n g^* (\nabla g(Tx_n)) \\
&\quad - \beta_n (1-\beta_n) \rho_{s_3}^* (\|\nabla g(x_n) - \nabla g(Tx_n)\|) \\
&\quad - (1-\beta_n) \langle z, \nabla g(x_n)\rangle - \beta_n \langle z, \nabla g(Tx_n)\rangle \\
&= (1-\beta_n) [g(z) + g^*(\nabla g(x_n)) - \langle z, \nabla g(x_n)\rangle] \\
&\quad + \beta_n [g(z) + g^*(\nabla g(Tx_n)) - \langle z, \nabla g(Tx_n)\rangle] \\
&\quad - \beta_n (1-\beta_n) \rho_{s_3}^* (\|\nabla g(x_n) - \nabla g(Tx_n)\|) \\
&= (1-\beta_n) [g(z) - g(x_n) \\
&\qquad + \langle x_n, \nabla g(x_n)\rangle - \langle z, \nabla g(x_n)\rangle] \\
&\quad + \beta_n [g(z) - g(Tx_n) \\
&\qquad + \langle Tx_n, \nabla g(Tx_n)\rangle - \langle z, \nabla g(Tx_n)\rangle] \\
&- \beta_n (1-\beta_n) \rho_{s_3}^* (\|\nabla g(x_n) - \nabla g(T_n x_n)\|) \\
&= (1-\beta_n) D(z, x_n) + \beta_n D(z, Tx_n) \\
&\quad - \beta_n (1-\beta_n) \rho_{s_3}^* (\|\nabla g(x_n) - \nabla g(Tx_n)\|) \\
&\le (1-\beta_n) D_g(z, x_n) + \beta_n D_g(z, x_n) \\
&\quad - \beta_n (1-\beta_n) \rho_{s_3}^* (\|\nabla g(x_n) - \nabla g(Tx_n)\|) \\
&= D(z, x_n) - \beta_n (1-\beta_n) \rho_{s_3}^* \\
&\quad \times (\|\nabla g(x_n) - \nabla g(Tx_n)\|) .
\end{aligned} \tag{93}$$

In view of Lemma 8 and (92), we obtain

$$\begin{aligned}
&D_g(z, x_{n+1}) \\
&\quad = D_g(z, \mathrm{proj}_C^g (\nabla g^* [\alpha_n \nabla g(u) + (1-\alpha_n) \nabla g(y_n)])) \\
&\quad \le D_g(z, \nabla g^* [\alpha_n \nabla g(u) + (1-\alpha_n) \nabla g(y_n)]) \\
&\quad = V(z, \alpha_n \nabla g(u) + (1-\alpha_n) \nabla g(y_n)) \\
&\quad \le \alpha_n V(z, \nabla g(u)) + (1-\alpha_n) V(z, \nabla g(y_n)) \\
&\quad = \alpha_n D_g(z, u) + (1-\alpha_n) D_g(z, y_n) \\
&\quad \le \alpha_n D_g(z, u) \\
&\qquad + (1-\alpha_n) \big[D_g(z, x_n) - \beta_n (1-\beta_n) \rho_{s_3}^* \\
&\qquad\qquad \times (\|\nabla g(x_n) - \nabla g(Tx_n)\|)\big] .
\end{aligned} \tag{94}$$

Let

$$\begin{aligned}
M_8 := \sup \Big\{ &\big|D_g(z, u) - D_g(z, x_n)\big| + \beta_n (1-\beta_n) \rho_{s_3}^* \\
&\times (\|\nabla g(x_n) - \nabla g(Tx_n)\|) : n \in \mathbb{N}\Big\} .
\end{aligned} \tag{95}$$

It follows from (94) that

$$\begin{aligned}
&\beta_n (1-\beta_n) \rho_{s_3}^* (\|\nabla g(x_n) - \nabla g(Tx_n)\|) \\
&\quad \le D_g(z, x_n) - D_g(z, x_{n+1}) + \alpha_n M_8 .
\end{aligned} \tag{96}$$

Let

$$z_n = \nabla g^* [\alpha_n \nabla g(u) + (1-\alpha_n) \nabla g(y_n)] . \tag{97}$$

Then $x_{n+1} = \mathrm{proj}_C^g(z_n)$ for all n in $\mathbb{N}$. In view of Lemma 8 and (92) we obtain

$$\begin{aligned}
&D_g(z, x_{n+1}) \\
&\quad = D_g(z, \mathrm{proj}_C^g (\nabla g^* [\alpha_n \nabla g(u) + (1-\alpha_n) \nabla g(y_n)])) \\
&\quad \le D_g(z, \nabla g^* [\alpha_n \nabla g(u) + (1-\alpha_n) \nabla g(y_n)]) \\
&\quad = V(z, \alpha_n \nabla g(u) + (1-\alpha_n) \nabla g(y_n)) \\
&\quad \le V(z, \alpha_n \nabla g(u) + (1-\alpha_n) \nabla g(y_n) \\
&\qquad\quad - \alpha_n (\nabla g(u) - \nabla g(z))) \\
&\qquad - \langle \nabla g^* [\alpha_n \nabla g(u) + (1-\alpha_n) \nabla g(y_n)] - z, \\
&\qquad\quad - \alpha_n (\nabla g(u) - \nabla g(z))\rangle \\
&\quad = V(z, \alpha_n \nabla g(z) + (1-\alpha_n) \nabla g(y_n)) \\
&\qquad + \alpha_n \langle z_n - z, \nabla g(u) - \nabla g(z)\rangle \\
&\quad \le \alpha_n V(z, \nabla g(z)) + (1-\alpha_n) V(z, \nabla g(y_n)) \\
&\qquad + \alpha_n \langle z_n - z, \nabla g(u) - \nabla g(z)\rangle \\
&\quad = \alpha_n D_g(z, z) + (1-\alpha_n) D_g(z, y_n) \\
&\qquad + \alpha_n \langle z_n - z, \nabla g(u) - \nabla g(z)\rangle \\
&\quad = (1-\alpha_n) D_g(z, x_n) \\
&\qquad + \alpha_n \langle z_n - z, \nabla g(u) - \nabla g(z)\rangle .
\end{aligned} \tag{98}$$

Step 3. We show that $x_n \to z$ as $n \to \infty$.

Case 1. If there exists n_0 in $\mathbb{N}$ such that $\{D_g(z, x_n)\}_{n=n_0}^{\infty}$ is nonincreasing, then $\{D_g(z, x_n)\}_{n\in\mathbb{N}}$ is convergent. Thus, we have $D_g(z, x_n) - D_g(z, x_{n+1}) \to 0$ as $n \to \infty$. This, together with (96) and conditions (a) and (c), implies that

$$\lim_{n\to\infty} \rho_{s_3}^* (\|\nabla g(x_n) - \nabla g(Tx_n)\|) = 0. \tag{99}$$

Therefore, from the property of $\rho_{s_3}^*$ we deduce that

$$\lim_{n\to\infty} \|\nabla g(x_n) - \nabla g(Tx_n)\| = 0. \tag{100}$$

Since $\nabla g^* = (\nabla g)^{-1}$ (Lemma 5) is uniformly norm-to-norm continuous on bounded subsets of E^* (see, e.g., [14]), we arrive at

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{101}$$

On the other hand, we have

$$\begin{aligned} D_g(Tx_n, y_n) &= D_g(Tx_n, \nabla g^*[\beta_n \nabla g(x_n) + (1-\beta_n)\nabla g(Tx_n)]) \\ &= V(Tx_n, \beta_n \nabla g(x_n) + (1-\beta_n)\nabla g(Tx_n)) \\ &\le \beta_n V(Tx_n, \nabla g(x_n)) \\ &\quad + (1-\beta_n) V(Tx_n, \nabla g(Tx_n)) \\ &= \beta_n D_g(Tx_n, x_n) + (1-\beta_n) D_g(Tx_n, Tx_n) \\ &= \beta_n D_g(Tx_n, x_n). \end{aligned} \tag{102}$$

This, together with Lemma 2 and (101), implies that

$$\lim_{n\to\infty} D_g(Tx_n, y_n) = 0. \tag{103}$$

Similarly, we have

$$\begin{aligned} D_g(y_n, z_n) &\le \alpha_n D_g(y_n, u) + (1-\alpha_n) D_g(y_n, y_n) \\ &= \alpha_n D_g(y_n, u) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{104}$$

In view of Lemma 2 and (101), we conclude that

$$\lim_{n\to\infty} \|y_n - Tx_n\| = 0, \qquad \lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{105}$$

Since $\{x_n\}_{n\in\mathbb{N}}$ is bounded, together with (33) we can assume that there exists a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{n_i} \rightharpoonup y \in F(T)$ (Proposition 12) and

$$\begin{aligned} \limsup_{n\to\infty} \langle x_n - z, \nabla g(u) - \nabla g(z)\rangle &= \lim_{i\to\infty} \left\langle x_{n_i} - z, \nabla g(u) - \nabla g(z)\right\rangle \\ &= \langle y - z, \nabla g(u) - \nabla g(z)\rangle \le 0. \end{aligned} \tag{106}$$

We thus conclude

$$\begin{aligned} \limsup_{n\to\infty} \langle z_n - z, \nabla g(u) - \nabla g(z)\rangle &= \limsup_{n\to\infty} \langle x_n - z, \nabla g(u) - \nabla g(z)\rangle \le 0. \end{aligned} \tag{107}$$

The desired result follows from Lemmas 2 and 23 and (98).

Case 2. Suppose there exists a subsequence $\{n_i\}_{i\in\mathbb{N}}$ of $\{n\}_{n\in\mathbb{N}}$ such that

$$D_g(z, x_{n_i}) < D_g(z, x_{n_i+1}) \tag{108}$$

for all i in $\mathbb{N}$. By Lemma 22, there exists a nondecreasing sequence $\{m_k\}_{k\in\mathbb{N}}$ of positive integers such that $m_k \to \infty$,

$$\begin{aligned} &D_g(z, x_{m_k}) < D_g(z, x_{m_k+1}), \\ &D_g(z, x_k) \le D_g(z, x_{m_k+1}), \quad \forall k \in \mathbb{N}. \end{aligned} \tag{109}$$

This, together with (96), implies that

$$\begin{aligned} \beta_{m_k}(1-\beta_{m_k})\rho^*_{s_3}\left(\left\|\nabla g(x_{m_k}) - \nabla g(Tx_{m_k})\right\|\right) &\le D_g(z, x_{m_k}) - D_g(z, x_{m_k+1}) \\ &\quad + \alpha_{m_k} M_8 \le \alpha_{m_k} M_8, \quad \forall k \in \mathbb{N}. \end{aligned} \tag{110}$$

Then, by conditions (a) and (c), we get

$$\lim_{k\to\infty} \rho^*_{s_3}\left(\left\|\nabla g(x_{m_k}) - \nabla g(Tx_{m_k})\right\|\right) = 0. \tag{111}$$

By the same argument, as in Case 1, we arrive at

$$\begin{aligned} \limsup_{k\to\infty} \left\langle z_{m_k} - z, \nabla g(u) - \nabla g(z)\right\rangle &= \limsup_{k\to\infty} \left\langle x_{m_k} - z, \nabla g(u) - \nabla g(z)\right\rangle \le 0. \end{aligned} \tag{112}$$

It follows from (98) that

$$\begin{aligned} D_g(z, x_{m_k+1}) \le &\left(1-\alpha_{m_k}\right) D_g(z, x_{m_k}) \\ &+ \alpha_{m_k}\left\langle z_{m_k} - z, \nabla g(u) - \nabla g(z)\right\rangle. \end{aligned} \tag{113}$$

Since $D_g(z, x_{m_k}) \le D_g(z, x_{m_k+1})$, we have that

$$\begin{aligned} \alpha_{m_k} D_g(z, x_{m_k}) &\le D_g(z, x_{m_k}) - D_g(z, x_{m_k+1}) \\ &\quad + \alpha_{m_k}\left\langle z_{m_k} - z, \nabla g(u) - \nabla g(z)\right\rangle \\ &\le \alpha_{m_k}\left\langle z_{m_k} - z, \nabla g(u) - \nabla g(z)\right\rangle. \end{aligned} \tag{114}$$

In particular, since $\alpha_{m_k} > 0$, we obtain

$$D_g(z, x_{m_k}) \le \left\langle z_{m_k} - z, \nabla g(u) - \nabla g(z)\right\rangle. \tag{115}$$

In view of (112), we deduce that

$$\lim_{k\to\infty} D_g(z, x_{m_k}) = 0. \tag{116}$$

This, together with (113), implies that

$$\lim_{k\to\infty} D_g(z, x_{m_k+1}) = 0. \tag{117}$$

On the other hand, we have $D_g(z, x_k) \le D_g(z, x_{m_k+1})$ for all k in $\mathbb{N}$. This ensures that $x_k \to z$ as $k \to \infty$ by Lemma 2. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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